47853 Assignment 1

Due Thursday, January 24, in class

The assignment will be out of 25 points. If you get more than 25 points, the extra points will be carried over to your future assignments with half the weight.

Q1. **(10 points)** Let $D = (V, A)$ be a directed graph and take distinct vertices s, t. An (s,t)-*path* is an st-path whose arcs are oriented from s to t. An (s,t)-*cut* is an arc subset of the form

$$\delta^+(U) := \{(u,v) \in A : u \in U, v \in V - U\}, \quad s \in U \subseteq V - \{t\}.$$

It is well-known that

> the maximum number of pairwise arc-disjoint (s,t)-paths is equal to the minimum cardinality of an (s,t)-cut.

Using this statement without proof, prove the following for a bipartite graph $G = (V, E)$:

(a) A *vertex cover* is a subset $U \subseteq V$ such that every edge is incident with at least one vertex in U. Prove that the minimum cardinality of a vertex cover is equal to the maximum cardinality of a matching.

(b) An *edge cover* is a subset $F \subseteq E$ such that every vertex is incident with at least one edge in F. Prove that the minimum cardinality of an edge cover is equal to the maximum cardinality of a stable set.

Q2. **(15 points)** A graph is *chordal* if every circuit of length at least 4 has a chord. Let $G = (V, E)$ be a chordal graph.

(a) A *clique cutset* is a nonempty subset $X \subseteq V$ such that $G[X]$ is a clique and $G - X$ has more connected components that G. Prove that G is either a clique or has a clique cutset. (**Hint.** Show that every minimal cutset, if any, is a clique.)

(b) Prove that the minimum number of stable sets needed to cover V is equal to the maximum cardinality of a clique.

(c) Prove that the minimum number of cliques needed to cover V is equal to the maximum cardinality of a stable set.

Q3. **(5 points)** A matrix M is *totally unimodular* if the determinant of every submatrix is either $0, 1, -1$. Let b be an integral vector. Prove that

(a) the polyhedron $\{x \geq \mathbf{0} : Mx \geq b\}$ is an integral polyhedron,

(b) the linear system $x \geq \mathbf{0}, Mx \geq b$ is totally dual integral.

Q4. **(10 points)** Let

$$A := \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Answer the following questions. Justify your answers.

(a) Is A totally unimodular?

(b) Is A balanced?

(c) Is the set covering polyhedron $\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$ integral?

(d) Is the set covering system $x \geq \mathbf{0}, Ax \geq \mathbf{1}$ totally dual integral?