47853 Assignment 2

Due Friday, February 1, at noon

The assignment will be out of 25 points. If you get more than 25 points, the extra points will be carried over to your future assignments with half the weight.

Q1. (**3 points**) Prove that a $0-1$ totally unimodular matrix is balanced.

Q2. (**7 points**) Prove the following statements:

(a) If M is a balanced matrix, then the linear system $x \geq \mathbf{0}, Mx \geq \mathbf{1}$ is totally dual integral.

(b) Let $G = (V, E)$ be a balanced hypergraph. A *vertex cover* is a subset of vertices that intersects every edge. Then the maximum cardinality of a matching is equal to the minimum cardinality of a vertex cover.

Q3. (**7 points**) Let M be a $0, \pm 1$ matrix. An *equitable bicoloring* of M is a partition of the columns into red columns and blue columns such that for every row, the sum of the blue entries and the sum of the red entries differ by at most one. Prove that if M is totally unimodular, then it has an equitable bicoloring.

Q4. (**3 points**) Let M be a $0, \pm 1$ matrix where every row has an even number of nonzero entries. Prove that if M has an equitable bicoloring, then its columns are linearly dependent.

Q5. (**15 points**) Let M be a $0, \pm 1$ matrix where every submatrix has an equitable bicoloring. In this question, we will prove that M is totally unimodular. To this end, let A be a nonsingular submatrix of M whose proper square submatrices are totally unimodular. It suffices to show that $\det(A) = \pm 1$.

(a) Prove that every entry of $\det(A) \cdot A^{-1}$ is $0, \pm 1$.

(b) Prove that $\det(A)$ is odd.

Hint. Let y be the first column of $\det(A) \cdot A^{-1}$. Then y is a $0, \pm 1$ vector satisfying $Ay = \det(A) \cdot e_1$, where e_1 is the first unit vector.

(c) Prove that $\det(A) = \pm 1$.

Q6. (**15 points**) Let A be a balanced matrix where every row has at least two nonzero entries. Consider the following algorithm that attempts to bicolor A:

Algorithm: while A has an uncolored column:

(i) If there is a row of A all of whose entries, except for exactly one, are colored the same and the remaining one is uncolored, then color the corresponding uncolored column, say e, with the opposite color. We say that e is *forcibly* colored.

(ii) Otherwise, take an arbitrary uncolored column f, and color it arbitrarily. We say that f is *freely* colored.

We will show that this simple algorithm successfully bicolors A. To this end, we will show that at no iteration do we create a row all of whose entries are colored the same. Suppose otherwise. Consider the first iteration, say iteration $k \geq 3$, of the algorithm where a row s has all entries colored the same. Let $1, 2, \dots, k$ be a labeling of the columns of A that have been assigned a color, where 1 is colored first, 2 is colored next, so on and so forth. Notice that column k has a non-zero entry in row s.

For each $i \in [k]$, define the *forcing sequence* $F(i)$ *of* i as follows: $F(i)$ is the sequence that alternates between columns and rows, its first element is a freely colored column, its last element is the column i, and each row in the sequence is followed by the column it forces the color of and preceded by the second to largest column with a non-zero entry in the row. It is obvious how to obtain $F(i)$ in reverse order, for each $i \in [k]$.

(a) Let $j \in [k-1]$ be the second to largest column with a non-zero entry in row s. Prove that $F(j)$ and $F(k)$ have the same initial columns.

Let B be a $0-1$ square $n \times n$ matrix, where $n \geq 3$ is odd. We say that B is *completely odd* if diagonals 1 and 2 are filled with 1s, and for each $t \in \{3, 5, \dots, n\}$, diagonal t is filled with 0s. (Diagonal $t \in [n]$ refers to the n entries $(1,t), (2,t+1), \dots, (n-t+1,n), (n-t+2,1), \dots, (n,t-1)$ of B.)

(b) Let i be the largest common column of $F(j)$ and $F(k)$. Let $F'(j)$ be the subsequence of $F(j)$ that starts from i and ends at j, and let $F'(k)$ be the subsequence of $F(k)$ that starts from i and ends at k. Let B be the submatrix of A obtained as follows: the first column of B is i, followed by the columns of $F'(j)$ (other than i), followed by the columns of $F'(k)$ in reverse order (other than i); the rows of B start with the rows of $F'(j)$, followed by row s, followed by the rows of $F'(k)$ in reverse order. Prove that B is completely odd.

The following part will yield the desired contradiction, as A is assumed to be balanced:

(c) Prove that a completely odd matrix has an odd cycle submatrix.