47853 Packing and Covering: Lecture 12

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February 26, 2019

9 Minimally nonideal clutters

Recall that a clutter is minimally nonideal (mni) if it is nonideal but every proper minor is ideal. There are two classes of mni clutters that behave quite differently from one another. Last time we introduced the first class.

9.1 The deltas

Take an integer $n \geq 3$. Recall that Δ_n is the clutter over ground set $[n]$ whose members are $\{1,2\}, \{1,3\}, \ldots, \{1,n\}, \{2,3,\ldots,n\}$. We proved the following theorem last time:

Theorem 9.2. *Take an integer $n \geq 3$. Then,*

(1) $b(\Delta_n) = \Delta_n$,

(2) $\min\{\mathbf{1}^\top x : M(\Delta_n)x \geq \mathbf{1}, x \geq \mathbf{0}\}$ has no integral optimal solution, and

(3) Δ_n is minimally nonideal.

The deltas form an important class of mni clutters that is tractable, in the sense that it is easy to see whether a clutter has a delta minor or not. To see why, we need the following result:

Theorem 9.3 (Abdi, Cornuéjols, Pashkovich 2018 [1])**.** *Take a clutter $\mathcal{C}$ over ground set E. If there exist an element e and distinct members C_1, C_2, C such that $e \in C_1 \cap C_2$, $e \notin C$ and $C_1 \cup C_2 \subseteq C \cup \{e\}$, then $\mathcal{C}$ has a delta minor through e that can be found in time $O(|E|^2|\mathcal{C}|^2)$.*

Proof. Let us call (C_1, C_2, C) a *bad triple through* e. We may assume that in every proper minor of $\mathcal{C}$ where e is present, no bad triple through e exists. We will prove that $\mathcal{C}$ itself is a delta. The minimality assumption implies that

(1) $C_1 \cap C_2 = \{e\}$,

because for $I := (C_1 \cap C_2) - \{e\}$, the minor $\mathcal{C}/I$ has the bad triple $(C_1 - I, C_2 - I, C - I)$ through e.

The minimality assumption also implies that

(2) $\{e\} \cup C = E$,

because for $J := E - (\{e\} \cup C)$, $\mathcal{C} \setminus J$ has the same bad triple (C_1, C_2, C) through e.

Next we claim that

(3) $|C_1| = |C_2| = 2$.

To see this, suppose for a contradiction that one of C_1, C_2, say C_1, has cardinality at least 3. Pick an element $h \in C_1 - \{e\}$, and note that $h \notin C_2$ by (1). Consider the minor $\mathcal{C}' := \mathcal{C}/h$, for which $C_1' := C_1 - \{h\}$ and $C' := C - \{h\}$ are members. Notice that C_2 contains a member C_2' of $\mathcal{C}'$, for which it is easy to see that $e \in C_2'$ and $C_2' \neq \{e\}$. But now $\mathcal{C}'$ has a bad triple (C_1', C_2', C') through e, a contradiction to our minimality assumption. This proves (3).

Now let $X := \big\{f \in E : \{e, f\} \text{ is a member}\big\}$. So $|X| \geq 2$ by (3), and $X \subseteq C$ by (2). Our last claim is that

(4) $X = C$.

For if not, pick an element $h \in C - X$, and note that $\mathcal{C}/h$ has a bad triple $(C_1, C_2, C - \{h\})$ through e, contradicting the minimality assumption. Thus $X = C$. Hence,

$$\mathcal{C} \supseteq \{\{e, f\} : f \in C\} \cup \{C\}.$$

Since $\{e\} \cup C = E$ by (2), and $\mathcal{C}$ is a clutter, equality must hold above, implying in turn that $\mathcal{C}$ indeed is a delta, as required. $\square$

We are now ready to prove the following:

Theorem 9.4 (Abdi, Cornuéjols, Pashkovich 2018 [1]). *There is an algorithm that given a clutter $\mathcal{C}$ over ground set E finds in time $O(|E|^4|\mathcal{C}|^4)$ a delta minor or certifies that none exists.*

Proof. We claim that the following statements are equivalent:

(i) $\mathcal{C}$ does not have a delta minor,

(ii) for all distinct members C_1, C_2 with $C_1 \cap C_2 \neq \emptyset$ and for all elements e, f, g with $e \in C_1 \cap C_2$, $f \in C_1 - C_2$, $g \in C_2 - C_1$, the following holds: for $X := (C_1 \cup C_2) - \{e, f, g\}$ and $\mathcal{C}' := \mathcal{C}/X$,

- $\{e, f\} \notin \mathcal{C}'$,
- $\{e, g\} \notin \mathcal{C}'$, or
- $\{f, g\}$ is not contained in a member of $\mathcal{C}'$.

(i) ⇒ (ii): Assume that (i) holds. Take $C_1, C_2, e, f, g, X, \mathcal{C}'$ as in (ii) where $\{e, f\} \in \mathcal{C}'$ and $\{e, g\} \in \mathcal{C}'$. Since $\mathcal{C}$ has no delta minor, neither does $\mathcal{C}'$, so by Theorem 9.3, $\{f, g\}$ is not contained in a member of $\mathcal{C}'$, so (ii) holds. **(ii) ⇒ (i)**: Assume that (i) does not hold. Suppose $\mathcal{C}$ has a delta minor obtained after deleting $I \subseteq E$ and contracting $J \subseteq E$. Pick elements $e, f, g \in E - (I \cup J)$ such that $\{e, f\}, \{e, g\}$ are members of the delta minor.

Notice that $\{f, g\}$ is contained in a member of the delta minor, so $\{f, g\}$ is contained in a member of $\mathcal{C}$. Let C_1, C_2 be members of $\mathcal{C}$ such that $\{e, f\} \subseteq C_1 \subseteq \{e, f\} \cup J$ and $\{e, g\} \subseteq C_2 \subseteq \{e, g\} \cup J$. It can be readily checked that C_1, C_2 and e, f, g do not satisfy (ii). Thus (ii) does not hold.

Since (ii) may be verified in time $O(|E|^4|\mathcal{C}|^4)$, and if (ii) does not hold, a delta minor can be found in time $O(|E|^2|\mathcal{C}|^2)$ using Theorem 9.3, we can find a delta minor or certify that none exists in time $O(|E|^4|\mathcal{C}|^4)$. $\square$

9.2 The other minimally nonideal clutters

We now move on to the mni clutters different from the deltas. Take an odd integer $n \geq 5$. Consider the clutter over ground set $[n]$ whose members are

$$\mathcal{C}_n^2 := \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}.$$

The clutter $\mathcal{C}_n^2$, and any clutter isomorphic to it, is called an *odd hole of dimension* n. It may be readily checked that odd holes are mni. In contrast to Theorem 9.4,

Theorem 9.5 (Ding, Feng, Zang 2008 [2])**.** *Let $\mathcal{C}$ be a clutter over ground set E. Then the problem*

Does $\mathcal{C}$ have an odd hole minor?

is NP-complete.

That is, unless P and NP are equal, there is no algorithm for finding an odd hole minor in a clutter $\mathcal{C}$ over ground set E, whose running time is polynomial in $|E|$ and $|\mathcal{C}|$. Theorems 9.4 and 9.5 highlight the difference between the deltas and the other mni clutters. There are many mni clutters: other than the infinite class $\{\mathcal{C}_{2n-1}^2 : n \geq 3\}$ of mni clutters and their blockers $\{b(\mathcal{C}_{2n-1}^2) : n \geq 3\}$, there are at least two other blocking infinite classes of mni clutters different from the deltas [5], as well as many sporadic examples. For instance, the clutter of the lines of the Fano plane

$$\mathbb{L}_7 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,7\},\{2,5,6\},\{3,4,6\},\{3,5,7\}\} = b(\mathbb{L}_7),$$

as well as $\mathcal{C}_9^2 \cup \{\{3,6,9\}\}$ are mni [4]. It may now seem that there is no good characterization of the mni clutters different from the deltas, but this is not the case! Alfred Lehman provided powerful geometric and combinatorial characterizations of these clutters [3]. Before getting to his characterizations, let us briefly study the geometric aspects of ideal clutters and of minor operations. First off, it is easier to work with polytopes rather than polyhedra:

Proposition 9.6. *Take a clutter $\mathcal{C}$ over ground set E. Then $\mathcal{C}$ is ideal if, and only if, $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$ is an integral polytope.*

Proof. This is left as an exercise. $\square$

For a clutter $\mathcal{C}$, denote by $P(\mathcal{C})$ the set covering polytope $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$. Notice that the covers of $\mathcal{C}$ are precisely the integer points of $P(\mathcal{C})$, and that every integer point of $P(\mathcal{C})$ is an extreme point. Moreover, the minors of $\mathcal{C}$ have a natural geometric interpretation in terms of $P(\mathcal{C})$:

Remark 9.7. *Let $\mathcal{C}$ be a clutter over ground set E, and take an element $e \in E$. Then the following statements hold:*

- *$P(\mathcal{C} \setminus e)$ is the restriction $P(\mathcal{C}) \cap \{x : x_e = 1\}$ after dropping coordinate x_e.*
- *$P(\mathcal{C}/e)$ is the restriction $P(\mathcal{C}) \cap \{x : x_e = 0\}$ after dropping coordinate x_e.*

We can now delve into Lehman's characterizations. First up is a lemma that will be very useful. Take an integer $n \geq 2$, and let A be an $n \times n$ matrix with $0-1$ entries and without a row or a column of all ones. We say that A is *cross regular* if whenever $A_{ij} = 0$, the number of ones in column j is equal to the number of ones in row i.

Lemma 9.8. *The following statements hold:*

(1) Take an integer $n \geq 2$, and let A be a $0-1$ $n \times n$ matrix without a row or a column of all ones, and whenever $A_{ij} = 0$, the number of ones in column j is greater than or equal to the number of ones in row i. Then A is cross regular.

(2) Cross regular matrices cannot differ in just one row.

Proof. **(1)** Suppose A is an $n \times n$ matrix. For each row $i \in [n]$ and column $j \in [n]$, denote by r_i the number of ones in row i and by c_j the number of ones in column j. Then

$$\sum_{j\in[n]} c_j = \sum_{j\in[n]} \sum_{i\in[n]:A_{ij}=0} \frac{c_j}{n-c_j} \geq \sum_{j\in[n]} \sum_{i\in[n]:A_{ij}=0} \frac{r_i}{n-r_i} = \sum_{i\in[n]} \sum_{j\in[n]:A_{ij}=0} \frac{r_i}{n-r_i} = \sum_{i\in[n]} r_i.$$

As the left- and right-hand side terms are equal, equality must hold throughout, implying in turn that whenever $A_{ij} = 0$, then $r_i = c_j$. Thus A is cross regular. **(2)** Suppose for a contradiction that $\begin{pmatrix} B \\ a \end{pmatrix}, \begin{pmatrix} B \\ a' \end{pmatrix}$ are cross regular matrices and $a \neq a'$. We may assume that $a_1 = 1$ and $a'_1 = 0$. Since $\begin{pmatrix} B \\ a \end{pmatrix}$ is cross regular, the first column of B has a zero entry, say it is the first entry. Let $k \geq 0$ be the number of ones in the first column of B. Then as $\begin{pmatrix} B \\ a \end{pmatrix}$ is cross regular, the first row of B has $k+1$ ones. However, as $\begin{pmatrix} B \\ a' \end{pmatrix}$ is also cross regular, the first row of B must have k ones, a contradiction. $\square$

Given a full-dimensional polytope $P \subseteq \mathbb{R}^n$ and a vertex $x^\star$, we say that $x^\star$ is *simple* if it belongs to exactly n facets. Notice that if $x^\star$ is simple, then there are exactly n edges emanating from $x^\star$, each of which is defined uniquely by $n-1$ many of the tight facets. As a result, if $x^\star$ is simple, then it has exactly n adjacent vertices. Lehman proved the following geometric characterization of the mni clutters different from the deltas:

Theorem 9.9 (Lehman 1990 [3]). *Let $\mathcal{C}$ be a minimally nonideal clutter over ground set E that is not a delta, and let $n := |E|$. Let $x^\star$ be a fractional extreme point of $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$. Then the following statements hold:*

(1) $\mathbf{0} < x^\star < \mathbf{1}$,

(2) $x^\star$ *lies on exactly* n *facets, that correspond to members* $C_1, \ldots, C_n \in \mathcal{C}$ *– so* $x^\star$ *is a simple vertex,*

(3) the n *neighbors of* $x^\star$ *are integral vertices, that correspond to covers* $B_1, \ldots, B_n$ *labeled so that for distinct* $i, j \in [n]$*,* $|C_i \cap B_i| > 1$ *and* $|C_i \cap B_j| = 1$*,*

(4) $B_1, \ldots, B_n$ *are minimal covers,*

(5) $C_1, \ldots, C_n$ *are precisely the minimum cardinality members of* $\mathcal{C}$*,*

(6) $x^\star$ *is the unique fractional extreme point of* $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$*,*

(7) there is an integer $d \geq 1$ *such that for each* $i \in [n]$*,* $|C_i \cap B_i| = 1 + d$*.*

In particular, $x^\star$ *is the unique fractional extreme point of* $\{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$*.*

Proof. Let $P := P(\mathcal{C}) = \{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$. Then for each element $e \in E$, the clutters $\mathcal{C}/e, \mathcal{C} \setminus e$ are ideal, so the polytopes $P \cap \{x : x_e = 0\}$ and $P \cap \{x : x_e = 1\}$ are integral by Remark 9.7, implying in turn for each fractional extreme point $x^\star$ that $0 < x^\star_e < 1$, so **(1)** holds. (The fact that $\mathcal{C}$ is different from a delta will be first used in Claim 4.)

Claim 1. *Let* $x^\star$ *be a fractional extreme point of* P*, and let* A *be an* $n \times n$ *nonsingular submatrix of* $M(\mathcal{C})$ *such that* $Ax^\star = \mathbf{1}$*. Then* A *is cross regular.*

Proof of Claim. Clearly, A has no all ones row, and since $x^\star$ is the unique solution to $Ax^\star = \mathbf{1}$, A has no all ones column either. To prove that A is cross regular, assume that $A_{11} = 0$. Let C be the member corresponding to the first row of A. By Lemma 9.8 (1), it suffices to show that the number of ones in the first column is greater than or equal to $|C|$. To this end, let $\widehat{x} := (1, x^\star_2, \ldots, x^\star_n) \in P \cap \{x : x_1 = 1\}$. Let F be the smallest face of the polytope $P \cap \{x : x_1 = 1\}$ containing $\widehat{x}$. Notice that $a^\top \widehat{x} = 1$ for every row a of A whose first entry is 0. As these rows are linearly independent, and as $\widehat{x}_1 = 1$, it follows that

$$\dim(F) \leq n - \text{ number of 0s in the first column } - 1 = \text{ number of 1s in the first column } - 1.$$

On the other hand, as $P \cap \{x : x_1 = 1\}$ is an integral polytope, F is also an integral polytope, so

$$\widehat{x} = \sum_{i=1}^{k} \lambda_i \chi_{B_i}$$

for some extreme points $\chi_{B_1}, \ldots, \chi_{B_k}$ of F and some $\lambda > \mathbf{0}$ such that $\sum_{i=1}^k \lambda_i = 1$. Notice for each $i \in [k]$ that B_i is a cover, and as $\widehat{x}(C) = 1$, we get that $|B_i \cap C| = 1$. Since $\widehat{x} > \mathbf{0}$, each element of C appears in at least one B_i, so the matrix whose rows are the χ_{B_i}'s has rank at least $|C|$, implying in turn that the affine dimension of the χ_{B_i}'s is at least $|C| - 1$. As a result,

$$\dim(F) \geq |C| - 1.$$

Putting the last two inequalities gives the desired inequality, as desired. $\Diamond$

To be continued . . . $\square$

References

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