47853 Packing and Covering: Lecture 6

Ahmad Abdi

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5 Perfect graphs

5.3 Star cutsets and antitwins

Let $G = (V, E)$ be a simple graph. Recall that a star cutset is a nonempty $X \subseteq V$ such that

- $G \setminus X$ has more connected components than G, and
- a vertex of X is adjacent to all the other vertices in X.

Last time we proved that,

Lemma 5.10 (Chvátal 1985 [2])**.** *A minimally imperfect graph does not have a star cutset.*

More generally, a *skew partition* is a partition of V into a pair (A, B) such that $G[A]$ is not connected and $\overline{G}[B]$ is not connected. Notice that if (A, B) is a skew partition for G, then it is a skew partition for $\overline{G}$. Notice further that if X is a star cutset and $|X| \geq 2$, then $(V - X, X)$ is a skew partition. In an attempt to generalize

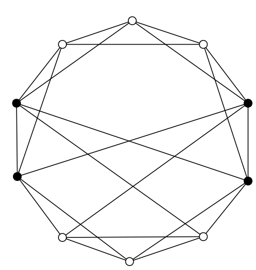

Figure 1: A graph with a skew partition.

Lemma 5.10, Chvátal 1985 [2] conjectured that a minimally imperfect graph does not have a skew partition. He also noticed that any proof of this conjecture would have to go deeper than that of the Star Cutset Lemma 5.10. To elaborate, observe that the proof of the Star Cutset Lemma 5.10 used only the following two properties of minimally imperfect graphs G:

- every proper induced subgraph has a proper $\omega(G)$-vertex-coloring,

- $\omega(G \setminus S) = \omega(G)$ for every stable set S of G.

However, both these properties are satisfied for the graph displayed in Figure 1, yet the graph has a skew partition.

The *length* of a path is the number of edges in it. A path of $\overline{G}$ is called an *antipath of* G. We say that a skew partition (A, B) is *balanced* if

- there is no induced odd path between non-adjacent vertices in B with interior in A,
- there is no induced odd antipath between adjacent vertices in A with interior in B.

Theorem 5.12 (Chudnovsky, Robertson, Seymour, Thomas 2006 [1])**.** *A minimally imperfect graph does not have a balanced skew partition.*

Let $G = (V, E)$ be a simple graph. Distinct vertices u, v are *antitwins* if every other vertex is adjacent to precisely one of u, v. Notice that if u, v are antitwins in G, then they are also antitwins in $\overline{G}$. The proof of the following lemma highlights the special role odd holes and odd antiholes have as minimally imperfect graphs.

Lemma 5.13 (Oraliu 1988 [5])**.** *A minimally imperfect graph does not have antitwins.*

Proof. Let $G = (V, E)$ be a minimally imperfect graph, and let $\omega := \omega(G)$. Suppose for a contradiction that G has antitwins u, v. Let $A \subseteq V - \{u, v\}$ be the neighbors of u other than possibly v, and let $B \subseteq V - \{u, v\}$ be the neighbors of v other than possibly u. Since u, v are antitwins, it follows that A, B partition $V - \{u, v\}$.

Claim 1. *B contains a clique of cardinality $\omega - 1$ that does not extend to a clique of cardinality ω in $A \cup B$.*

Proof of Claim. Let $f : V - \{v\} \to [\omega]$ be an ω-vertex-coloring of $G \setminus v$, and let $S := \{w \in V - \{v\} : f(w) = f(u)\}$. Notice that $u \in S \subseteq \{u\} \cup B$. Recall that $G \setminus S$ has a clique K of cardinality ω. As the vertices of $G \setminus v \setminus S$ are $(\omega - 1)$-vertex-colored, it follows that

- $v \in K$, implying in turn that $K - \{v\} \subseteq B$,
- $K - \{v\}$ does not extend to a clique of cardinality ω in $A \cup B$.

$K - \{v\}$ is the desired clique. $\diamondsuit$

Let $\alpha := \alpha(G)$. By Corollary 5.8, $\overline{G}$ is also minimally imperfect. Thus, since u, v are also antitwins in $\overline{G}$, Claim 1 applied to $\overline{G}$ implies that,

Claim 2. *A contains a stable set of cardinality $\alpha - 1$ that does not extend to a stable set of cardinality α in $A \cup B$.*

Let $C \subseteq B$ be the clique from Claim 1, and let $S \subseteq A$ be the stable set from Claim 2. Among all the vertices in C, pick one x with the least number of neighbors in S. Since S does not extend to a stable set in $A \cup B$, it follows that x has a neighbor $y \in S$. Since C does not extend to a clique in $A \cup B$, y has a non-neighbor $z \in C$. As z has at least as many neighbors in S as x does, there is a vertex $t \in S$ that is a neighbor of z but is not a neighbor of x. Observe now that $\{u, y, x, z, t\}$ induces an odd hole (and an odd antihole), which is imperfect, thereby contradicting the minimality of G. $\square$

Let $G = (V, E)$ be a simple graph. Take disjoint nonempty subsets $A, B \subseteq V$ such that $|A| + |B| \geq 3$ and $|V - (A \cup B)| \geq 2$. The pair (A, B) is *homogeneous* if for each $v \in V - (A \cup B)$,

- if v is adjacent to a vertex of A, then it is adjacent to all of A, and
- if v is adjacent to a vertex of B, then it is adjacent to all of B.

Note that if (A, B) is homogeneous for G, then it is homogeneous for $\overline{G}$. Observe that if $|V(G)| \geq 5$ and u, v are antitwins both of which have a neighbor in $V(G) - \{u, v\}$, then $(N(u) - \{v\}, N(v) - \{u\})$ is homogeneous, where $N(u), N(v)$ denote the neighbors of u, v, respectively. The following theorem generalizes the Antitwin Lemma 5.13:

Theorem 5.14 (Chvátal and Sbihi 1987 [3])**.** *A minimally imperfect graph does not have a homogeneous pair.*

Let $G = (V, E)$ be a simple graph. A 2*-join* is a partition of V into parts V_1, V_2 and nonempty disjoint subsets $A_1, B_1 \subseteq V_1$ and $A_2, B_2 \subseteq V_2$ such that

- $|V_1| \geq 3$ and $|V_2| \geq 3$,
- all the vertices in A_1 are adjacent to all the vertices in A_2, and all the vertices in B_1 are adjacent to all the vertices in B_2,
- there are no other adjacencies between V_1 and V_2.

Notice that an odd circuit of length at least 7 has a 2-join.

Theorem 5.15 (Cornuéjols and Cunningham 1985 [4])**.** *Let G be a minimally imperfect graph. If G has a* 2*-join, then it is an odd hole, and if* $\overline{G}$ *has a* 2*-join, then G is an odd antihole.*

5.4 The strong perfect graph theorem

Let $G = (V, E)$ be a simple graph. We say that G is *Berge* if it has no odd hole and no odd antihole. Clearly, the complement of a Berge graph, as well as its induced subgraphs, are also Berge. By Remark 5.9, a perfect graph is always Berge. Conversely, the strong perfect graph theorem proves that a Berge graph is always perfect. The main idea behind the proof is that Berge graphs are a very small (yet rich) class of graphs, and a lot more than just perfection can be said about them. It is shown that apart from a few basic classes of graphs that happen to be perfect, Berge graphs enjoy properties that we saw in the preceding section do not hold for minimally imperfect graphs.

As for the basic classes of Berge graphs, we need a definition. We say that a simple graph G is a *double split graph* if $V(G)$ can be partitioned into four parts $\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_m\}, \{c_1, \ldots, c_n\}$ and $\{d_1, \ldots, d_n\}$ for some $m, n \geq 2$ such that

- for each $i \in [m]$, a_i and b_i are adjacent, and for each $j \in [n]$, c_j and d_j are not adjacent,

- for $1 \leq i < i' \leq m$, there are no edges between $\{a_i, b_i\}, \{a_{i'}, b_{i'}\}$, and for $1 \leq j < j' \leq n$, the four edges betwen $\{c_j, d_j\}, \{c_{j'}, d_{j'}\}$ are present,
- for $i \in [m]$ and $j \in [n]$, there are precisely two edges between $\{a_i, b_i\}, \{c_j, d_j\}$, and these two edges have no vertex in common.

Notice that if a graph is a double split graph, then so is its complement. We leave the following as an exercise:

Proposition 5.16. *Double split graphs are perfect.*

Let us say that a simple graph G is *basic* if either

- G or $\overline{G}$ is bipartite,
- G or $\overline{G}$ is the line graph of a bipartite graph, or
- G is a double split graph.

Clearly, if a graph is basic, then so is its complement. Notice that by Corollary 5.4 and Proposition 5.16, basic graphs are perfect, and so they are Berge. The following theorem is the main piece to proving that Berge graphs are perfect:

Theorem 5.17 (Chudnovsky, Robertson, Seymour, Thomas 2006 [1])**.** *Let G be a Berge graph that is not basic. Then either G has a balanced skew partition, or G has a homogeneous pair, or one of $G, \overline{G}$ has a 2-join.*

Combining this result with the results from the previous section, we get the strong perfect graph theorem:

Theorem 5.18. *A graph is perfect if, and only if, it has no odd hole and no odd antihole.*

Proof. Let G be a simple graph. $(\Rightarrow)$ If G is perfect, then by Remark 5.9, G has no odd hole and no odd antihole. $(\Leftarrow)$ Suppose conversely that G has no odd hole and no odd antihole, that is, G is Berge. Suppose for a contradiction that G is not perfect. We may assume that G is minimally imperfect. Since G is imperfect, it follows that G is not basic. Thus, by Theorem 5.17, either G has a balanced skew partition, or G has a homogeneous pair, or one of $G, \overline{G}$ has a 2-join. It follows from Theorems 5.12 and 5.14 that one of $G, \overline{G}$ has a 2-join. But then Theorem 5.15 implies that G is either an odd hole or an odd antihole, a contradiction as G is Berge. Thus G is perfect. □

As a consequence,

Corollary 5.19. *Every simple graph G satisfies at least one of the following statements:*

- $\chi(G) = \omega(G)$, *or*
- *G has an odd hole or an odd antihole.*

6 Perfect matrices

Let $G = (V, E)$ be a perfect graph. Let A be the $0-1$ matrix whose columns are labeled by V and whose rows are the incidence vectors of the stable sets of G. Take weights $c \in \mathbb{Z}_+^V$. Consider the set packing primal-dual pair

$$(P) \quad \begin{array}{ll} \max & c^\top x \\ \text{s.t.} & Ax \leq \mathbf{1} \\ & x \geq \mathbf{0} \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{ll} \min & \mathbf{1}^\top y \\ \text{s.t.} & A^\top y \geq c \\ & y \geq \mathbf{0}. \end{array}$$

We can rewrite the primal as

$$(P) \quad \begin{array}{ll} \max & \sum (c_v x_v : v \in V) \\ \text{s.t.} & \sum (x_v : v \in S) \leq 1 \quad \forall \text{ stable sets } S \\ & x_v \geq 0 \quad \forall v \in V. \end{array}$$

Observe that a clique gives a feasible solution to this program. So the maximum weight of a clique is a lower bound on the optimal value of (P). To make this precise, let G_c be the graph obtained from G after replacing each vertex v by c_v duplicates. (If $c_v = 0$ then delete v.) Notice that by Theorem 5.11, G_c is also a perfect graph. Observe that the maximum weight of a clique of G is equal to the maximum cardinality $\omega(G_c)$ of a clique of G_c. Thus, $\omega(G_c)$ is a lower bound on the optimal value of (P). Let us next rewrite the dual as

$$(D) \quad \begin{array}{ll} \min & \sum (y_S : \text{ stable sets } S) \\ \text{s.t.} & \sum (y_S : \text{ stable sets } S \text{ such that } v \in S) \geq c_v \quad \forall v \in V \\ & y_S \geq 0 \quad \forall \text{ stable sets } S. \end{array}$$

Observe that a covering of $V(G_c)$ using stable sets gives a feasible solution to (D). Thus, the minimum number of stable sets needed to cover $V(G_c)$, which is $\chi(G_c)$, is an upper bound on the optimal value of (D). Since G_c is perfect, we have $\chi(G_c) = \omega(G_c)$, implying in turn that,

Corollary 6.1. *Let G be a perfect graph. Then the set packing system corresponding to the stable sets of G is totally dual integral. In particular, the set packing polytope*

$$\left\{x \in \mathbb{R}_+^V : \sum (x_v : v \in S) \leq 1 \quad \forall \textit{ stable sets } S\right\}$$

is integral.

In fact, we will see that these are essentially the *only* examples of integral set packing polytopes and totally dual integral set packing systems!

References

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