47853 Packing and Covering: Lecture 7

Ahmad Abdi

February 7, 2019

6 Perfect matrices

Let A be a $0-1$ matrix without a column of all zeros. We say that A is *perfect* if the associated set packing polytope $\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$ is integral. Last time we proved the following:

Corollary 6.1. *Let G be a perfect graph. Then the set packing system corresponding to the stable sets of G is totally dual integral. In particular, the incidence matrix of the stable sets of G is a perfect matrix.*

Today we will see that these are essentially the *only* examples of integral set packing polytopes and totally dual integral set packing systems!

6.1 Perfection implies total dual integrality.

From Corollary 6.1 it seems more natural to call a matrix perfect when the corresponding set packing system is totally dual integral. The following amazing result justifies our choice of terminology:

Theorem 6.2 (Fulkerson 1972 [3])**.** *Let A be a perfect matrix. Then the linear system $x \geq \mathbf{0}, Ax \leq \mathbf{1}$ is totally dual integral.*

Proof. Denote by E the column labels of A. Consider the set packing primal-dual pair

$$(P)\quad \begin{array}{ll} \max & c^\top x \\ \text{s.t.} & Ax \leq \mathbf{1} \\ & x \geq \mathbf{0} \end{array} \qquad \text{and} \qquad (D)\quad \begin{array}{ll} \min & \mathbf{1}^\top y \\ \text{s.t.} & A^\top y \geq c \\ & y \geq \mathbf{0} \end{array} \qquad c \in \mathbb{Z}^E.$$

As A is perfect, (P) has an integral optimal solution for all $c \in \mathbb{Z}^E$. We will prove by induction on the optimal value $\omega \in \mathbb{Z}_+$ of (P) that (D) has an integral dual solution for all $c \in \mathbb{Z}^E$. If $\omega = 0$ for some $c \in \mathbb{Z}^E$, then as A has no column of all zeros, it follows that $c \leq \mathbf{0}$, implying in turn that $\mathbf{0}$ is an optimal solution for (D). For the induction step, assume that $\omega \geq 1$ for some $c \in \mathbb{Z}^E$. Take an arbitrary row a of A such that

$$a^\top x^\star = 1 \qquad \text{for all optimal solutions } x^\star \text{ of } (P).$$

(To find this row, take an optimal dual solution $y^\star$, and pick a so that $y^\star_a > 0$; apply the complementary slackness conditions.) We may assume that a is the first row of A. Consider the set packing primal-dual pair

$$(P') \quad \begin{array}{ll} \max & (c-a)^\top x \\ \text{s.t.} & Ax \leq \mathbf{1} \\ & x \geq \mathbf{0} \end{array} \qquad \text{and} \qquad (D') \quad \begin{array}{ll} \min & \mathbf{1}^\top y \\ \text{s.t.} & A^\top y \geq c-a \\ & y \geq \mathbf{0} \end{array}$$

Clearly, the optimal value of (P') is at most ω, and our choice of a implies that it is exactly $\omega - 1$. Thus, by the induction hypothesis, (D') has an integral optimal solution $\bar{y} = (\bar{y}_1, \bar{z})$ of value $\omega - 1$. Let $y^\star := (\bar{y}_1 + 1, \bar{z})$. Then $y^\star$ is an integral feasible solution for (D) and has value ω, so it is optimal. This completes the induction step. $\square$

6.2 The pluperfect graph theorem

In an attempt to prove it, Ray Fulkerson proposed and proved a polyhedral analogue of the weak perfect graph theorem, and he called it the *pluperfect graph theorem*. To prove his theorem, we will need two ingredients. Let A be a nonnegative matrix without a column of all zeros. Let

$$P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}.$$

The *antiblocker of* P is the set

$$a(P) := \{y \geq \mathbf{0} : x^\top y \leq 1 \ \ \forall x \in P\}.$$

Proposition 6.3. *Let A be a nonnegative matrix without a column of all zeros. Let B be the matrix whose rows are the extreme points of $P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$. Then B is nonnegative, has no column of all zeros, and*

$$\begin{aligned} a(P) &= \{y \geq \mathbf{0} : By \leq \mathbf{1}\} \\ a(a(P)) &= P. \end{aligned}$$

Proof. Clearly, B is a nonnegative matrix. Since A has no column of all zeros, P is a polytope, so every point of P can be written as a convex combination of the rows of B – this has two consequences. First, as $\epsilon\mathbf{1} \in P$ for a sufficiently small $\epsilon > 0$, B cannot have a column of all zeros. Secondly, $\{y \geq \mathbf{0} : By \leq \mathbf{1}\} \subseteq a(P)$. As the reverse inclusion holds trivially, we see that $a(P) = \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$. For the next equation, by definition

$$a(a(P)) = \{x \geq \mathbf{0} : y^\top x \leq 1 \ \ \forall y \in a(P)\}.$$

So clearly, $P \subseteq a(a(P))$. To prove the reverse inclusion, it suffices to show that every row a of A belongs to $a(P)$. Since $a \geq \mathbf{0}$ and $Ba \leq \mathbf{1}$, the result follows. $\square$

Next we study the extreme points of the antiblocker. Let's see an example first. Consider the matrix

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then the extreme points of $P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$ are the rows of the matrix

$$B := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Proposition 6.3, the antiblocker of P is the polytope $a(P) = \{x \geq \mathbf{0} : Bx \leq \mathbf{1}\}$. Aside from the three rows of A, the extreme points of $a(P)$ are $(1\,0\,0), (0\,1\,0), (0\,0\,1), (0\,0\,0)$, which are all orthogonal projections of the rows of A. We will show that this is true in general. Given vectors x, y of the same dimension, if x is obtained from y after setting some of the coordinates to 0, then we say that x is a *projection* of y.

Proposition 6.4. *Let A be a nonnegative matrix and let $P := \{x \in \mathbb{R}^n_+ : Ax \leq \mathbf{1}\}$. Then the following statements hold:*

(1) Let $\bar{x}$ be an extreme point of P for which

$$\bar{x} \leq \sum_{i=1}^{k} \lambda_i x^i$$

for some points $x^1, \ldots, x^k \in P$ and scalars $\lambda_1, \ldots, \lambda_k > 0$ with $\sum_{i=1}^k \lambda_i = 1$. Then $\bar{x}$ is a projection of each x^i.

(2) Suppose that A has no column of all zeros. Then every extreme point of $a(P)$ is a (possibly trivial) projection of a row of A.

Proof. **(1)** If $\bar{x} = \mathbf{0}$, then we are done. Otherwise, after possibly rearranging the coordinates, we have $\bar{x} = (\bar{z}, \mathbf{0})$ for some $\ell \geq 1$ and $\bar{z} \in \mathbb{R}^\ell$ such that $\bar{z} > \mathbf{0}$. For each $i \in [k]$, denote by z^i the vector consisting of the first ℓ coordinates of x^i. Then

$$\bar{z} \leq \sum_{i=1}^{k} \lambda_i z^i =: z.$$

Notice that z consists of the first ℓ coordinates of $\sum_{i=1}^k \lambda_i x^i$. As $\bar{x}$ is an extreme point of P, there is an $\ell \times \ell$ nonsingular submatrix E of A such that $E\bar{z} = \mathbf{1}$. On the one hand, as E is nonnegative and $z \geq \bar{z}$, it follows that $Ez \geq E\bar{z} = \mathbf{1}$. On the other hand, as $Ax \leq \mathbf{1}$, it follows that $Ez \leq \mathbf{1}$. Thus, $Ez = E\bar{z} = \mathbf{1}$, implying in turn that $z = \bar{z}$. As a result,

$$\bar{x} = (\bar{z}, \mathbf{0}) = (z, \mathbf{0}) = \sum_{i=1}^{k} \lambda_i (z^i, \mathbf{0}).$$

Since $\bar{x}$ is an extreme point, and each $(z^i, \mathbf{0})$ belongs to P, it follows that $\bar{x} = (z^1, \mathbf{0}) = \cdots = (z^k, \mathbf{0})$, as required.

(2) Denote by B the matrix whose rows are the extreme points of the polytope P. Then by Proposition 6.3, B is a nonnegative matrix without a column of all zeros, and $a(P) = \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$. Denote by A' the matrix whose rows are the extreme points of the polytope $a(P)$. Then by Proposition 6.3,

$$\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\} = a(a(P)) = \{x \geq \mathbf{0} : A'x \leq \mathbf{1}\}.$$

Take an extreme point a' of $a(P)$, which is also a row of A'. Since ${a'}^\top x \leq \mathbf{1}$ is valid for $\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$, it follows that a' is bounded above by a convex combination of the rows of A. Applying (1) to $a(P)$, we see that a' must be a projection of a row of A, as required. □

We are now ready for the *pluperfect graph theorem*:

Theorem 6.5 (Fulkerson 1972 [3]). *Let A be a nonnegative matrix without a column of all zeros, and let B be the matrix whose rows are the extreme points of $\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$. If A is a $0-1$ perfect matrix, then so is B.*

Proof. Suppose that A is a $0-1$ perfect matrix, that is, the set packing polytope $P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$ is integral. So B is a $0-1$ matrix. By Proposition 6.3, B has no column of all zeros and $a(P) = \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$. Therefore, by Proposition 6.4 (2), every extreme point of $\{y \geq \mathbf{0} : By \leq \mathbf{1}\}$ is a projection of a row of A. In particular, $\{y \geq \mathbf{0} : By \leq \mathbf{1}\}$ is integral, that is, B is perfect. □

6.3 Clutters and antiblockers

Let V be a finite set of *elements*, and let $\mathcal{A}$ be a family of subsets of V, called *members*. We say that $\mathcal{A}$ is a *clutter* over *ground set* V if no member is contained in another one (Edmonds and Fulkerson 1970 [2]).[1] The *incidence matrix* of $\mathcal{A}$, denoted $M(\mathcal{A})$, is the $0-1$ matrix whose columns are labeled by V and whose rows are the incidence vectors of the members.

Remark 6.6. *Let $\mathcal{A}_1, \mathcal{A}_2$ be clutters over the same ground set, where every member of $\mathcal{A}_1$ contains a member of $\mathcal{A}_2$, and every member of $\mathcal{A}_2$ contains a member of $\mathcal{A}_1$. Then $\mathcal{A}_1 = \mathcal{A}_2$.*

Proof. Take $A_1 \in \mathcal{A}_1$. Then A_1 contains a member A of $\mathcal{A}_2$, and A contains a member of $\mathcal{A}_1$. As $\mathcal{A}_1$ is a clutter, it must be that $A_1 \subseteq A \subseteq A_1$, implying in turn that $A = A_1$. Thus, $\mathcal{A}_1 \subseteq \mathcal{A}_2$. Similarly, $\mathcal{A}_2 \subseteq \mathcal{A}_1$, so $\mathcal{A}_1 = \mathcal{A}_2$. □

Let $\mathcal{A}$ be a clutter over ground set V, where every element is contained in a member. Consider the set packing polytope associated with $\mathcal{A}$:

$$\left\{x \in \mathbb{R}^V_+ : \sum (x_v : v \in A) \leq 1 \ \ \forall A \in \mathcal{A}\right\} = \{x \geq \mathbf{0} : M(\mathcal{A})x \leq \mathbf{1}\}.$$

Notice that the $0-1$ points of $P(\mathcal{A})$ correspond to the sets in

$$\{B \subseteq V : |B \cap A| \leq 1 \ \ \forall A \in \mathcal{A}\}.$$

Moreover, every $0-1$ point of $P(\mathcal{A})$ is in fact an extreme point (the proof of which is left as an exercise for the reader). We say that $\mathcal{A}$ is a *perfect clutter* if the associated set packing polytope is integral, that is, when the associated incidence matrix $M(\mathcal{A})$ is perfect. Notice that an arbitrary $0-1$ matrix A is perfect if, and only if, the clutter corresponding to the maximal rows of A is perfect. As a consequence, studying perfect clutters is just as general as studying perfect matrices.

[1] Clutters are also referred to as *Sperner families*.

Let $\mathcal{A}$ be a clutter over ground set V. The maximal sets of $\{B \subseteq V : |B \cap A| \leq 1 \ \forall A \in \mathcal{A}\}$ form another clutter over the same ground set, called the *antiblocker of* $\mathcal{A}$ and denoted $a(\mathcal{A})$. If every element is used in a member of $\mathcal{A}$, then the members of $a(\mathcal{A})$ are precisely the maximal integral points contained in the set packing polytope. For instance,

$$\text{the antiblocker of } \{\{1,2\},\{2,3\},\{3,1\}\} = \{\{1\},\{2\},\{3\}\}$$
$$\text{the antiblocker of } \{\{1\},\{2\},\{3\}\} = \{\{1,2,3\}\}$$
$$\text{the antiblocker of } \{\{1,2,3\}\} = \{\{1\},\{2\},\{3\}\}.$$

One natural question to ask is, when do we have $a(a(\mathcal{A})) = \mathcal{A}$? Perhaps surprisingly, the answer is very simple:

Proposition 6.7 (Fulkerson 1971 [4])**.** *Let $\mathcal{A}$ be a clutter over ground set V. Then the following statements are equivalent:*

(i) $a(a(\mathcal{A})) = \mathcal{A}$,

(ii) *$\mathcal{A}$ consists of the maximal stable sets of a graph over vertex set V.*

Proof. **(ii)** $\Rightarrow$ **(i)**: Suppose $\mathcal{A}$ consists of the maximal stable sets of $G = (V, E)$. Then a vertex set intersects every stable set at most once if, and only if, it is a clique. This implies that $a(\mathcal{A})$ consists of the maximal cliques of G. Applying the same argument to $\overline{G}$ implies that $a(a(\mathcal{A}))$ consists of the maximal stable sets of G, so $a(a(\mathcal{A})) = \mathcal{A}$. **(i)** $\Rightarrow$ **(ii)**: Suppose $a(a(\mathcal{A})) = \mathcal{A}$. Let G be the graph over vertex set V, where distinct vertices u, v are non-adjacent if there is a member containing both u, v. Clearly, every member of $\mathcal{A}$ is a stable set of G. Conversely, let $S \subseteq V$ be a stable set of G. We claim that

$$(\star) \qquad |S \cap B| \leq 1 \quad \forall B \in a(\mathcal{A}).$$

Suppose otherwise. Then for distinct vertices u, v of G, $\{u, v\} \subseteq S \cap B$. However, as u and v are non-adjacent, $\{u, v\} \subseteq A$ for some member $A \in \mathcal{A}$, but then $\{u, v\} \subseteq A \cap B$, a contradiction as $B \in a(\mathcal{A})$. This proves $(\star)$, implying in turn that S is contained in a member of $a(a(\mathcal{A})) = \mathcal{A}$. Remark 6.6 implies that $\mathcal{A}$ consists of the maximal stable sets of G, as required. □

As a consequence,

Theorem 6.8 (Padberg 1973 [5])**.** *If a clutter is perfect, then its members are the maximal stable sets of a simple graph.*

Proof. Let $\mathcal{A}$ be a perfect clutter over ground set V, and let A be the corresponding incidence matrix. Let B be the matrix whose rows are the extreme points of $P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$, and let $Q := \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$. Then by Proposition 6.3, $a(P) = Q$ and $a(Q) = P$. Moreover, since the clutter $\mathcal{A}$ is perfect, the matrix A is perfect, so by Theorem 6.5, B is a perfect matrix. Let $\mathcal{B}$ be the clutter over ground set V whose members correspond to the maximal rows of B. Notice that $a(\mathcal{A})$ corresponds to the maximal integral extreme points of P, so $a(\mathcal{A}) = \mathcal{B}$. Similarly, $a(\mathcal{B})$ corresponds to the maximal integral extreme points of Q, so $a(\mathcal{B}) = \mathcal{A}$. It therefore follows from Proposition 6.7 that $\mathcal{A}$ consists of the maximal stable sets of a graph, as required. □

In fact, the simple graph above is always perfect, the proof of which is left as an exercise.

Theorem 6.9 (Chvátal 1975 [1])**.** *Let $G = (V, E)$ be a simple graph. If the clutter of the maximal stable sets of G is perfect, then G is a perfect graph.*

Summarizing the results of this section and the previous one, we get the following characterization of when the set packing polytope is integral:

Corollary 6.10. *The following statements hold:*

(1) Let A be a $0-1$ matrix without a column of all zeros whose set packing polytope $\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$ is integral. Then the linear system $x \geq \mathbf{0}, Ax \leq \mathbf{1}$ is totally dual integral, the maximal rows of A correspond to the maximal stable sets of a simple graph, and the graph is perfect.

(2) Let G be a simple graph. Then G is perfect if, and only if, it has no odd hole and no odd antihole.

References

[1] Chvátal, V.: On certain polytopes related associated with graphs. J. Combin. Theory Ser. B **18**(2), 138–154 (1975)

[2] Edmonds, J. and Fulkerson, D.R.: Bottleneck extrema. J. Combin. Theory Ser. B **8**, 299–306 (1970)

[3] Fulkerson, D.R.: Anti-blocking polyhedra. J. Combin. Theory Ser. B **12**(1), 50–71 (1972)

[4] Fulkerson, D.R.: Blocking and anti-blocking pairs of polyhedra. Math. Program. **1**, 168–194 (1971)

[5] Padberg, M.W.: On the facial structure of set packing polyhedra. Math. Program. **5**, 199–215 (1973)