47853 Packing and Covering: Lecture 8

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7 Integral and totally dual integral set covering programs

Let $\mathcal{C}$ be a clutter over ground set E. Consider the set covering polyhedron associated with $\mathcal{C}$:

$$\left\{x \in \mathbb{R}^E_+ : \sum (x_e : e \in C) \geq 1 \quad \forall C \in \mathcal{C}\right\} = \{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}.$$

A *cover* is a subset of E that intersects every member ([6], Volume C, §77.5).[1] Notice that the covers of $\mathcal{C}$ correspond precisely to the $0-1$ points of the associated set covering polyhedron. If a set is a cover then so is every superset of it, so not all covers are interesting. A cover is *minimal* if it does not contain another cover.

7.1 Clutters and blockers

Let $\mathcal{C}$ be a clutter over ground set E. The *blocker of* $\mathcal{C}$, denoted $b(\mathcal{C})$, is the clutter over ground set E whose members are the minimal covers of $\mathcal{C}$ (Edmonds and Fulkerson 1970 [2]).[2] Unlike antiblockers,

Theorem 7.1 (Isbell 1958 [4], Edmonds and Fulkerson 1970 [2])**.** *Given a clutter $\mathcal{C}$, we have $b(b(\mathcal{C})) = \mathcal{C}$. That is, if $\mathcal{B}$ is the blocker of $\mathcal{C}$, then $\mathcal{C}$ is the blocker of $\mathcal{B}$.*

Proof. Denote by E the ground set of $\mathcal{C}$. We need to show that the minimal covers of $b(\mathcal{C})$ are precisely the members of $\mathcal{C}$. By Remark 6.6, it suffices to show that (a) every member of $\mathcal{C}$ is a cover of $b(\mathcal{C})$, and (b) every minimal cover of $b(\mathcal{C})$ contains a member of $\mathcal{C}$.

(a) Take $C \in \mathcal{C}$. Since $C \cap B \neq \emptyset$ for every $B \in b(\mathcal{C})$, we get that C is a cover of $b(\mathcal{C})$.

(b) Take a minimal cover C' of $b(\mathcal{C})$. Then $E - C'$ cannot contain a member of $b(\mathcal{C})$, so $E - C'$ is not a cover of $\mathcal{C}$, implying in turn that $E - C'$ is disjoint from a member of $\mathcal{C}$. Consequently, C' contains a member of $\mathcal{C}$.

Thus, $b(b(\mathcal{C})) = \mathcal{C}$. □

Let us see some examples of blocking pairs of clutters:

[1] In the literature, a cover is also referred to as a *hitting set*, a *blocking set*, a *transversal*, etc.

[2] Berge 1989 [1] referred to $b(\mathcal{C})$ as the *transversal of* $\mathcal{C}$ and denoted it $\mathrm{Tr}(\mathcal{C})$.

Remark 7.2. *The following statements hold:*

(1) Let G be a graph and take distinct vertices s, t. Over ground set $E(G)$, the clutter of st-paths and the clutter of minimal st-cuts are blockers.

(2) Let G be a simple graph. Over ground set $V(G)$, the clutter of edges and the clutter of minimal vertex covers are blockers.

(3) Consider the clutter of the triangles of K_4 over ground set $E(K_4)$:

$$Q_6 := \big\{\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\}\big\}.$$

Its blocker consists of the triangles, and the perfect matchings:

$$b(Q_6) = \big\{\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\},\{1,2\},\{3,4\},\{5,6\}\big\}.$$

Proof. **(1)** Let $\mathcal{C}$ be the clutter of st-paths over ground set $E(G)$. Clearly, every st-cut is a cover for $\mathcal{C}$. Let B be a minimal cover of $\mathcal{C}$. By definition, $E(G) - B$ does not contain an st-path of G, implying in turn that in $G \setminus B$ the vertices s, t are disconnected, so $G \setminus B$ has an empty st-cut, implying in turn that B contains an st-cut of G. Thus, $b(\mathcal{C})$ consists of the minimal st-cuts, as required. **(2)** follows from the definition of a vertex cover. **(3)** We leave this as an easy exercise. □

7.2 Packing and covering parameters

To each clutter $\mathcal{C}$, we can associate two dual parameters. A *packing* is a collection of pairwise disjoint members.[3] The *packing number*, denoted $\nu(\mathcal{C})$, is the maximum cardinality of a packing. The *covering number*, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover. Since a cover picks up a different element from each member of a packing, we see that

$$\tau(\mathcal{C}) \geq \nu(\mathcal{C}).$$

For instance, for the clutter $\big\{\{1,2\},\{2,3\},\{3,1\}\big\}$, the packing number is 1, while the covering number is 2 – so the two parameters are not always equal. We say that $\mathcal{C}$ *packs* if $\tau(\mathcal{C}) = \nu(\mathcal{C})$.

Proposition 7.3. *The following statements hold:*

(1) Given a graph G with distinct vertices s, t, the clutter of st-paths packs, and the clutter of minimal st-cuts packs.

(2) Given a bipartite simple graph G, the clutter of edges packs, and the clutter of minimal vertex covers packs.

(3) Q_6 does not pack, and $b(Q_6)$ packs.

[3] A packing is also referred to as a *matching*.

Proof. **(1)** By Theorem 1.1, the maximum number of edge-disjoint st-paths is equal to the minimum cardinality of an st-cut, so the clutter of st-paths packs. By Theorem 1.2, the maximum number of edge-disjoint st-cuts is equal to the minimum cardinality of an st-path, so the clutter of minimal st-cuts packs. **(2)** It follows from Theorem 5.2 that the maximum cardinality of a matching in G is equal to the minimum cardinality of a vertex cover of G, so the clutter of edges of G packs. We leave it as an easy exercise to prove that the clutter of minimal vertex covers of G packs. **(3)** Q_6 does not pack as $\tau(Q_6) = 2 > 1 = \nu(Q_6)$. On the other hand, $b(Q_6)$ packs as $\tau(b(Q_6)) = 3$ and $b(Q_6)$ has disjoint members $\{1,2\}, \{3,4\}, \{5,6\}$. $\square$

Let $\mathcal{C}$ be a clutter over ground set E. Take nonnegative weights $w \in \mathbb{R}^E_+$. A *weighted packing* is a collection of members such that every element e is contained in at most w_e of the members. Denote by $\nu(\mathcal{C}, w)$ the maximum cardinality of a weighted packing. Notice that a member may be taken more than once in a weighted packing, and if so, it would contribute more than one unit to $\nu(\mathcal{C}, w)$. Given a cover B, its *weight* is $w(B) := \sum_{e \in B} w_e$. Denote by $\tau(\mathcal{C}, w)$ the minimum weight of a cover. Notice that for weights $\mathbf{1}$, weighted packings are precisely packings and cover weights are precisely cover cardinalities, so $\nu(\mathcal{C}, \mathbf{1}) = \nu(\mathcal{C})$ and $\tau(\mathcal{C}, \mathbf{1}) = \tau(\mathcal{C})$.

Remark 7.4. *Given a clutter $\mathcal{C}$ over ground set E and weights $w \in \mathbb{R}^E_+$,*

$$\tau(\mathcal{C}, w) \geq \nu(\mathcal{C}, w).$$

Proof. Take a cover B and a weighted packing $C_1, \ldots, C_k$. Then

$$w(B) = \sum_{e \in B} w_e \geq \sum_{e \in B} |\{i \in [k] : e \in C_i\}| = \sum_{i \in [k]} |\{e \in B : e \in C_i\}| = \sum_{i \in [k]} |B \cap C_i| \geq k.$$

Since this is true for all covers and weighted packings, the inequality $\tau(\mathcal{C}, w) \geq \nu(\mathcal{C}, w)$ follows. $\square$

Consider the associated set covering program

$$(P) \qquad \begin{array}{ll} \min & w^\top x \\ \text{s.t.} & \sum (x_e : e \in C) \geq 1 \quad \forall C \in \mathcal{C} \\ & x \geq \mathbf{0}. \end{array}$$

As the $0-1$ solutions of (P) are precisely the covers, it follows that $\tau(\mathcal{C}, w)$ computes the optimal value of a $0-1$ solution, and hence an integral solution, to (P). Consider the dual program

$$(D) \qquad \begin{array}{ll} \max & \sum (y_C : C \in \mathcal{C}) \\ \text{s.t.} & \sum (y_C : C \in \mathcal{C}, e \in C) \leq w_e \quad \forall e \in E \\ & y \geq \mathbf{0}. \end{array}$$

As the integral solutions of (D) are precisely the weighted packings, we get that $\nu(\mathcal{C}, w)$ computes the optimal value of an integral solution to (D). In particular, LP Weak Duality offers an alternate proof of the inequality $\tau(\mathcal{C}, w) \geq \nu(\mathcal{C}, w)$. We will refer to each solution of (D) as a *fractional* weighted packing, and its *value* is the objective value of the solution.

We say that $\mathcal{C}$ is *Mengerian* if for all weights $w \in \mathbb{Z}^E_+$, the minimum weight of a cover is equal to the maximum cardinality of a weighted packing, i.e. $\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w)$. We say that $\mathcal{C}$ is *ideal* if for all weights

$w \in \mathbb{Z}_+^E$, the minimum weight of a cover is equal to the maximum value of a fractional weighted packing. Equivalently, by LP Strong Duality, $\mathcal{C}$ is ideal if for all weights $w \in \mathbb{Z}_+^E$, the set covering program (P) has an integral optimal solution, i.e. the optimal value of (P) is $\tau(\mathcal{C}, w)$. Clearly if a clutter is Mengerian then it is ideal.

Observe that $\mathcal{C}$ is Mengerian if, and only if, the corresponding set covering system $M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}$ is totally dual integral.[4] Observe further that $\mathcal{C}$ is ideal if, and only if, the set covering polyhedron

$$\left\{x \in \mathbb{R}_+^E : M(\mathcal{C})x \geq \mathbf{1}\right\}$$

is integral.

Studying Mengerian and ideal clutters is just as general as studying integral and totally dual integral set covering systems:

Remark 7.5. *Take a $0-1$ matrix A with column labels E. Let $\mathcal{C}$ be the clutter over ground set E whose members correspond to the minimal rows of A. Then the following statements hold:*

- *$x \geq \mathbf{0}, Ax \geq \mathbf{1}$ is totally dual integral if, and only if, $\mathcal{C}$ is Mengerian,*
- *$\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$ is integral if, and only if, $\mathcal{C}$ is ideal.*

As mentioned already, a Mengerian clutter is always ideal. In contrast to Theorem 6.2 in the set packing case, an ideal clutter is not necessarily Mengerian:

Remark 7.6. *The following statements hold:*

(1) Q_6 is an ideal clutter that is not Mengerian,

(2) $b(Q_6)$ is a Mengerian clutter.

Proof. We leave this as an exercise for the reader. □

This remark also shows that being Mengerian is not closed under taking blockers. However, much like the pluperfect graph theorem, Theorem 6.5, in the set packing case, being ideal *is* closed under taking blockers.

7.3 The width-length inequality

The following *width-length inequality* is the analogue of the max-max inequality, Theorem 5.5, for set covering polyhedra. Alfred Lehman proved this inequality and wrote it up in 1963, taught it to Ray Fulkerson in 1965 at RAND Corporation, but the result was not published until much later in 1979:

Theorem 7.7 (Lehman 1979 [5], Fulkerson 1970 [3]). *Let $\mathcal{C}$ be a clutter over ground set E. Then $\mathcal{C}$ is ideal if, and only if, for all $w, \ell \in \mathbb{R}_+^E$,*

$$\min\{w(C) : C \in \mathcal{C}\} \cdot \min\{\ell(B) : B \in b(\mathcal{C})\} \leq w^\top \ell.$$

[4] In the literature, the Mengerian property is also referred to as the *max-flow min-cut* property.

Proof. Suppose first that $\mathcal{C}$ is ideal. Take $w, \ell \in \mathbb{R}_+^E$. Let $\tau := \tau(\mathcal{C}, \ell) = \min\{\ell(B) : B \in b(\mathcal{C})\}$. Since $\mathcal{C}$ is ideal, there is a fractional ℓ-weighted packing $y \in \mathbb{R}_+^{\mathcal{C}}$ of value τ:

$$\sum \left(y_C : C \in \mathcal{C}\right) = \tau$$
$$\sum \left(y_C : e \in C \in \mathcal{C}\right) \leq \ell_e \quad \forall e \in E.$$

Now we have

$$\begin{aligned} w^\top \ell = \sum_{e \in E} w_e \ell_e \geq \sum_{e \in E} w_e \left[\sum \left(y_C : e \in C \in \mathcal{C}\right)\right] &= \sum_{C \in \mathcal{C}} y_C \cdot w(C) \\ &\geq \min\left\{w(C) : C \in \mathcal{C}\right\} \cdot \sum_{C \in \mathcal{C}} y_C \\ &= \min\left\{w(C) : C \in \mathcal{C}\right\} \cdot \tau \\ &= \min\left\{w(C) : C \in \mathcal{C}\right\} \cdot \min\left\{\ell(B) : B \in b(\mathcal{C})\right\}, \end{aligned}$$

as required. Suppose conversely that the width-length inequality holds for all $w, \ell \in \mathbb{R}_+^E$. We will show that $\mathcal{C}$ is ideal. To this end, take an arbitrary $\ell \in \mathbb{R}_+^E$, and let $x^\star$ be an optimal solution to

$$\begin{array}{lll} \min & \ell^\top x & \\ \text{s.t.} & x(C) \geq 1 & \forall\, C \in \mathcal{C} \\ & x \geq \mathbf{0}. & \end{array}$$

We will show that

$$\ell^\top x^\star = \min\left\{\ell(B) : B \in b(\mathcal{C})\right\},$$

thereby finishing the proof. Well, it is clear that $\leq$ holds above. We will prove that $\geq$ holds as well. By the width-length inequality,

$$\begin{aligned} \ell^\top x^\star &\geq \min\left\{\ell(B) : B \in b(\mathcal{C})\right\} \cdot \min\left\{x^\star(C) : C \in \mathcal{C}\right\} \\ &\geq \min\left\{\ell(B) : B \in b(\mathcal{C})\right\}. \end{aligned}$$

as required. □

As an immediate consequence, we get the following analogue of the pluperfect graph theorem, Theorem **??**:

Theorem 7.8. *If a clutter is ideal, then so is its blocker.*

7.4 Deletions, contractions and minors

Let $\mathcal{C}$ be a clutter over ground set E, and take an element $e \in E$. We will define two clutters over ground set $E - \{e\}$. The *deletion* is the clutter

$$\mathcal{C} \setminus e := \{C \in \mathcal{C} : e \notin C\}$$

while the *contraction* is the clutter

$$\mathcal{C}/e := \text{ the minimal sets of } \{C - \{e\} : C \in \mathcal{C}\}.$$

Notice that deletion and contraction are blocking operations:

Proposition 7.9. *Let $\mathcal{C}$ be a clutter over ground set E. Then for $e \in E$, $b(\mathcal{C} \setminus e) = b(\mathcal{C})/e$ and $b(\mathcal{C}/e) = b(\mathcal{C}) \setminus e$.*

Proof. Let us first prove that $b(\mathcal{C} \setminus e) = b(\mathcal{C})/e$. If B' is a cover of $\mathcal{C} \setminus e$ then $B' \cup \{e\}$ is a cover of $\mathcal{C}$. So every member of $b(\mathcal{C} \setminus e)$ contains a member of $b(\mathcal{C})/e$. For the reverse inclusion, if B is a cover of $\mathcal{C}$ then $B - \{e\}$ is a cover of $\mathcal{C} \setminus e$. So every member of $b(\mathcal{C})/e$ contains a member of $b(\mathcal{C} \setminus e)$. Remark 6.6 implies that $b(\mathcal{C} \setminus e) = b(\mathcal{C})/e$. To prove the second equation, let us apply the first equation to $b(\mathcal{C})$:

$$b(b(\mathcal{C}) \setminus e) = b(b(\mathcal{C}))/e = \mathcal{C}/e.$$

Taking blockers yields $b(\mathcal{C}) \setminus e = b(\mathcal{C}/e)$, thereby proving the second equation. $\square$

For disjoint subsets $I, J \subseteq E$, the following clutter over ground set $E - (I \cup J)$,

$$\mathcal{C} \setminus I/J := \text{ the minimal sets of } \{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}$$

is a *minor* of $\mathcal{C}$ obtained after deleting I and contracting J. If $I \cup J \neq \emptyset$, then $\mathcal{C} \setminus I/J$ is a *proper* minor. By the proposition above, $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$. From an optimization point of view, minors operations are quite natural:

Remark 7.10. *Take a clutter $\mathcal{C}$ over ground set E, and disjoint subsets $I, J \subseteq E$. Then the linear programs*

$$\min\{w^\top x : M(\mathcal{C} \setminus I/J)x \geq \mathbf{1}, x \geq \mathbf{0}\} = \max\{\mathbf{1}^\top y : M(\mathcal{C} \setminus I/J)^\top y \leq w, y \geq \mathbf{0}\}$$

for $w \in \mathbb{R}_+^{E-(I \cup J)}$, are equivalent to the linear programs

$$\min\{w^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\} = \max\{\mathbf{1}^\top y : M(\mathcal{C})^\top y \leq w, y \geq \mathbf{0}\}$$

for $w \in \mathbb{R}_+^E$ such that $w_e = 0$ for all $e \in I$ and $w_f = +\infty$ for all $f \in J$.

As an immediate consequence,

Remark 7.11 (Seymour 1977 [7])**.** *If a clutter is ideal (resp. Mengerian), then so is any minor of it.*

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