CO 750-1 Assignment 1

Due Thursday, May 25, in class

The assignment will be out of 30 points. If you get more than 30 points, the extra points will be carried over to your future assignments with half the weight.

Q1. **(5 points)** Let $(E, \leq)$ be a partially ordered set. Prove that the minimum number of antichains needed to cover E is equal to the maximum cardinality of a chain.

Q2. **(5 points)** The following theorem is a variant of the max-flow min-cut theorem of Ford and Fulkerson (1956):

> In a directed graph $D = (V, A)$ with distinct vertices s, t, the maximum number of pairwise arc-disjoint (s, t)-paths is equal to the minimum cardinality of an (s, t)-cut.

(Our proof of Menger's theorem can be modified every easily to prove this statement.) Use this result (without proof) to prove the following result of Kőnig (1931):

> In a bipartite loopless graph, the maximum size of a matching is equal to the minimum cardinality of a vertex cover.

Q3. **(10 points)** Let $D = (V, A)$ be a directed loopless graph, and take a *root* $r \in V$. An r-*arborescence* is a subset $T \subseteq A$ such that $D[T]$ is a tree containing r and rooted away from r. An r-arborescence is *spanning* if it spans every vertex. An r-*cut* is a cut of the form $\delta^+(U)$, for some $r \in U \subsetneq V$. Prove that the maximum number of disjoint spanning r-arborescences is equal to the minimum cardinality of an r-cut.

Hint. Let τ be the minimum cardinality of an r-cut. Take an inclusionwise maximal r-arborescence $T \subseteq A$ such that

> $(\star)$ for every r-cut $\delta^+(U)$, $|\delta^+(U) \setminus T| \geq \tau - 1$.

As a first step, show that T is spanning. To prove this, call an r-cut $\delta^+(U)$ *tight* if $|\delta^+(U) \setminus T| = \tau - 1$ and $(V - U) - V(T) \neq \emptyset$. Show that if $\delta^+(U_1)$ and $\delta^+(U_2)$ are tight, and $U_1 \cup U_2 \neq V$, then so are $\delta^+(U_1 \cap U_2)$ and $\delta^+(U_1 \cup U_2)$. If T is not spanning, what can you say about the tight cut $\delta^+(U)$ where U is maximal?

Q4. **(5 points)** A matrix M is *totally unimodular* if the determinant of every submatrix is either $0, 1, -1$. Let b be an integral vector. Prove that

(a) the polyhedron $\{x \geq \mathbf{0} : Mx \geq b\}$ is integral,

(b) the linear system $x \geq \mathbf{0}, Mx \geq b$ is totally dual integral.

Q5. **(8 points)** Let

$$A := \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Answer the following questions. Justify your answers.

(a) Is A totally unimodular?

(b) Is A balanced?

(c) Is the set covering polyhedron $\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$ integral?

(d) Is the set covering system $x \geq \mathbf{0}, Ax \geq \mathbf{1}$ totally dual integral?

Q6. **(7 points)**

(a) Let A be a balanced matrix. Prove that the linear system $x \geq \mathbf{0}, Ax \geq \mathbf{1}$ is totally dual integral.

(b) Let $G = (V, E)$ be a balanced hypergraph. A *vertex cover* is a subset of vertices that intersects every edge. Prove that the maximum cardinality of a matching is equal to the minimum cardinality of a vertex cover.