CO 750-1 Assignment 3

Due Tuesday, June 27, in class

The assignment will be out of 36 points. If you get more than 36 points, the extra points will be carried over to your future assignments with half the weight. You may use the results in a problem for proving another problem, even if you didn't solve the first problem.

1. Let $\mathcal{C}$ be a clutter over ground set E.

   (a) **(3 points)** Let $R \cup B$ be a partition of E. Prove that either R contains a member of $\mathcal{C}$ or B contains a member of $b(\mathcal{C})$, but not both.

   (b) **(3 points)** Prove that for every member C and element $e \in C$, there is a minimal cover B such that $C \cap B = \{e\}$.

2. Let $G = (V, E)$ be a graph.

   (a) **(4 points)** Take a subset $\Sigma \subseteq E$. The pair (G, Σ) is called a *signed graph*. An *odd circuit* is a circuit C such that $|C \cap \Sigma|$ is odd. A *signature* is a set of the form $\Sigma \triangle \delta(U), U \subseteq V$. Prove that the clutter of minimal signatures and the clutter of odd circuits are blockers.

   (b) **(4 points)** Let $\mathcal{C}$ be the clutter over ground set E whose members are the complements of spanning trees. Prove that $b(\mathcal{C})$ is the clutter of circuits.

3. **(5 points)** Let $\mathcal{C}$ be a clutter that has two members that intersect. Prove that either $\{\{1,2\},\{1,3\}\}$ or $\{\{1,2\},\{1,3\},\{2,3\}\}$ is a minor.

4. **(5 points)** Let $\mathcal{C}$ be a clutter over groundset E and let $w \in \mathbb{R}^E$. Prove that

$$\max_{C \in \mathcal{C}} \min_{x \in C} w_x = \min_{B \in b(\mathcal{C})} \max_{x \in B} w_x.$$

   (**Hint**. You may find it helpful to think of $\mathcal{C}$ as a clutter of st-paths.)

5. **(5 points)** Let $\mathcal{C}$ be a clutter over ground set E. Prove that $\{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$ is integral if, and only if, $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$ is integral.

6. **(7 points)** Let $G = (V, E)$ be a simple graph, and let $\mathcal{C}$ be the clutter of the minimal vertex covers of G. Prove that the following statements are equivalent:

(i) $\mathcal{C}$ is ideal,

(ii) G is bipartite,

(iii) $\mathcal{C}$ is Mengerian.

7. **(5 points)** Let $\mathcal{C}$ be a clutter whose incidence matrix is balanced. Prove that $b(\mathcal{C})$ is Mengerian.

8. **(5 points)** Let D be a directed planar graph. A *feedback arc set* is an arc subset whose deletion makes the directed graph acyclic. Prove that the minimum cardinality of a feedback arc set is equal to the maximum number of arc-disjoint directed circuits.

9. **(5 points)** Let $G = (V, E)$ be a loopless graph where $|V|$ is even. Prove that the convex hull of

$$\{\chi_M : M \subseteq E \text{ is a perfect matching}\}$$

is equal to

$$\left\{ \mathbf{1} \geq x \geq \mathbf{0} : \sum_{e \in E} x_e = \frac{|V|}{2}, \sum (x_e : e \in \delta(U)) \geq 1 \;\; \forall\, U \subseteq V, |U| \text{ is odd} \right\}.$$

10. **(5 points)** Take an integer $r \geq 3$, and take an r-regular graph $G = (V, E)$ where $|V|$ is even, and for each odd cardinality $U \subseteq V$, $|\delta(U)| \geq r$. Prove that every edge of G belongs to a perfect matching.