# CO 750-1 Assignment 4

Due Tuesday, July 18, in class

The assignment will be out of 35 points. If you get more than 35 points, the extra points will be added to your previous assignments with half the weight. You may use the results in a problem for proving another problem, even if you didn't solve the first problem.

1. **(6 points)** Let $\mathcal{C}$ be a minimally non-ideal clutter.

   (a) Assume that the minimum cardinality of a member is two. Describe all the minimum cardinality members.

   (b) Assume that $\tau(\mathcal{C}) = 2$. Describe all the minimum cardinality members (not covers).

2. **(5 points)** What are all the minimally non-ideal clutters of dimension at most 6? (Do not forget the deltas!)

3. **(4 points)** Take an odd integer $n \geq 5$ and let $\mathcal{C}$ be a clutter over ground set $[n]$ whose minimum cardinality members are

   $$\{1, 2\}, \{2, 3\}, \{3, 4\}, \ldots, \{n-1, n\}, \{n, 1\}.$$

   We will refer to $\mathcal{C}$, and any clutter isomorphic to it, as an *extended odd hole*. Prove that an extended odd is non-ideal.

4. **(5 points)** Let $\mathcal{C}$ be a clutter whose incidence matrix $M(\mathcal{C})$ has complementary columns (i.e. a pair of columns that add up to $\mathbf{1}$). Prove that $\mathcal{C}$ cannot be minimally non-ideal. (**Hint.** Q1(b).)

5. **(10 points)** Let $\mathcal{C}$ be a clutter over ground set $E$, and take a set $T \subseteq E$. Let $\mathcal{C}'$ be the clutter of the minimal sets of $\mathcal{C} \cup \{T\}$. Assume that $\mathcal{C}$ is ideal and $\mathcal{C}'$ is minimally non-ideal. Prove that $\mathcal{C}'$ is either a delta or an extended odd hole.

6. **(5 points)** Let $(G, \Sigma)$ be a signed graph. A *blocking vertex* is a vertex that is used in every odd circuit. A *blocking pair* is a pair of vertices $u, v$ such that every odd circuits uses at least one of $u, v$.

   (a) Prove that if $u$ is a blocking vertex, then there exists a signature $\Gamma$ such that $\Gamma \subseteq \delta(u)$.

(b) Prove that if $u, v$ is a blocking pair, then there exists a signature $\Gamma$ such that $\Gamma \subseteq \delta(u) \cup \delta(v)$.

7. **(5 points)** Prove that if $(G, \Sigma)$ has a blocking vertex, then its clutter of odd circuits packs. (**Hint.** Menger's theorem.)

8. **(5 points)** Prove that if $(G, \Sigma)$ has a blocking pair, then it is weakly bipartite.

9. **(5 points)** Prove that if $G$ is planar, then $(G, \Sigma)$ is weakly bipartite. (You may not use the fact that a signed graph without an odd-$K_5$ minor is weakly bipartite.)

10. **(10 points)** Consider the linear binary system

$$Ax \equiv b \pmod 2$$

for a $0,1$ matrix $A$ and a $0,1$ vector $b$. Prove that if this system has no $0,1$ solution, then there exists a $0,1$ vector $c$ such that

$$c^\top A \equiv \mathbf{0} \quad \text{and} \quad c^\top b \equiv 1 \pmod 2.$$

11. **(15 points)** Let $A$ be a $0,1$ matrix whose columns are labeled by $E$. We will refer to the vectors of

$$\{x \in \{0,1\}^E : Ax \equiv \mathbf{0} \pmod 2\}$$

as *cycles*. Every vector in the row space of $A$ modulo 2 is called a *cocycle*, that is, the cocycles are the vectors of

$$\left\{A^\top y \text{ modulo } 2 : y \in \{0,1\}^m\right\},$$

where $m$ is number of rows of $A$.

(a) Prove that the cycles form a vector space over $GF(2)$.

(b) Prove that the cocycles form a vector space over $GF(2)$.

(c) A subset of $E$ is called a *cycle* if its incidence vector is a cycle, and it is called a *cocycle* if its incidence vector is a cocycle. Take an element $e \in E$. Prove that

$$\mathcal{C} := \text{ the minimal sets of } \{C - \{e\} : C \subseteq E \text{ is a cycle such that } e \in C\}$$

and

$$\mathcal{B} := \text{ the minimal sets of } \{B - \{e\} : B \subseteq E \text{ is a cocycle such that } e \in B\}$$

are blockers. (**Hint.** Q10.)

(d) Use (c) to prove that the clutter of minimal $T$-joins and the clutter of minimal $T$-cuts are blockers.

(e) Use (c) to prove that in a signed graph, the clutter of odd circuits and the clutter of minimal signatures are blockers.