CO 750 Packing and Covering: Lecture 11

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7 Integral and totally dual integral set covering programs

Let $\mathcal{C}$ be a clutter over ground set E. Consider the set covering polyhedron associated with $\mathcal{C}$:

$$\left\{x \in \mathbb{R}^E_+ : \sum (x_e : e \in C) \geq 1 \quad \forall C \in \mathcal{C}\right\} = \{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}.$$

A *cover* is a subset of E that intersects every member.[1] Notice that the covers of $\mathcal{C}$ correspond precisely to the $0-1$ points of the associated set covering polyhedron. If a set is a cover then so is every superset of it, so not all covers are interesting.

7.1 Clutters and blockers

Let $\mathcal{C}$ be a clutter over ground set E. The *blocker of* $\mathcal{C}$, denoted $b(\mathcal{C})$, is the clutter over ground set E whose members are the minimal covers of $\mathcal{C}$.[2] Unlike antiblockers,

Theorem 7.1 (Isbell 1958, Edmonds and Fulkerson 1970)**.** *Given a clutter $\mathcal{C}$, we have $b(b(\mathcal{C})) = \mathcal{C}$.*

Proof. Denote by E the ground set of $\mathcal{C}$. We need to show that the minimal covers of $b(\mathcal{C})$ are precisely the members of $\mathcal{C}$. By Remark 6.6, it suffices to show that (a) every member of $\mathcal{C}$ is a cover of $b(\mathcal{C})$, and (b) every minimal cover of $b(\mathcal{C})$ contains a member of $\mathcal{C}$.

(a) Take $C \in \mathcal{C}$. Since $C \cap B \neq \emptyset$ for every $B \in b(\mathcal{C})$, we get that C is a cover of $b(\mathcal{C})$.

(b) Take a minimal cover C' of $b(\mathcal{C})$. Then $E - C'$ cannot contain a member of $b(\mathcal{C})$, so $E - C'$ is not a cover of $\mathcal{C}$, implying in turn that $E - C'$ is disjoint from a member of $\mathcal{C}$. Consequently, C' contains a member of $\mathcal{C}$.

Thus, $b(b(\mathcal{C})) = \mathcal{C}$. □

That is, if $\mathcal{B}$ is the blocker of $\mathcal{C}$, then $\mathcal{C}$ is the blocker of $\mathcal{B}$. Let us see some examples of blocking pairs of clutters:

Remark 7.2. *The following statements hold:*

[1] In the literature, a cover is also referred to as a *hitting set*, a *blocking set*, a *transversal*, etc.

[2] Berge (1989) referred to $b(\mathcal{C})$ as the *transversal of* $\mathcal{C}$ and denoted it $\mathrm{Tr}(\mathcal{C})$.

(1) Let G be a graph and take distinct vertices s, t. Over ground set $E(G)$, the clutter of st-paths and the clutter of minimal st-cuts are blockers.

(2) Let G be a simple graph. Over ground set $V(G)$, the clutter of edges and the clutter of minimal vertex covers are blockers.

(3) Consider the clutter of the triangles of K_4 over ground set $E(K_4)$:

$$Q_6 := \big\{\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\}\big\}.$$

Its blocker consists of the triangles, as well as the perfect matchings:

$$b(Q_6) = \big\{\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\},\{1,2\},\{3,4\},\{5,6\}\big\}.$$

Proof. **(1)** Let $\mathcal{C}$ be the clutter of st-paths over ground set $E(G)$. Clearly, every st-cut is a cover for $\mathcal{C}$. Let B be a minimal cover of $\mathcal{C}$. By definition, $E(G) - B$ does not contain an st-path of G, implying in turn that in $G \setminus B$ the vertices s, t are disconnected, so $G \setminus B$ has an empty st-cut, implying in turn that B contains an st-cut of G. Thus, $b(\mathcal{C})$ consists of the minimal st-cuts, as required. **(2)** follows from the definition of a vertex cover. **(3)** We leave this as an easy exercise. □

7.2 Packing and covering parameters

To each clutter $\mathcal{C}$, we can associate two dual parameters. A *packing* is a collection of pairwise disjoint members.[3] The *packing number*, denoted $\nu(\mathcal{C})$, is the maximum size of a packing. The *covering number*, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover. Since a cover picks up a different element from each member of a packing, we see that

$$\tau(\mathcal{C}) \geq \nu(\mathcal{C}).$$

For instance, for the clutter $\big\{\{1,2\},\{2,3\},\{3,1\}\big\}$, the packing number is 1, while the covering number is 2 – so the two parameters are not always equal. We say that $\mathcal{C}$ *packs* if $\tau(\mathcal{C}) = \nu(\mathcal{C})$.

Proposition 7.3. *The following statements hold:*

(1) Given a graph G with distinct vertices s, t, the clutter of st-paths packs, and the clutter of minimal st-cuts packs.

(2) Given a bipartite simple graph G, the clutter of edges packs, and the clutter of minimal vertex covers packs.

(3) Q_6 does not pack, and $b(Q_6)$ packs.

Proof. **(1)** By Theorem 1.1 (Menger), the maximum number of edge-disjoint st-paths is equal to the minimum cardinality of an st-cut, so the clutter of st-paths packs. Denote by $\mathcal{C}$ the clutter of minimal st-cuts of G. We may

[3] A packing is also referred to as a *matching*.

assume that G has no empty st-cut, so G has at least one st-path. Notice that $\tau(\mathcal{C})$, the minimum cardinality of an st-path, is simply the distance between s,t. To prove that $\mathcal{C}$ packs, it suffices to exhibit $\tau(\mathcal{C})$ disjoint st-cuts. To this end, for each $i \in [\tau(\mathcal{C})]$, denote by $U_i \subseteq V(G)$ the set of vertices within distance $i-1$ from s. Notice that $s = U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_{\tau(\mathcal{C})} \subseteq V(G) - \{t\}$, and that $\delta(U_1), \delta(U_2), \ldots, \delta(U_{\tau(\mathcal{C})})$ are disjoint st-cuts, as required.[4] **(2)** It follows from Theorem 5.2 (Kőnig) that the maximum cardinality of a matching in G is equal to the minimum cardinality of a vertex cover of G, so the clutter of edges of G packs. We leave it as an easy exercise to prove that the clutter of minimal vertex covers of G packs. **(3)** Q_6 does not pack as $\tau(Q_6) = 2 > 1 = \nu(Q_6)$. On the other hand, $b(Q_6)$ packs as $\tau(b(Q_6)) = 3$ and $b(Q_6)$ has disjoint members $\{1,2\},\{3,4\},\{5,6\}$. □

Let $\mathcal{C}$ be a clutter over ground set E. Take non-negative weights $w \in \mathbb{R}^E_+$. A *weighted packing* is a collection of members such that every element e is contained in at most w_e of the members. (Notice that a member may be taken more than once.) Denote by $\nu(\mathcal{C}, w)$ the maximum size of a weighted packing. Given a cover B, its *weight* is $w(B) := \sum_{e \in B} w_e$. Denote by $\tau(\mathcal{C}, w)$ the minimum weight of a cover. Notice that for weights $\mathbf{1}$, weighted packings are precisely packings and cover weights are precisely cover cardinalities, so $\nu(\mathcal{C}, \mathbf{1}) = \nu(\mathcal{C})$ and $\tau(\mathcal{C}, \mathbf{1}) = \tau(\mathcal{C})$.

Remark 7.4. *Given a clutter $\mathcal{C}$ over ground set E and weights $w \in \mathbb{R}^E_+$,*

$$\tau(\mathcal{C}, w) \geq \nu(\mathcal{C}, w).$$

Proof. Take a cover B and a weighted packing $C_1, \ldots, C_k$. Then

$$w(B) = \sum_{e \in B} w_e \geq \sum_{e \in B} |\{i \in [k] : e \in C_i\}| = \sum_{i \in [k]} |\{e \in B : e \in C_i\}| = \sum_{i \in [k]} |B \cap C_i| \geq k.$$

Since this is true for all covers and weighted packings, the inequality $\tau(\mathcal{C}, w) \geq \nu(\mathcal{C}, w)$ follows. □

Consider the associated set covering program

$$(P) \qquad \begin{array}{ll} \min & w^\top x \\ \text{s.t.} & \sum (x_e : e \in C) \geq 1 \quad \forall C \in \mathcal{C} \\ & x \geq \mathbf{0} \end{array}$$

As the $0-1$ solutions of (P) are precisely the covers, it follows that $\tau(\mathcal{C}, w)$ computes the optimal value of a $0-1$ solution, and hence an integral solution, to (P). Consider the dual program

$$(D) \qquad \begin{array}{ll} \max & \sum (y_C : C \in \mathcal{C}) \\ \text{s.t.} & \sum (y_C : C \in \mathcal{C}, e \in C) \leq w_e \quad \forall e \in E \\ & y \geq \mathbf{0} \end{array}$$

As the integral solutions of (D) are precisely the weighted packings, we get that $\nu(\mathcal{C}, w)$ computes the optimal value of an integral solution to (D). In particular, linear program duality offers an alternate proof of the inequality $\tau(\mathcal{C}, w) \geq \nu(\mathcal{C}, w)$. We will refer to each solution of (D) as a *fractional* weighted packing, and its *value* is the objective value of the solution.

[4] We just proved the min-work max-potential theorem of Duffin (1962).

We say that $\mathcal{C}$ is *Mengerian* if for all weights $w \in \mathbb{Z}_+^E$, the minimum weight of a cover is equal to the maximum size of a weighted packing:

$$\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w).$$

The discussion we just had implies that $\mathcal{C}$ is Mengerian if, and only if, the corresponding set covering program (P) is totally dual integral.[5] As we know, total dual integrality is a notion stronger than primal integrality. We say that $\mathcal{C}$ is *ideal* if for all weights $w \in \mathbb{Z}_+^E$, the minimum weight of a cover is equal to the maximum value of a fractional weighted packing. Equivalently, by LP Strong Duality, $\mathcal{C}$ is ideal if for all weights $w \in \mathbb{Z}_+^E$, the set covering program (P) has an integral optimal solution, i.e. the optimal value of (P) is $\tau(\mathcal{C}, w)$. Recall that $\mathcal{C}$ is ideal if, and only if, the set covering polyhedron $\left\{x \in \mathbb{R}_+^E : M(\mathcal{C})x \geq \mathbf{1}\right\}$ is integral. Studying Mengerian and ideal clutters is just as general as studying integral and totally dual integral set covering systems:

Remark 7.5. *Take a $0-1$ matrix A with column labels E. Let $\mathcal{C}$ be the clutter over ground set E whose members correspond to the minimal rows of A. Then the following statements hold:*

- $x \geq \mathbf{0}, Ax \geq \mathbf{1}$ *is totally dual integral if, and only if, $\mathcal{C}$ is Mengerian,*
- $\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$ *is integral if, and only if, $\mathcal{C}$ is ideal.*

Notice that a Mengerian clutter is always ideal.

[5] In the literature, the Mengerian property is also referred to as the *max-flow min-cut* property.