CO 750 Packing and Covering: Lecture 12

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7.2 Packing and covering parameters

Last time we defined ideal and Mengerian clutters. Recall that a clutter is ideal if the associated set covering program is integral, and it is Mengerian if the associated set covering program is totally dual integral. So clearly, a Mengerian clutter is always ideal. However, in contrast to Theorem 6.2 in the set packing case, an ideal clutter is not necessarily Mengerian:

Remark 7.6. *The following statements hold:*

(1) Q_6 *is an ideal clutter that is not Mengerian,*

(2) $b(Q_6)$ *is a Mengerian clutter.*

Proof. **(1)** We saw in Assignment 1 that Q_6 is ideal. On the other hand, as Q_6 does not pack, it is not Mengerian. **(2)** We leave this as an exercise. □

This remark also shows that being Mengerian is not closed under taking blockers. However, much like the pluperfect graph theorem – Theorem 6.5 – in the set packing case, being ideal *is* closed under taking blockers.

7.3 The width-length inequality

The following "width-length" inequality is the analogue of the max-max inequality, Theorem 5.6, for set covering polyhedra. Alfred Lehman proved this inequality and wrote it up in 1963, taught it to Ray Fulkerson in 1965 at RAND Corporation, but the result was not published until much later in 1979:

Theorem 7.7 (Lehman 1979)**.** *Let* $\mathcal{C}$ *be a clutter over ground set* E*. Then* $\mathcal{C}$ *is ideal if, and only if, for all* $w, \ell \in \mathbb{R}^E_+$,

$$\min\{w(C) : C \in \mathcal{C}\} \cdot \min\{\ell(B) : B \in b(\mathcal{C})\} \leq w^\top \ell.$$

Proof. Suppose first that $\mathcal{C}$ is ideal. Take $w, \ell \in \mathbb{R}^E_+$. Let $\tau := \tau(\mathcal{C}, \ell) = \min\left\{\ell(B) : B \in b(\mathcal{C})\right\}$. Since $\mathcal{C}$ is ideal, there is a fractional ℓ-weighted packing $y \in \mathbb{R}^{\mathcal{C}}_+$ of value τ:

$$\sum \left(y_C : C \in \mathcal{C}\right) = \tau$$
$$\sum \left(y_C : e \in C \in \mathcal{C}\right) \leq \ell_e \quad \forall e \in E.$$

Now we have

$$\begin{aligned} w^\top \ell = \sum_{e\in E} w_e \ell_e \geq \sum_{e\in E} w_e \left[\sum \left(y_C : e \in C \in \mathcal{C}\right)\right] &= \sum_{C\in\mathcal{C}} y_C \cdot w(C) \\ &\geq \min\left\{w(C) : C \in \mathcal{C}\right\} \cdot \sum_{C\in\mathcal{C}} y_C \\ &= \min\left\{w(C) : C \in \mathcal{C}\right\} \cdot \tau \\ &= \min\left\{w(C) : C \in \mathcal{C}\right\} \cdot \min\left\{\ell(B) : B \in b(\mathcal{C})\right\}, \end{aligned}$$

as required. Suppose conversely that the width-length inequality holds for all $w, \ell \in \mathbb{R}^E_+$. We will show that $\mathcal{C}$ is ideal. To this end, take an arbitrary $\ell \in \mathbb{R}^E_+$, and let $x^\star$ be an optimal solution to

$$\begin{array}{lll} \min & \ell^\top x & \\ \text{s.t.} & x(C) \geq 1 & \forall\, C \in \mathcal{C} \\ & x \geq \mathbf{0}. & \end{array}$$

We will show that

$$\ell^\top x^\star = \min\left\{\ell(B) : B \in b(\mathcal{C})\right\},$$

thereby finishing the proof. Well, it is clear that $\leq$ holds above. We will prove that $\geq$ holds as well. By the width-length inequality,

$$\begin{aligned} \ell^\top x^\star &\geq \min\left\{\ell(B) : B \in b(\mathcal{C})\right\} \cdot \min\left\{x^\star(C) : C \in \mathcal{C}\right\} \\ &\geq \min\left\{\ell(B) : B \in b(\mathcal{C})\right\}. \end{aligned}$$

as required. □

As an immediate consequence, we get the following analogue of the pluperfect graph theorem, Theorem 6.5:

Theorem 7.8 (Lehman 1979)**.** *If a clutter is ideal, then so is its blocker.*

7.4 Deletions, contractions and minors

Let $\mathcal{C}$ be a clutter over ground set E, and take an element $e \in E$. We will define two clutters over ground set $E - \{e\}$. The *deletion* is the clutter

$$\mathcal{C} \setminus e := \{C \in \mathcal{C} : e \notin C\}$$

while the *contraction* is the clutter

$$\mathcal{C}/e := \text{ the minimal sets of } \{C - \{e\} : C \in \mathcal{C}\}.$$

Notice that deletion and contraction are blocking operations:

Proposition 7.9. *Let $\mathcal{C}$ be a clutter over ground set E. Then for $e \in E$, $b(\mathcal{C} \setminus e) = b(\mathcal{C})/e$ and $b(\mathcal{C}/e) = b(\mathcal{C}) \setminus e$.*

Proof. Let us first prove that $b(\mathcal{C} \setminus e) = b(\mathcal{C})/e$. If B' is a cover of $\mathcal{C} \setminus e$ then $B' \cup \{e\}$ is a cover of $\mathcal{C}$. So every member of $b(\mathcal{C} \setminus e)$ contains a member of $b(\mathcal{C})/e$. For the reverse inclusion, if B is a cover of $\mathcal{C}$ then $B - \{e\}$ is a cover of $\mathcal{C} \setminus e$. So every member of $b(\mathcal{C})/e$ contains a member of $b(\mathcal{C} \setminus e)$. Remark **??** implies that $b(\mathcal{C} \setminus e) = b(\mathcal{C})/e$. To prove the second equation, let us apply the first equation to $b(\mathcal{C})$:

$$b(b(\mathcal{C}) \setminus e) = b(b(\mathcal{C}))/e = \mathcal{C}/e.$$

Taking blockers yields $b(\mathcal{C}) \setminus e = b(\mathcal{C}/e)$, thereby proving the second equation. □

For disjoint subsets $I, J \subseteq E$, the following clutter over ground set $E - (I \cup J)$,

$$\mathcal{C} \setminus I/J := \text{ the minimal sets of } \{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}$$

is a *minor* of $\mathcal{C}$ obtained after deleting I and contracting J. If $I \cup J \neq \emptyset$, then $\mathcal{C} \setminus I/J$ is a *proper* minor. By the proposition above, $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$. From an optimization point of view, minors operations are quite natural:

Remark 7.10. *Take a clutter $\mathcal{C}$ over ground set E, and disjoint subsets $I, J \subseteq E$. Then the linear programs*

$$\min\{w^\top x : M(\mathcal{C} \setminus I/J)x \geq \mathbf{1}, x \geq \mathbf{0}\} = \max\{\mathbf{1}^\top y : M(\mathcal{C} \setminus I/J)^\top y \leq w, y \geq \mathbf{0}\}$$

for $w \in \mathbb{R}_+^{E-(I\cup J)}$, are equivalent to the linear programs

$$\min\{w^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\} = \max\{\mathbf{1}^\top y : M(\mathcal{C})^\top y \leq w, y \geq \mathbf{0}\}$$

for $w \in \mathbb{R}_+^E$ such that $w_e = 0$ for all $e \in I$ and $w_f = +\infty$ for all $f \in J$.

As an immediate consequence,

Remark 7.11 (Seymour 1977)**.** *If a clutter is ideal (resp. Mengerian), then so is any minor of it.*

8 Ideal clutters

We will see two rich classes of ideal clutters that are quite different in nature, suggesting that ideal clutters form a much richer class than perfect clutters. Unfortunately for us, it also suggests that studying general ideal clutters is more complicated than perfect clutters. Indeed, this is confirmed by a negative complexity result on detecting idealness that we will mention at the end of this section.

8.1 Dicuts and dijoins

Let $D = (V, A)$ be a digraph. We say that D is *strongly connected* if for all distinct vertices $s, t \in V$, there is an (s,t)-dipath. Take a non-empty and proper subset U of V. We say that the cut $\delta^+(U)$ is a *dicut* if $\delta^-(U) = \emptyset$; that is, $\delta^+(U)$ is a dicut if it has no in-coming arc; we will refer to U as an *out-shore* of $\delta^+(U)$.

Remark 8.1. *A digraph is strongly connected if, and only if, it has no dicut.*

Proof. Take a digraph $D = (V, A)$. Suppose first that D is strongly connected. Let $\delta^+(U)$ be a cut, and take vertices $t \in U$ and $s \in V - U$. Since there is an (s,t)-dipath, it follows that $\delta^-(U) \neq \emptyset$, implying in turn that $\delta^+(U)$ is not a dicut. Suppose conversely that D is not strongly connected. Then there are distinct vertices s, t without an (s,t)-dipath. Let $\overline{U}$ be the set of all vertices that can be reached from s. Clearly, $s \in \overline{U}$ and $t \notin \overline{U}$, and by construction, $\delta^-(U) = \delta^+(\overline{U}) = \emptyset$, so $\delta^+(U)$ is a dicut. □

Given a digraph, what is the minimum number of arcs whose contraction makes the digraph strongly connected? By the remark above, we can rephrase the question as, what is the covering number of the clutter of dicuts of a digraph? In this section, we will answer this question by showing that in a digraph, the clutter of dicuts packs.