CO 750 Packing and Covering: Lecture 13

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8.1 Dicuts and dijoins

Let $D = (V, A)$ be a digraph. Take a non-empty and proper subset U of V. Recall that the cut $\delta^+(U)$ is a *dicut* if $\delta^-(U) = \emptyset$; that is, $\delta^+(U)$ is a dicut if it has no in-coming arc; we will refer to U as an *out-shore* of $\delta^+(U)$. Last time, we proved the following:

Remark 8.1. *A digraph is strongly connected if, and only if, it has no dicut.*

Given a digraph, what is the minimum number of arcs whose contraction makes the digraph strongly connected? By the remark above, we can rephrase the question as, what is the covering number of the clutter of dicuts of a digraph? In this section, we will answer this question by showing that in a digraph, the clutter of dicuts packs. To prove this, we will need a coloring lemma.

Let V be a finite set, and let $\mathcal{S}$ be a family of subsets of V (some subsets may be equal). We say that two sets $S, S' \in \mathcal{S}$ are *crossing* if the four sets $S_1 \cap S_2, S_1 - S_2, S_2 - S_1, V - (S_1 \cup S_2)$ are non-empty. Notice that if S_1, S_2 are crossing, then so are $S_1, \overline{S_2}$. We say that $\mathcal{S}$ is *cross-free* if it has no crossing sets, that is, for all $S_1, S_2 \in \mathcal{S}$, either $S_1 \cap S_2 = \emptyset, S_1 \subseteq S_2, S_2 \subseteq S_1$ or $S_1 \cup S_2 = V$. Observe that if $\mathcal{S}$ is cross-free, then so is any family obtained from $\mathcal{S}$ after complementing some sets. We will need the following *dicut coloring lemma*: [1]

Lemma 8.2 (Lucchesi and Younger 1976)**.** *Let $D = (V, A)$ be a digraph, and $\mathcal{F}$ a family of (possibly equal) dicuts whose out-shores form a cross-free family. Take an integer $k \geq 1$. If every arc appears in at most k dicuts of $\mathcal{F}$, then the dicuts of $\mathcal{F}$ can be k-colored so that dicuts of the same color are arc-disjoint.*

Proof. Denote by $\mathcal{S}$ the family of the out-shores of $\mathcal{F}$. By definition, $\mathcal{S}$ is a cross-free family. In particular, if an arc belongs to dicuts $\delta^+(U_1), \delta^+(U_2) \in \mathcal{F}$, then either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. As a result,

($\star$) given the dicuts of $\mathcal{F}$ containing a fixed arc, their out-shores are nested.

This observation is crucial to the proof. Take an arbitrary vertex $r \in V$, and let $\mathcal{S}'$ be obtained from $\mathcal{S}$ after complementing each out-shore containing r. Clearly, $\mathcal{S}'$ is a cross-free family, and as no set contains r, it follows that for all $S_1, S_2 \in \mathcal{S}'$, either $S_1 \cap S_2 = \emptyset$, $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. That is, $\mathcal{S}'$ is a laminar family. We may

[1] Lucchesi and Younger (1976) called this the *disjunctive partition property*.

therefore represent $\mathcal{S}'$ by an r-arborescence T' whose arcs are in a one-to-one correspondence with the sets of $\mathcal{S}'$. Let T be the directed tree obtained from T' as follows: for every set $S' \in \mathcal{S}'$ obtained by complementing an out-shore of $\mathcal{S}$, flip the arc of T' corresponding to S'. Notice the one-to-one correspondence between the arcs of T and the out-shores of $\mathcal{S}$. Notice further that by $(\star)$, the dicuts of $\mathcal{F}$ containing a fixed arc correspond to a directed path in T of length at most k. Thus, to prove the lemma, it suffices to k-color the arcs of T so that in every directed path of length at most k, the arcs get different colors. To this end, partition the vertices of T into layers $L_0, L_1, L_2, \ldots$ so that each arc of T goes from some layer L_{i+1} to the layer L_i. Color the arcs going from layer L_{i+1} to layer L_i with color $i \pmod k$, for each $i \geq 0$. It is then easy to see that the arcs of a directed path of length at most k get different colors, as required. $\square$

Let $D = (V, A)$ be a digraph. A *dijoin of* D is an arc subset B such that D/B is strongly connected. Notice that by Remark 8.1, an arc subset is a dijoin if and only if it intersects every dicut. In other words, the dijoins of D are precisely the covers of the clutter of dicuts. The proof of the following theorem is due to Lovász (1976).

Theorem 8.3 (Lucchesi and Younger 1976)**.** *In a digraph, the maximum number of disjoint dicuts is equal to the minimum cardinality of a dijoin. That is, the clutter of dicuts of a digraph packs.*

Proof. Let $D = (V, A)$ be a digraph. We will prove by induction on $|A| \geq 1$ that the clutter of dicuts packs. The base case $|A| = 1$ is trivial. For the induction step, assume that $|A| \geq 2$. We may assume that the underlying undirected graph of D is connected, and that D is not strongly connected. Let ν be the maximum size of a packing of dicuts. Let us say that an arc is *essential* if it is used in every maximum packing of dicuts.

Claim. *D has an an essential arc.*

Proof of Claim. Suppose otherwise. Then for each arc, we have a packing of ν disjoint dicuts of D excluding the arc. Doing this for every arc of D, we get a family $\mathcal{F}$ such that

> $(\star)$ $\mathcal{F}$ is a family of dicuts of D such that $|\mathcal{F}| = |A| \cdot \nu$, and every arc of D is used in at most $|A| - 1$ dicuts of $\mathcal{F}$.

We will recursively update the family $\mathcal{F}$ so that each intermediate family satisfies $(\star)$, and at the end, the out-shores form a cross-free family. If the out-shores of $\mathcal{F}$ form a cross-free family, then we are done. Otherwise, take dicuts $\delta^+(U_1), \delta^+(U_2) \in \mathcal{F}$ where U_1, U_2 are crossing. Then $\delta^+(U_1 \cap U_2), \delta^+(U_1 \cup U_2)$ are also dicuts such that

$$\delta^+(U_1 \cap U_2) \cap \delta^+(U_1 \cup U_2) \subseteq \delta^+(U_1) \cap \delta^+(U_2) \quad \text{and} \quad \delta^+(U_1 \cap U_2) \cup \delta^+(U_1 \cup U_2) \subseteq \delta^+(U_1) \cup \delta^+(U_2).$$

We update $\mathcal{F}$ by replacing the dicuts $\delta^+(U_1), \delta^+(U_2)$ by the dicuts $\delta^+(U_1 \cap U_2), \delta^+(U_1 \cup U_2)$. The inclusions above imply that $\mathcal{F}$ still satisfies $(\star)$. Since at each iteration, the potential $\sum_{\delta^+(U) \in \mathcal{F}} |U|^2$ strictly increases, we will eventually reach a family $\mathcal{F}$ satisfying $(\star)$ whose out-shores form a cross-free family. Therefore, by the Dicut Coloring Lemma 8.2, we may $(|A| - 1)$-color the dicuts of $\mathcal{F}$ so that each color class is a packing of dicuts. One of the color classes has cardinality at least $\frac{|A| \cdot \nu}{|A| - 1} > \nu$, implying in turn that D has a packing of $\nu + 1$ dicuts, a contradiction. Thus, D has an essential arc. $\Diamond$

Let e be an essential arc of D, and let $C_1, \dots, C_\nu$ be a maximum packing of dicuts such that $e \in C_\nu$. To complete the induction step, it suffices to exhibit a dijoin of cardinality ν. As e is essential, the dicuts $C_1, \dots, C_{\nu-1}$ give a maximum packing of dicuts of D/e. Thus, by the induction hypothesis, D/e has a dijoin B' of cardinality $\nu - 1$. Notice that $B' \cup \{e\}$ is a dijoin of D of cardinality ν, as required. This finishes the proof. $\square$

Using this result, we can prove the following:

Corollary 8.4. *The clutter of dicuts of a digraph is Mengerian, and therefore ideal.*

Proof. Let $\mathcal{C}$ be the clutter of dicuts of digraph $D = (V, A)$. To prove that $\mathcal{C}$ is Mengerian, take weights $w \in \mathbb{Z}_+^A$. We need to show that $\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w)$, that is, the minimum weight of a dijoin is equal to the maximum size of a weighted packing of dicuts. Construct a digraph D' starting from D as follows: for each arc e with $w_e = 0$ contract arc e, and for each arc w with $w_e \geq 1$ replace arc e by w_e arcs in series (forming a directed path). Then $\tau(\mathcal{C}, w)$ is equal to the minimum cardinality of a dijoin of D', while $\nu(\mathcal{C}, w)$ is equal to the maximum number of disjoint dicuts of D'. Therefore, Theorem 8.3 implies that $\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w)$, as required. $\square$

Together with Theorem 7.8, this result implies that,

Corollary 8.5. *The clutter of dijoins of a digraph is ideal.*

Schrijver (1980) showed that in contrast to dicuts, the clutter of dijoins is *not* necessarily Mengerian. Nevertheless, Woodall (1978) conjectures that the clutter of dijoins always packs. (Why would Woodall's conjecture not imply that the clutter of dijoins is Mengerian?)