CO 750 Packing and Covering: Lecture 16

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8.3 Testing idealness is co-NP-complete.

We saw two rich classes of ideal clutters, namely the clutter of dicuts of a digraph and the clutter of T-joins of a graph. This suggests that studying general ideal clutters is more complicated than perfect clutters. Indeed, this is confirmed by a negative complexity result on detecting idealness that we will mention here. Let A be a $0-1$ matrix. Consider the following problem:

Is A an ideal matrix?

This is a co-NP problem: to certify that A is non-ideal, all we need is a fractional point $x^\star \in Q(A) = \{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$ along with a full-rank row subsystem $A'x \geq b'$ of $\begin{pmatrix} A \\ I \end{pmatrix} x \geq \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$ such that $A'x^\star = b'$. In fact, as the following result claims, this problem is one of the most difficut problems in the co-NP class:

Theorem 8.15 (Ding, Feng, Zang 2008)**.** *Let A be a $0-1$ matrix, where every column has exactly two 1s. Then the problem*

Is A an ideal matrix?

is co-NP-complete.

In other words, given a general $0-1$ matrix that is a priori ideal, we cannot convince an adversary in polynomial time that A is indeed an ideal matrix, unless P and co-NP are equal. This means that unlike perfect clutters, ideal clutters do not admit a polynomial characterization in this model. (The authors above proved that “Is A a Mengerian matrix?” is a also co-NP-complete problem.) Let us study ideal clutters from a different angle.

9 Minimally non-ideal clutters

By Remark 7.11, we know that if a clutter is ideal, then so is any minor of it. In other words, the class of ideal clutters is minor-closed. As a result, we may indirectly study the class by characterizing the excluded minors defining the class. We say that a clutter is *minimally non-ideal (mni)* if it is non-ideal, and every proper minor of it is ideal. It follows from Remark 7.11 and Theorem 7.8 that,

Remark 9.1. *The following statements hold:*

- *a non-ideal clutter is minimally non-ideal if every single deletion and contraction minor is ideal,*
- *a clutter is ideal if, and only if, it has no minimally non-ideal minor,*
- *if a clutter is minimally non-ideal, then so is its blocker.*

As we will see, mni clutters split into two classes that behave quite differently from one another. We will study each class independently.

9.1 The deltas

Given a clutter $\mathcal{C}$, we may obtain another clutter $\mathcal{C}'$ by relabeling the elements of $\mathcal{C}$. We will say that $\mathcal{C}, \mathcal{C}'$ are *isomorphic* and write $\mathcal{C} \cong \mathcal{C}'$. Take an integer $n \geq 3$. Consider the clutter over ground set $[n] := \{1, 2, 3, \ldots, n\}$ whose members are

$$\Delta_n := \big\{\{1,2\}, \{1,3\}, \ldots, \{1,n\}, \{2,3,\ldots,n\}\big\}$$

and whose incidence matrix is

$$M(\Delta_n) = \begin{pmatrix} 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \\ & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

We refer to Δ_n, and any clutter isomorphic to it, as a *delta of dimension* n. Notice that the elements and members of Δ_n correspond to the points and lines of a degenerate projective plane.[1]

Theorem 9.2. *Take an integer $n \geq 3$. Then,*

(1) $b(\Delta_n) = \Delta_n$*,*

(2) $\min\{\mathbf{1}^\top x : M(\Delta_n)x \geq \mathbf{1}\}$ *has no integral optimal solution, and*

(3) Δ_n *is minimally non-ideal.*

Proof. **(1)** As Δ_n does not have disjoint members, every member is also a cover, so every member of Δ_n contains a member of $b(\Delta_n)$. Conversely, let B be a minimal cover of Δ_n. If $1 \notin B$, then as B intersects each one of $\{1,2\}, \{1,3\}, \ldots, \{1,n\}$, it follows that $\{2,3,\ldots,n\} \subseteq B$. If $1 \in B$, then as B intersects $\{2,3,\ldots,n\}$, it follows that $\{1,i\} \subseteq B$ for some $i \in \{2,3,\ldots,n\}$. In both cases, we see that B contains a member, so every member of $b(\Delta_n)$ contains a member of Δ_n. It therefore follows from Remark 6.6 that $b(\Delta_n) = \Delta_n$. **(2)** In particular, $\tau(\mathcal{C}) = 2$. Consider now the fractional feasible solution $x^\star := \begin{pmatrix} \frac{n-2}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \end{pmatrix}$. The objective

[1] In the literature, a delta of dimension n is called a degenerate projective plane of order $n-1$. However, as there are other degenerate projective planes, we refrain from using this terminology.

value of this solution is $1 + \frac{n-2}{n-1} < 2 = \tau(\mathcal{C})$, so (2) holds. **(3)** It follows from (2) that Δ_n is non-ideal. To prove that Δ_n is mni, we need to show for each $e \in [n]$ that $\Delta_n \setminus e$ and Δ_n / e are ideal clutters. In fact, since

$$\Delta_n \setminus e = b(b(\Delta_n \setminus e)) = b(b(\Delta_n)/e) = b(\Delta_n/e)$$

by (1), it suffices by Theorem 7.8 to show that one of $\Delta_n \setminus e, \Delta_n/e$ is ideal. By the symmetry between the elements $2, 3, \ldots, n$, we may assume that $e \in \{1, n\}$. Observe that

$$\Delta_n \setminus 1 = \{\{2, 3, \ldots, n\}\}$$

and

$$\Delta_n/n = \{\{1\}, \{2, \ldots, n-1\}\}.$$

We leave it as an exercise for the reader to see that these clutters are indeed ideal. Thus, Δ_n is mni. □

The deltas form an important class of mni clutters that is tractable in the sense that it is easy to see whether a clutter has a delta minor or not. To see why, we need the following result:

Theorem 9.3 (Abdi, Cornuéjols, Pashkovich 2017)**.** *Take a clutter $\mathcal{C}$ over ground set E and an element $e \in E$. If there are distinct members C_1, C_2, C such that $e \in C_1 \cap C_2$, $e \notin C$ and $(C_1 \cup C_2) - \{e\} \subseteq C$, then $\mathcal{C}$ has a delta minor that can be found in time $O(|E||\mathcal{C}|)$.*

Proof. Let us call (C_1, C_2, C) a *bad triple through* e. We may assume that in every proper minor of $\mathcal{C}$ where e is present, no bad triple through e exists. We will prove that $\mathcal{C}$ itself is a delta. The minimality assumption implies that

(1) $C_1 \cap C_2 = \{e\}$,

because for $I := (C_1 \cap C_2) - \{e\}$, the minor $\mathcal{C}/I$ has the bad triple $(C_1 - I, C_2 - I, C - I)$ through e.

The minimality assumption also implies that

(2) $\{e\} \cup C = E$,

because for $J := E - (\{e\} \cup C)$, $\mathcal{C} \setminus J$ has the same bad triple (C_1, C_2, C) through e.

Next we claim that

(3) $|C_1| = |C_2| = 2$.

To see this, suppose for a contradiction that one of C_1, C_2, say C_1, has cardinality at least 3. Pick an element $h \in C_1 - \{e\}$, and note that by (1), $h \notin C_2$. Consider the minor $\mathcal{C}' := \mathcal{C}/h$, for which $C_1' := C_1 - \{h\}$ and $C' := C - \{h\}$ are still members. Notice that C_2 contains a member C_2' of $\mathcal{C}'$, for which it is easy to see that $e \in C_2'$ and $C_2' \neq \{e\}$. But now $\mathcal{C}'$ has a bad triple (C_1', C_2', C') through e, a contradiction to our minimality assumption. This proves (3).

Now let $X := \big\{f \in E : \{e, f\} \text{ is a member}\big\}$. So $|X| \geq 2$ by (3), and $X \subseteq C$ by (2). Our last claim is that

(4) $X = C$.

For if not, pick an element $h \in C - X$, and note that $\mathcal{C}/h$ has a bad triple $(C_1, C_2, C - \{h\})$ through e, contradicting the minimality assumption. Thus, $X = C$. Hence,

$$\mathcal{C} \supseteq \{\{e, f\} : f \in C\} \cup \{C\}.$$

Since $\{e\} \cup C = E$ by (2), and $\mathcal{C}$ is a clutter, equality must hold above, implying in turn that $\mathcal{C}$ indeed is a delta, as required. $\square$

We will use this in the next lecture to give a polynomial time algorithm for certifying whether or not a clutter has a delta minor.