CO 750 Packing and Covering: Lecture 17

Ahmad Abdi

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9.1 The deltas, continued

Take an integer $n \geq 3$. Recall that a delta of dimension n is (any clutter isomorphic to) the clutter over ground set $[n]$ whose members are

$$\Delta_n = \big\{\{1,2\},\{1,3\},\ldots,\{1,n\},\{2,3,\ldots,n\}\big\}.$$

We showed last time that the deltas are mni. We also proved the following tool for finding a delta minor:

Theorem 9.3 (Abdi, Cornuéjols, Pashkovich 2017)**.** *Take a clutter $\mathcal{C}$ over ground set E and an element $e \in E$. If there are distinct members C_1, C_2, C such that $e \in C_1 \cap C_2$, $e \notin C$ and $(C_1 \cup C_2) - \{e\} \subseteq C$, then $\mathcal{C}$ has a delta minor that can be found in time $O(|E||\mathcal{C}|)$.*

Let us say that two elements of a clutter are *exclusive* if they are never used together in a member. That is, distinct elements f, g of a clutter are exclusive if no member contains $\{f, g\}$. Notice that exclusive elements remain exclusive in every minor that they are present in. The preceding result has the following immediate consequence:

Corollary 9.4. *Let $\mathcal{C}$ be a clutter without a delta minor, and take distinct elements e, f, g. If $\{e, f\}, \{e, g\}$ are members, then f, g are exclusive.*

We are now ready to prove the following:

Theorem 9.5 (Abdi, Cornuéjols, Pashkovich 2017)**.** *Let $\mathcal{C}$ be a clutter over ground set E. Then in time $O(|E|^3|\mathcal{C}|^3)$, one can find a delta minor or certify that none exists.*

Proof. We claim that the following statements are equivalent:

(i) $\mathcal{C}$ does not have a delta minor,

(ii) for all distinct members C_1, C_2 with $C_1 \cap C_2 \neq \emptyset$ and for all elements e, f, g with $e \in C_1 \cap C_2, f \in C_1 - C_2, g \in C_2 - C_1$, the following holds: for $X := (C_1 \cup C_2) - \{e, f, g\}$ and $\mathcal{C}' := \mathcal{C}/X$, either $\{e, f\} \notin \mathcal{C}'$ or $\{e, g\} \notin \mathcal{C}'$ or f, g are exclusive elements of $\mathcal{C}'$.

(ii) $\Rightarrow$ (i): Assume that (i) does not hold. Suppose $\mathcal{C}$ has a delta minor obtained after deleting $I \subseteq E$ and contracting $J \subseteq E$. Pick elements $e, f, g \in E - (I \cup J)$ such that $\{e, f\}, \{e, g\}$ are members of the delta minor. Notice that f, g are not exclusive elements in the delta minor, and so they are not exclusive in $\mathcal{C}$. Let C_1, C_2 be members of $\mathcal{C}$ such that $\{e, f\} \subseteq C_1 \subseteq \{e, f\} \cup J$ and $\{e, g\} \subseteq C_2 \subseteq \{e, g\} \cup J$. It can be readily checked that C_1, C_2 and e, f, g do not satisfy (ii). Thus, (ii) does not hold. **(i) $\Rightarrow$ (ii)**: Assume that (i) holds. Take $C_1, C_2, e, f, g, X, \mathcal{C}'$ as in (ii) where $\{e, f\} \in \mathcal{C}'$ and $\{e, g\} \in \mathcal{C}'$. Since $\mathcal{C}$ has no delta minor, neither does $\mathcal{C}'$, so by Corollary 9.4, f and g are exclusive elements of $\mathcal{C}'$, so (ii) holds. Hence, (i) and (ii) are equivalent. Since (ii) may be verified in time $O(|E|^3|\mathcal{C}|^3)$, and if (ii) does not hold, a delta minor can be found in time $O(|E||\mathcal{C}|)$ using Theorem 9.3, we can find a delta minor or certify that none exists in time $O(|E|^3|\mathcal{C}|^3)$. $\square$

9.2 The other minimally non-ideal clutters

We now move on to the mni clutters different from the deltas. Take an odd integer $n \geq 5$. Consider the clutter over ground set $[n]$ whose members are

$$\mathcal{C}_n^2 := \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}.$$

The clutter $\mathcal{C}_n^2$, and any clutter isomorphic to it, is called an *odd hole of dimension* n. It may be readily checked that odd holes are mni. In contrast to Theorem 9.5,

Theorem 9.6 (Ding, Feng, Zang 2008)**.** *Finding an odd hole minor in a clutter is an NP-complete problem.*

That is, unless P and NP are equal, there is no algorithm for finding an odd hole minor in a clutter $\mathcal{C}$ over ground set E, whose running time is polynomial in $|E|$ and $|\mathcal{C}|$. Theorems 9.5 and 9.6 highlight the difference between the deltas and the other mni clutters. There are many mni clutters: other than the two infinite classes $\{\mathcal{C}_{2n-1}^2 : n \geq 3\}$ and $\{b(\mathcal{C}_{2n-1}^2) : n \geq 3\}$, there are at least two other infinite classes of mni clutters different from the deltas, as well as many sporadic examples. For instance, the clutter of the lines of the Fano plane

$$\mathbb{L}_7 = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 7\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 5, 7\}\} = b(\mathbb{L}_7),$$

as well as $\mathcal{C}_9^2 \cup \{\{3, 6, 9\}\}$ are mni. It may now seem that there is no good characterization of the mni clutters different from the deltas, but this is not the case – Alfred Lehman (1990) provided powerful geometric and combinatorial characterizations of these clutters. Before getting to his characterizations, let us briefly study the geometric aspects of ideal clutters and of minor operations. First off, it is easier to work with polytopes rather than polyhedra:

Proposition 9.7. *Take a clutter $\mathcal{C}$ over ground set E. Then $\mathcal{C}$ is ideal if, and only if, $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$ is an integral polytope.*

Proof. Let $Q := \{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$ and $P := \{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$. If Q is not integral, it has a fractional extreme point $x^\star$, and as $x^\star \leq \mathbf{1}$, it follows that $x^\star$ is also an extreme point of P, so P is not integral.

Conversely, assume that P is not integral, and let $x^\star$ be a fractional extreme point. Let

$$I_{x^\star} := \{e \in E : x^\star_e = 1\}.$$

We prove by induction on $|I_{x^\star}| \geq 0$ that Q has a fractional extreme point. If $I_{x^\star} = \emptyset$, then $x^\star$ is also an extreme point of Q, so we are done. For the induction step, we assume that $|I_{x^\star}| \geq 1$. If for each $e \in I$, there is a member C such that $C \cap I_{x^\star} = \{e\}$, then $x^\star$ is an extreme point of Q also, so we are done. Otherwise, for some $f \in I_{x^\star}$ there is no member C such that $C \cap I_{x^\star} = \{f\}$. That is, there is no member C such that $f \in C$ and $x^\star(C) = 1$. Thus, we may strictly decrease the f^{th} coordinate of $x^\star$ until we get another fractional extreme point $\bar{x}$ of P. Clearly, $I_{\bar{x}} = I_{x^\star} - \{f\}$, so by the induction hypothesis, Q has a fractional extreme point. This completes the induction step. □

For a clutter $\mathcal{C}$, denote by $P(\mathcal{C})$ the set covering polytope $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$. Notice that the covers of $\mathcal{C}$ are precisely the integer extreme points of $P(\mathcal{C})$. (Every integer point of $P(\mathcal{C})$ is also an extreme point.) Moreover, the minors of $\mathcal{C}$ have a natural geometric interpretation in terms of $P(\mathcal{C})$:

Remark 9.8. *Let $\mathcal{C}$ be a clutter over ground set E, and take an element $e \in E$. Then the following statements hold:*

- *$P(\mathcal{C} \setminus e)$ is the restriction $P(\mathcal{C}) \cap \{x : x_e = 1\}$ after dropping coordinate x_e.*
- *$P(\mathcal{C}/e)$ is the restriction $P(\mathcal{C}) \cap \{x : x_e = 0\}$ after dropping coordinate x_e.*

We can now dive into Lehman's characterizations. First up is a brilliant lemma that will be very useful. Take an integer $n \geq 2$, and let A be an $n \times n$ matrix with $0-1$ entries and without a row or a column of all ones. We say that A is *cross regular* if whenever $A_{ij} = 0$, the number of ones in column j is equal to the number of ones in row i.

Lemma 9.9 (Lehman 1990)**.** *The following statements hold:*

(1) Take an integer $n \geq 2$, and let A be a $0-1$ $n \times n$ matrix without a row or a column of all ones, and whenever $A_{ij} = 0$, the number of ones in column j is greater than or equal to the number of ones in row i. Then A is cross regular.

(2) Cross regular matrices cannot differ in just one row.

Proof. **(1)** Suppose A is an $n \times n$ matrix. For each row $i \in [n]$ and column $j \in [n]$, let r_i denote the number of ones in row i and let c_j denote the number of ones in column j. Then

$$\sum_{j\in[n]} c_j = \sum_{j\in[n]} \sum_{i\in[n]:A_{ij}=0} \frac{c_j}{n-c_j} \geq \sum_{j\in[n]} \sum_{i\in[n]:A_{ij}=0} \frac{r_i}{n-r_i} = \sum_{i\in[n]} \sum_{j\in[n]:A_{ij}=0} \frac{r_i}{n-r_i} = \sum_{i\in[n]} r_i.$$

As the left- and right-hand side terms are equal, equality must hold throughout, implying in turn that whenever $A_{ij} = 0$, then $r_i = c_j$. Thus, A is cross regular. **(2)** Suppose for a contradiction that $\begin{pmatrix} B \\ a \end{pmatrix}, \begin{pmatrix} B \\ a' \end{pmatrix}$ are cross regular matrices and $a \neq a'$. We may assume that $a_1 = 1$ and $a'_1 = 0$. Since $\begin{pmatrix} B \\ a \end{pmatrix}$ is cross regular, the first column of B has a zero entry, say it is the first entry. Let $k \geq 0$ be the number of ones in the first column of B. Then as $\begin{pmatrix} B \\ a \end{pmatrix}$ is cross regular, the first row of B has $k+1$ ones. However, as $\begin{pmatrix} B \\ a' \end{pmatrix}$ is also cross regular, the first row of B must have k ones, a contradiction. □

Given a full-dimensional polytope $P \subseteq \mathbb{R}^n$ and a vertex $x^\star$, we say that $x^\star$ is *simple* if it belongs to exactly n facets. Recall that if $x^\star$ is simple, then there are exactly n edges emanating from $x^\star$, each of which is defined uniquely by $n-1$ many of the tight facets. As a result, if $x^\star$ is simple, then it has exactly n adjacent vertices. Lehman proved the following geometric characterization of the mni clutters different from the deltas:

Theorem 9.10 (Lehman 1990). *Let $\mathcal{C}$ be a minimally non-ideal clutter over ground set E that is not a delta, and let $n := |E|$. Let $x^\star$ be a fractional extreme point of $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$. Then the following statements hold:*

(1) $\mathbf{0} < x^\star < \mathbf{1}$*,*

(2) $x^\star$ lies on exactly n facets, that correspond to members $C_1, \ldots, C_n \in \mathcal{C}$, so $x^\star$ is a simple vertex,

(3) the n neighbors of $x^\star$ are integral vertices, that correspond to covers $B_1, \ldots, B_n$ labeled so that for distinct $i, j \in [n]$, $|C_i \cap B_i| > 1$ and $|C_i \cap B_j| = 1$,

(4) $B_1, \ldots, B_n$ are minimal covers,

(5) $C_1, \ldots, C_n$ are precisely the minimum cardinality members of $\mathcal{C}$,

(6) $x^\star$ is the unique fractional extreme point of $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$,

(7) there is an integer $d \geq 1$ such that for each $i \in [n]$, $|C_i \cap B_i| = 1 + d$.

In particular, $x^\star$ is the unique fractional extreme point of $\{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$.

We will prove this theorem next time.