CO 750 Packing and Covering: Lecture 18

Ahmad Abdi

July 4, 2017

9.2 The minimally non-ideal clutters different from the deltas, continued

Take an integer $n \geq 2$, and let A be an $n \times n$ matrix with $0-1$ entries and without a row or a column of all ones. Recall that A is cross regular if whenever $A_{ij} = 0$, the number of ones in column j is equal to the number of ones in row i. Last time, we proved the following lemma:

Lemma 9.9 (Lehman 1990)**.** *The following statements hold:*

(1) Take an integer $n \geq 2$, and let A be a $0-1$ $n \times n$ matrix without a row or a column of all ones, and whenever $A_{ij} = 0$, the number of ones in column j is greater than or equal to the number of ones in row i. Then A is cross regular.

(2) Cross regular matrices cannot differ in just one row.

We are now ready to prove the following geometric characterization of the mni clutters different from the deltas:

Theorem 9.10 (Lehman 1990)**.** *Let $\mathcal{C}$ be a minimally non-ideal clutter over ground set E that is not a delta, and let $n := |E|$. Let $x^\star$ be a fractional extreme point of $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$. Then the following statements hold:*

(1) $\mathbf{0} < x^\star < \mathbf{1}$,

(2) $x^\star$ lies on exactly n facets, that correspond to members $C_1, \ldots, C_n \in \mathcal{C}$ – so $x^\star$ is a simple vertex,

(3) the n neighbors of $x^\star$ are integral vertices, that correspond to covers $B_1, \ldots, B_n$ labeled so that for distinct $i, j \in [n]$, $|C_i \cap B_i| > 1$ and $|C_i \cap B_j| = 1$,

(4) $B_1, \ldots, B_n$ are minimal covers,

(5) $C_1, \ldots, C_n$ are precisely the minimum cardinality members of $\mathcal{C}$,

(6) $x^\star$ is the unique fractional extreme point of $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$,

(7) there is an integer $d \geq 1$ such that for each $i \in [n]$, $|C_i \cap B_i| = 1 + d$.

In particular, $x^\star$ is the unique fractional extreme point of $\{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$.

Proof. Let $P := P(\mathcal{C}) = \{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$. Then for each element $e \in E$, the clutters $\mathcal{C}/e, \mathcal{C} \setminus e$ are ideal, so the polytopes $P \cap \{x : x_e = 0\}$ and $P \cap \{x : x_e = 1\}$ are integral by Remark 9.8, implying in turn for each fractional extreme point $x^\star$ that $0 < x^\star_e < 1$, so **(1)** holds. (The fact that $\mathcal{C}$ is different from a delta will be first used in Claim 4.)

Claim 1. *Let $x^\star$ be a fractional extreme point of P, and let A be an $n \times n$ non-singular submatrix of $M(\mathcal{C})$ such that $Ax^\star = \mathbf{1}$. Then A is cross regular.*

Proof of Claim. Clearly, A has no all ones row, and since $x^\star$ is the unique solution to $Ax^\star = \mathbf{1}$, A has no all ones column either. To prove that A is cross regular, assume that $A_{11} = 0$. Let C be the member corresponding to the first row of A. By Lemma 9.9 (1), it suffices to show that the number of ones in the first column is greater than or equal to $|C|$. To this end, let $\widehat{x} := (1, x^\star_2, \ldots, x^\star_n) \in P \cap \{x : x_1 = 1\}$. Let F be the smallest face of the polytope $P \cap \{x : x_1 = 1\}$ containing $\widehat{x}$. Notice that $a^\top \widehat{x} = 1$ for every row a of A whose first entry is 0. As these rows are linearly independent, and as $\widehat{x}_1 = 1$, it follows that

$$\dim(F) \leq n - \text{ number of 0s in the first column } - 1 = \text{ number of 1s in the first column } - 1.$$

On the other hand, as $P \cap \{x : x_1 = 1\}$ is an integral polytope, F is also an integral polytope, so

$$\widehat{x} = \sum_{i=1}^{k} \lambda_i \chi_{B_i}$$

for some extreme points $\chi_{B_1}, \ldots, \chi_{B_k}$ of F and some $\lambda > \mathbf{0}$ such that $\sum_{i=1}^k \lambda_i = 1$. Notice for each $i \in [k]$ that B_i is a cover, and as $\widehat{x}(C) = 1$, we get that $|B_i \cap C| = 1$. Since $\widehat{x} > \mathbf{0}$, each element of C appears in at least one B_i, so the matrix whose rows are the χ_{B_i}'s has rank at least $|C|$, implying in turn that the affine dimension of the χ_{B_i}'s is at least $|C| - 1$. As a result,

$$\dim(F) \geq |C| - 1.$$

Putting the last two inequalities gives the desired inequality, as desired. $\Diamond$

Claim 2. *Every fractional extreme point of P is simple, that is, it lies on exactly n facets. Thus,* **(2)** *holds.*

Proof of Claim. Suppose for a contradiction that P has a non-simple fractional extreme point $x^\star$. Let A be an $n \times n$ non-singular submatrix of $M(\mathcal{C})$ such that $Ax^\star = \mathbf{1}$. As $x^\star$ is non-simple, there is another row a' of $M(\mathcal{C})$ such that $a'^\top x^\star = 1$. Pick a row a of A such that the matrix A' obtained by replacing a and a' is non-singular. (To find a, write a' as a linear combination of the rows of A, and pick the row a whose coefficient is non-zero.) Then by Claim 1, both A and A' are cross regular, a contradiction to Lemma 9.9 (2) as A and A' differ in exactly one row. $\Diamond$

Claim 3. *P does not have neighboring fractional extreme points. Thus,* **(3)** *holds.*

Proof of Claim. Suppose for a contradiction that P has neighboring fractional extreme points $x^\star, y^\star$. Then there are $n \times n$ non-singular submatrices A, B of $M(\mathcal{C})$ that differ in exactly one row such that $Ax^\star = \mathbf{1} = By^\star$. By Claim 1, both A and B are cross regular, a contradiction to Lemma 9.9 (2). $\diamondsuit$

Now pick a fractional extreme point $x^\star$ of P. By Claims 2 and 3, $x^\star$ lies on n facets and has precisely n neighbors, all of which are integral. Let $C_1, \ldots, C_n \in \mathcal{C}$ be the members corresponding to the facets $x^\star$ sits on, and let $B_1, \ldots, B_n$ be the covers corresponding to the neighbors of $x^\star$, where our labeling satisfies for $i, j \in [n]$ the following:

$$|C_i \cap B_j| \begin{cases} > 1 & \text{if } i = j \\ = 1 & \text{if } i \neq j. \end{cases}$$

Let A (resp. B) be the $0-1$ matrix whose columns are labeled by E and whose rows are the incidence vectors of $C_1, \ldots, C_n$ (resp. $B_1, \ldots, B_n$). Then the equations above imply that

$$AB^\top = J + \mathrm{diag}\left(|C_1 \cap B_1| - 1, \ldots, |C_n \cap B_n| - 1\right).$$

In particular, $AB^\top$ is non-singular, implying in turn that B is non-singular. Moreover, by Claim 1, A is cross regular. Let G be the bipartite representation of A, where column e and row C are adjacent if $e \notin C$. Since A is cross regular, it follows that adjacent vertices of G have the same degree. In particular, every component of G is regular and so it has the same number of vertices in each bipartition.

Claim 4. *G is connected.*

Proof of Claim. Suppose for a contradiction that G is not connected. Then there exist a partition of the rows of A into non-empty parts X_1, X_2 and a partition of the columns of A into non-empty parts $Y_1, Y_2 \subseteq E$ such that $|X_1| = |Y_1|$, $|X_2| = |Y_2|$, and the (X_2, Y_1) and (X_1, Y_2) blocks of A are submatrices of all ones. If $|Y_1| = 1$ or $|Y_2| = 1$, then A has a row with $n-1$ ones, so $\mathcal{C}$ has a delta minor by Theorem 9.3, implying in turn by minimality that $\mathcal{C}$ is a delta, a contradiction as $\mathcal{C}$ is not a delta. Otherwise, $|X_1| = |Y_1| \geq 2$ and $|X_2| = |Y_2| \geq 2$. As a result, for each $i \in [n]$, $|B_i \cap Y_1| = |B_i \cap Y_2| = 1$, implying in turn that the columns of B corresponding to Y_1 have the same sum as the columns of B corresponding to Y_2, a contradiction as B is non-singular. $\diamondsuit$

In particular, G is a regular graph, implying in turn that for some integer $r \geq 2$, every row and every column of A has exactly r ones – this has two consequences. Firstly, each B_i is a minimal cover. For if not, then $B_i - \{e\}$ is a cover for some $e \in B_i$, implying in turn that column e of A has at least $n-1$ zero entries, implying in turn that $r \leq 1$, which is not the case. Thus **(4)** holds. Secondly, since A is non-singular, it follows that $x^\star = \left(\frac{1}{r}\ \frac{1}{r}\ \cdots\ \frac{1}{r}\right)$. As a result, as $x^\star \in P$, every row of $M(\mathcal{C})$ has at least r ones, and as $x^\star$ is simple, every row of $M(\mathcal{C})$ not in A has at least $r+1$ ones, so **(5)** holds. In particular, we cannot run this argument for another fractional extreme point, so $x^\star$ is the unique fractional extreme point of P, so **(6)** holds. Finally, for each $i \in [n]$, let $d_i := |C_i \cap B_i| - 1 \in \{1, \ldots, r-1\}$, and let $D := \mathrm{diag}(d_1, \ldots, d_n)$. Then

$$(n + d_1, n + d_2, \ldots, n + d_n) = \mathbf{1}^\top (J + D) = \mathbf{1}^\top (AB^\top) = (\mathbf{1}^\top A) B^\top = r \cdot (B\mathbf{1})^\top.$$

Since there is at most one multiple of r in $\{n+1,\dots,n+r-1\}$, it follows that $d := d_1 = d_2 = \cdots = d_n$, implying in turn that **(7)** holds, thereby finishing the proof. $\square$

We will use this next time to prove a combinatorial characterization of the mni clutters different from the deltas.