CO 750 Packing and Covering: Lecture 19

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9.2 The minimally non-ideal clutters different from the deltas, continued

Last time, we proved the following theorem:

Theorem 9.10 (Lehman 1990)**.** *Let $\mathcal{C}$ be a minimally non-ideal clutter over ground set E that is not a delta, and let $n := |E|$. Let $x^\star$ be a fractional extreme point of $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$. Then the following statements hold:*

(1) $\mathbf{0} < x^\star < \mathbf{1}$*,*

(2) $x^\star$ *lies on exactly* n *facets, that correspond to members* $C_1, \ldots, C_n \in \mathcal{C}$ *– so* $x^\star$ *is a simple vertex,*

(3) the n *neighbors of* $x^\star$ *are integral vertices, that correspond to covers* $B_1, \ldots, B_n$ *labeled so that for distinct* $i, j \in [n]$*,* $|C_i \cap B_i| > 1$ *and* $|C_i \cap B_j| = 1$*,*

(4) $B_1, \ldots, B_n$ *are minimal covers,*

(5) $C_1, \ldots, C_n$ *are precisely the minimum cardinality members of* $\mathcal{C}$*,*

(6) $x^\star$ *is the unique fractional extreme point of* $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$*,*

(7) there is an integer $d \geq 1$ *such that for each* $i \in [n]$*,* $|C_i \cap B_i| = 1 + d$*.*

In particular, $x^\star$ *is the unique fractional extreme point of* $\{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$*.*

Parts (3) and (7) of this theorem lead to square $0-1$ matrices whose product is of the form $J + dI$ for an integer $d \geq 1$ – Bridges and Ryser (1969) studied such matrices and proved nice properties about them. For an integer $k \geq 1$, a square $0-1$ matrix is *k-regular* if every row and every column has exactly k ones.

Theorem 9.11 (Bridges and Ryser 1969)**.** *Take an integer* $n \geq 3$*, and let* A, B *be* $n \times n$ *matrices with* $0-1$ *entries such that*

$$AB = J + dI$$

for some integer $d \geq 1$*. Then* A, B *are non-singular matrices that commute*

$$BA = J + dI,$$

and for some integers $r, s \geq 2$ *such that* $rs = n + d$*,* A *is* r*-regular and* B *is* s*-regular.*

Proof. As $J + dI$ is non-singular, it follows that both A, B are non-singular matrices. In particular, neither A nor B has a zero row or a zero column. We have

$$I = (J + dI)\left(\frac{1}{d}I - \frac{1}{d(n+d)}J\right) = (AB)\left(\frac{1}{d}I - \frac{1}{d(n+d)}J\right) = A\left(\frac{1}{d}B - \frac{1}{d(n+d)}BJ\right),$$

so A and $\frac{1}{d}B - \frac{1}{d(n+d)}BJ$ are inverses of one another. Thus,

$$I = \left(\frac{1}{d}B - \frac{1}{d(n+d)}BJ\right)A = \frac{1}{d}BA - \frac{1}{d(n+d)}(B\mathbf{1})(A^\top \mathbf{1})^\top,$$

so

$$BA = \frac{1}{n+d}(B\mathbf{1})(A^\top \mathbf{1})^\top + dI.$$

For each $i \in [n]$, denote by $s_i \in \{1, 2, \dots, n\}$ the number of ones in row i of B, and by $r_i \in \{1, 2, \dots, n\}$ the number of ones in column i of A. Then the previous equation implies that

(1) for all $i, j \in [n]$, $n + d \mid s_i r_j$.

As $\mathrm{trace}(AB) = \mathrm{trace}(BA)$, it follows that

$$n + nd = \frac{1}{n+d}\sum_{i=1}^{n} s_i r_i + nd,$$

so

$$n(n+d) = \sum_{i=1}^{n} s_i r_i \geq n(n+d),$$

implying in turn that

(2) for each $i \in [n]$, $n + d = s_i r_i$.

(1) and (2) imply that $r := r_1 = r_2 = \cdots = r_n$ and $s := s_1 = s_2 = \cdots = s_n$. As a consequence,

$$BA = \frac{1}{n+d}(B\mathbf{1})(A^\top \mathbf{1})^\top + dI = J + dI = AB.$$

Analyzing the equation $AB = J + dI$, we proved that every row of B has the same s number of ones, and every column of A has the same r number of ones. The same argument on the equation $BA = J + dI$ implies that every row of A has the same number of ones, and the number inevitably has to be r, while every column of B has the same number of ones, and the number inevitably has to be s. In particular, A is r-regular and B is s-regular. As $rs = n + d$ and $r, s < n + d$, it follows that $r, s \geq 2$, thereby finishing the proof. □

We are now ready for Lehman's combinatorial characterization of the mni clutters different from the deltas:

Theorem 9.12 (Lehman 1990)**.** *Suppose $\mathcal{C}$ is a minimally non-ideal clutter over ground set E that is not a delta, and let $\mathcal{B} := b(\mathcal{C})$. Denote by $\overline{\mathcal{C}}, \overline{\mathcal{B}}$ the clutters over ground set E of the minimum cardinality members of $\mathcal{C}, \mathcal{B}$, respectively. Then*

(1) $M(\overline{\mathcal{C}})$ and $M(\overline{\mathcal{B}})$ are square and non-singular matrices,

(2) for some integers $r \geq 2$ and $s \geq 2$, $M(\overline{\mathcal{C}})$ is r-regular and $M(\overline{\mathcal{B}})$ is s-regular,

(3) for $n := |E|$, $rs \geq n+1$,

(4) after possibly permuting the rows of $M(\overline{\mathcal{B}})$, we have

$$M(\overline{\mathcal{C}})M(\overline{\mathcal{B}})^\top = J + (rs-n)I = M(\overline{\mathcal{B}})^\top M(\overline{\mathcal{C}}),$$

that is, there is a labeling $C_1, \ldots, C_n$ of the members of $\overline{\mathcal{C}}$ and a labeling $B_1, \ldots, B_n$ of the members of $\overline{\mathcal{B}}$ such that for all $i, j \in [n]$,

$$|C_i \cap B_j| = \begin{cases} rs - n + 1 & \text{if } i = j \\ 1 & \text{if } i \neq j, \end{cases}$$

and for all elements $g, h \in E(\mathcal{C})$,

$$\left|\{i \in [n] : g \in C_i, h \in B_i\}\right| = \begin{cases} rs - n + 1 & \text{if } g = h \\ 1 & \text{if } g \neq h. \end{cases}$$

Proof. Let $x^\star \in [0,1]^E$ be a fractional extreme point of $P(\mathcal{C})$. After applying Theorem 9.10 to the mni clutter $\mathcal{C}$, we get the following implications. The point $x^\star \in [0,1]^E$ is the unique fractional extreme point of $P(\mathcal{C})$, $\mathbf{1} > x^\star > \mathbf{0}$ and $x^\star$ is simple. Let A be the submatrix of $M(\mathcal{C})$ such that $Ax^\star = \mathbf{1}$. We have that $A = M(\overline{\mathcal{C}})$. Let $B_1, \ldots, B_n$ be the minimal covers that correspond to the neighbors of $x^\star$, and let B be the matrix whose rows are the incidence vectors of $B_1, \ldots, B_n$. Then after possibly permuting the rows of B, $AB^\top = J + dI$ for some integer $d \geq 1$.

It now follows from Theorem 9.11 that A, B are non-singular matrices such that $AB^\top = J + dI = B^\top A$, and for some integers $r, s \geq 2$ such that $rs = n + d$, A is r-regular and B is s-regular. To finish the proof, it remains to show that $B = M(\overline{\mathcal{B}})$. To this end, notice that $x^\star$ is equal to $(\frac{1}{r} \ \cdots \ \frac{1}{r})$, and the neighbors of $x^\star$ lie on the hyperplane $\sum_{i=1}^n x_i = s$. Therefore, the inequality $\sum_{i=1}^n x_i \geq s$ is valid for all the integer extreme points of P, implying in turn that every member of $\mathcal{B}$ has cardinality at least s. As a result, $(\frac{1}{s} \ \cdots \ \frac{1}{s})$ is a fractional extreme point of $P(\mathcal{B})$. Applying Theorem 9.10 to the mni clutter $\mathcal{B}$, we see that $(\frac{1}{s} \ \cdots \ \frac{1}{s})$ must be the unique fractional extreme point of $P(\mathcal{B})$ and $B = M(\overline{\mathcal{B}})$, as required. □

9.3 Immediate applications

The first application of Theorem 9.12 is that the deltas (with the exception of Δ_3) are the only mni clutters requiring unequal weights to violate the width-length inequality. The following application is the true analogue of the max-max inequality, Theorem 5.6:

Theorem 9.13. *A clutter without a delta minor is ideal if, and only if, for each minor $\mathcal{C}$,*

$$\min\{|C| : C \in \mathcal{C}\} \cdot \min\{|B| : B \in b(\mathcal{C})\} \leq |E(\mathcal{C})|.$$

Proof. If the clutter is ideal, then the inequality follows from the width-length inequality of Theorem **??**. Conversely, it suffices to prove that for an mni clutter $\mathcal{C}$ that is not a delta,

$$\min\{|C| : C \in \mathcal{C}\} \cdot \min\{|B| : B \in b(\mathcal{C})\} > |E(\mathcal{C})|.$$

Let n, r, s be the parameters as in Theorem 9.12. Then the inequality $rs \geq n+1$ implies the inequality above, as required. □

(Notice that the theorem can be extended to clutters without a minor in $\{\Delta_n : n \geq 4\}$.) A second application of Theorem 9.12 is the following truly remarkable result that, to test integrality of an n-dimensional set covering polyhedron, it is sufficient to test just 3^n directions:

Theorem 9.14. *If $\mathcal{C}$ is a minimally non-ideal clutter, then*

$$\min\{\mathbf{1}^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\}$$

has no integral optimal solution. As a consequence, if $\mathcal{C}$ is a non-ideal clutter over ground set E, then there exists $w \in \{0, 1, +\infty\}^E$ such that

$$\min\{w^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\}$$

has no integral optimal solution.

Proof. If $\mathcal{C}$ is a delta, then the result follows from Theorem 9.2 (2). Otherwise, $\mathcal{C}$ is not a delta, and let n, r, s be as in Theorem 9.12. As every member has cardinality at least r, it follows that $x^\star := \left(\frac{1}{r}\ \frac{1}{r}\ \cdots\ \frac{1}{r}\right)$ is a feasible solution, and its objective value is $\frac{n}{r} \leq \frac{rs-1}{r} < s$. However, the minimum cardinality of a cover is s, so $\min\{\mathbf{1}^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\}$ has no integral optimal solution. The second part follows from the first part after applying Remark 7.10. □

A clutter $\mathcal{C}$ *fractionally packs* if it has a fractional packing of value $\tau(\mathcal{C})$. It follows from the preceding theorem that an mni clutter does not fractionally pack. Thus,

Theorem 9.15. *A clutter is ideal if, and only if, every minor fractionally packs.*

We say that a clutter has the *packing property* if every minor packs. An immediate consequence of the preceding theorem is that,

Corollary 9.16. *If a clutter has the packing property, then it is ideal.*

Conforti and Cornuéjols (1993) conjecture that if a clutter has the packing property, then it must be Mengerian.