CO 750 Packing and Covering: Lecture 20

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July 11, 2017

10 Weakly bipartite graphs

Let $G = (V, E)$ be a graph. A subset $F \subseteq E$ is *bipartite* if the vertices can be bicolored so that every edge of F gets both colors. Let P be the convex hull of the incidence vectors

$$\{\chi_F : F \text{ is bipartite}\} \subseteq \{0,1\}^E.$$

Notice that the polytope P carries information about the cuts of G. For instance, for $w \in \mathbb{R}^E_+$, the optimization problem $\max\{w^\top x : x \in P\}$ seeks the maximum weight of a cut of G. As the latter is a fundamental NP-complete optimization problem, studying the polytope P is certainly worthwhile. We will be after a polyhedral description of P. Observe that an edge subset is bipartite if, and only if, it contains no odd-length circuit. As a result,

$$P \subseteq \left\{x \in [0,1]^E : \sum (x_e : e \in C) \leq |C| - 1 \;\; \forall \text{ odd-length circuits } C\right\}.$$

Observe that equality holds above if, and only if, the polytope on the right is integral. Following Grötschel and Pulleyblank (1981), a graph $G = (V, E)$ is *weakly bipartite* if the polytope

$$\left\{x \in [0,1]^E : \sum (x_e : e \in C) \leq |C| - 1 \;\; \forall \text{ odd-length circuits } C\right\}$$

is integral. After a change of variables $x \mapsto \mathbf{1} - x$, we see that G is weakly bipartite if, and only if, the set covering polytope

$$\left\{x \in [0,1]^E : \sum (x_e : e \in C) \geq 1 \;\; \forall \text{ odd-length circuits } C\right\}$$

is integral. Hence, by Proposition 9.7, a graph is weakly bipartite if, and only if, its clutter of odd-length circuits is ideal. Bipartite graphs are vacuously weakly bipartite. A non-trivial example is provided below:

Theorem 10.1 (Hadlock 1975, Barahona 1980)**.** *Planar graphs are weakly bipartite.*

Proof. Let $G = (V, E)$ be a plane graph. Notice that

> $(\star)$ every circuit has an inside and an outside; the circuit can be written as the symmetric difference of the facial circuits that are inside (or outside); the circuit is odd-length if and only if the number of odd-length facial circuits used in the sum is odd.

Consider now the plane dual $G^\star = (V^\star, E)$, and let $T \subseteq V^\star$ denote the odd-degree vertices. Observe that T is simply the odd-length facial circuits of G. Notice that the cycles of G are the cuts of $G^\star$, and so the circuits of G are the minimal cuts of $G^\star$. Moreover, it follows from ($\star$) that the odd-length circuits of G are the minimal T-cuts of $G^\star$. As the clutter of minimal T-cuts of $G^\star$ is ideal by Theorem 8.11 (3), it follows that the clutter of odd-length circuits of G is ideal, so G is weakly bipartite. $\square$

Thus the class of weakly bipartite graphs is quite rich. Let us analyze the two non-planar graphs $K_{3,3}$ and K_5. As $K_{3,3}$ is bipartite, it is also weakly bipartite. K_5 however is not weakly bipartite. To see this, let us look at the set covering polytope associated with the odd-length circuits of K_5:

$$\left\{x \in [0,1]^{E(K_5)} : \sum (x_e : e \in C) \geq 1 \;\; \forall \text{ odd-length circuits } C \text{ of } K_5\right\}.$$

Observe that K_5 has precisely 10 triangles, whose incidence vectors are linearly independent in $\mathbb{R}^{E(K_5)} \cong \mathbb{R}^{10}$, and that the other odd-length circuits all have length 5. As a result, the fractional point $(\frac{1}{3}\; \frac{1}{3}\; \cdots\; \frac{1}{3}) \in \mathbb{R}^{E(K_5)}$ belongs to the polytope and is an extreme point. Consequently, the clutter of odd-length circuits of K_5 is non-ideal, so K_5 is not weakly bipartite.

What are the weakly bipartite graphs? Whatever the class is, it must contain both bipartite and planar graphs. These rich classes suggest that a complete classification of the weakly bipartite graphs is a difficult problem, and indeed, this is still an open problem. We may however take another approach. The question we asked may be posed as, when is the clutter of odd-length circuits of a graph ideal? The advantage with this question is that idealness is a minor-closed property, so we may look for an excluded minor characterization. Let $G = (V, E)$ be a graph, and let $\mathcal{C}$ be its clutter of odd-length circuits. Take an edge $e \in E$. What do the minors $\mathcal{C} \setminus e, \mathcal{C}/e$ correspond to in terms of G? Recall that $\mathcal{C} \setminus e$ is the clutter of odd-length circuits of G avoiding e, so it is the clutter of odd-length circuits of $G \setminus e$. However,

$$\mathcal{C}/e = \text{ the minimal sets of } \{C - \{e\} : C \text{ is an odd-length circuit of } G\}$$

is *not* the clutter of odd-length circuits of G/e. (For instance, we could have that G is bipartite while G/e is non-bipartite.) It is not clear what $\mathcal{C}/e$ corresponds to in terms of the graph G. To make sense of this, we will need to change our framework.

10.1 Signed graphs

Let $G = (V, E)$ be a graph, and take a subset $\Sigma \subseteq E$. The pair (G, Σ) is called a *signed graph*. In (G, Σ), an *odd cycle* is a cycle $C \subseteq E$ such that $|C \cap \Sigma|$ is odd, and an *even cycle* is a cycle $C \subseteq E$ such that $|C \cap \Sigma|$ is even. Observe that for sets $C_1, C_2 \subseteq E$ we have

$$(C_1 \triangle C_2) \cap \Sigma = (C_1 \cap \Sigma) \triangle (C_2 \cap \Sigma).$$

In particular, if C_1, C_2 are cycles of parities $p_1, p_2 \in \{0, 1\}$, then $C_1 \triangle C_2$ is a cycle of parity $p_1 + p_2 \pmod 2$. In (G, Σ), an *odd circuit* is a circuit $C \subseteq E$ such that $|C \cap \Sigma|$ is odd, and an *even circuit* is a circuit $C \subseteq E$ such that $|C \cap \Sigma|$ is even. We leave the following as an exercise:

Remark 10.2. *Let (G, Σ) be a signed graph, and take a subset $C \subseteq E(G)$. The following statements are equivalent:*

- *C is a even cycle,*
- *C is a disjoint union of circuits, an even number of which are odd circuits,*

and the following statements are equivalent:

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- *C is a disjoint union of circuits, an odd number of which are odd circuits.*

We will use this useful observation without reference. To *resign* (G, Σ) is to replace it by the signed graph $(G, \Sigma \triangle \delta(U))$ for some $U \subseteq V$.

Remark 10.3. *Resigning preserves the parity of a cycle.*

Proof. Let (G, Σ) be a signed graph, and let $(G, \Sigma \triangle \delta(U))$ be a resigning. Let C be a cycle. As $|C \cap \delta(U)|$ is even, it follows that

$$|C \cap (\Sigma \triangle \delta(U))| = |(C \cap \Sigma) \triangle (C \cap \delta(U))| \equiv |C \cap \Sigma| + |C \cap \delta(U)| \equiv |C \cap \Sigma| \pmod 2.$$

Thus, C has the same parity in both (G, Σ) and $(G, \Sigma \triangle \delta(U))$, thereby finishing the proof. □

A *signature* of (G, Σ) is any set of the form $\Sigma \triangle \delta(U), U \subseteq V$.

Proposition 10.4 (Zaslavsky 1982)**.** *If (G, Σ) has no odd cycle, then $\emptyset$ is a signature.*

Proof. Let A be the $0-1$ matrix whose columns are labeled by the edges, and whose first $|V|$ many rows are the incidence vectors of $\delta(v), v \in V$ and whose last row is the incidence vector of Σ. Let b be the column vector whose first $|V|$ many coordinates are 0 and whose last entry is 1. As (G, Σ) has no odd cycle, the system

$$Ax \equiv b \pmod 2$$

has no $0-1$ solution. By Farkas' lemma for binary spaces, there is a certificate $c \in \{0,1\}^V \times \{0,1\}$ such that

$$c^\top A \equiv \mathbf{0} \quad \text{and} \quad c^\top b \equiv 1 \pmod 2.$$

The second equation implies that the last entry of c is 1. Pick $U \subseteq V$ such that $c = (\chi_U \; 1)$. Then the first equation implies that $\Sigma = \delta(U)$, so $\emptyset = \Sigma \triangle \delta(U)$ is a signature. □

As a consequence,

Theorem 10.5. *For a signed graph, the clutter of odd circuits and the clutter of minimal signatures are blockers.*

Proof. Let $\mathcal{C}$ be the clutter of odd circuits of (G, Σ). By Remark 10.3, every minimal signature intersects every odd circuit in an odd number of edges, so every minimal signature is a cover of $\mathcal{C}$. Conversely, let B be a minimal cover of $\mathcal{C}$. Then the signed graph $(G \setminus B, \Sigma - B)$ has no odd circuit by definition, implying in turn that it has no odd cycle. It therefore follows from Proposition 10.4 that $\Sigma - B = \delta_{G\setminus B}(U)$ for some $U \subseteq V$. Then $\Sigma \triangle \delta(U) \subseteq B$, so B contains a signature of (G, Σ). It follows Remark 6.6 that $b(\mathcal{C})$ is the clutter of minimal signatures, as required. □

Take disjoint edge subsets I, J of (G, Σ). By Theorem 10.5, J does not contain an odd cycle if, and only if, there is a signature disjoint from J. Let

$$(G,\Sigma) \setminus I/J := \begin{cases} (G \setminus I/J, \emptyset) & \text{if } J \text{ contains an odd cycle,} \\ (G \setminus I/J, B - I) & B \text{ is a signature disjoint from } J. \end{cases}$$

We refer to $(G, \Sigma) \setminus I/J$ as a *minor* of (G, Σ) obtained after deleting I and contracting J.[1] Observe that $(G, \Sigma) \setminus I/J$ is defined up to resigning. In contrast to the unsigned graph case, we have the following:

Proposition 10.6. *Let (G, Σ) be a signed graph and $\mathcal{C}$ the clutter of its odd circuits. Take disjoint edge subsets I, J such that J does not contain an odd cycle. Then $\mathcal{C} \setminus I/J$ is the clutter of odd circuits of $(G, \Sigma) \setminus I/J$.*

Proof. Let B be a signature of (G, Σ) disjoint from J. Then $(G \setminus I/J, B - I) = (G, \Sigma) \setminus I/J$. By Remark 6.6, it suffices to show that every odd circuit of $(G \setminus I/J, B - I)$ contains a member of $\mathcal{C} \setminus I/J$, and every member of $\mathcal{C} \setminus I/J$ contains an odd circuit of $(G \setminus I/J, B - I)$.

Let C' be an odd circuit of $(G \setminus I/J, B - I)$. Then there is a circuit C of (G, Σ) such that $C' \subseteq C \subseteq C' \cup J$. As B is a signature of (G, Σ) disjoint from J, it follows that $B \cap C = B \cap C' = (B - I) \cap C'$, so $|B \cap C|$ is odd, implying in turn that C is an odd circuit of (G, Σ). As $C - J$ contains a member of $\mathcal{C} \setminus I/J$, it follows that C' contains a member of $\mathcal{C} \setminus I/J$.

Conversely, let C be an odd circuit of (G, Σ) such that $C \cap I = \emptyset$. Then $C - J$ is a cycle of $(G \setminus I/J, B - I)$. Since $|C \cap B|$ is odd, we get that $|(C - J) \cap (B - I)|$ is odd, so $C - J$ is an odd cycle of $(G \setminus I/J, B - I)$, so it contains an odd circuit of $(G \setminus I/J, B - I)$, as required. □

We say that a signed graph is *weakly bipartite* if its clutter of odd circuits is ideal. Observe that a graph $G = (V, E)$ is weakly bipartite if, and only if, the signed graph (G, E) is weakly bipartite. Hence, as the graph K_5 is not weakly bipartite, it follows that the signed graph $(K_5, E(K_5))$ is not weakly bipartite. We will refer to the signed graph $(K_5, E(K_5))$ as an *odd-K_5*. It follows from Remark 7.11 and Proposition 10.6 that,

Remark 10.7. *If a signed graph is weakly bipartite, then it has no odd-K_5 minor.*

Seymour (1977) conjectured that the converse of this remark also holds. Over 20 years later, in his PhD thesis, Guenin (2001) proved this conjecture! His proof made a spectacular use of Lehman's powerful result, Theorem 9.12.

[1] In this setting, to contract a loop is to delete it.