CO 750 Packing and Covering: Lecture 21

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10.1 Signed graphs, continued

Let (G, Σ) be a signed graph. Recall that for disjoint edge subsets I, J, we defined the minor

$$(G,\Sigma) \setminus I/J = \begin{cases} (G \setminus I/J, \emptyset) & \text{if } J \text{ contains an odd cycle,} \\ (G \setminus I/J, B - I) & \text{otherwise, for some signature } B \text{ disjoint from } J. \end{cases}$$

Consider now the case when $\Sigma = E(G)$, $I = \emptyset$ and J forms a cut of G. As $E(G) - J = E(G) \triangle J$ is a signature for $(G, E(G))$, it follows that

$$(G, E(G))/J = (G/J, E(G/J)).$$

This observation will be useful throughout the rest of this section.

Recall that a signed graph is weakly bipartite if its clutter of odd circuits is ideal. We showed last time that the signed graph $(K_5, E(K_5))$, called an odd-K_5, is not weakly bipartite. We also showed in Remark 10.7 that if a signed graph is weakly bipartite, then it has no odd-K_5 minor. Seymour (1977) conjectured that the converse of this remark also holds. Over 20 years later, in his PhD thesis, Guenin (2001) proved this conjecture. His proof made a spectacular use of Lehman's powerful result, Theorem 9.12. To prove the conjecture, we will need a lemma due to Schrijver (2002).

10.2 The whirlpool lemma and pseudo-odd-K_5's

The signed graph $(K_4, E(K_4))$ is called an *odd-K_4*. Schrijver (2002) found a very nice way to find an odd-K_4 minor in a signed graph. To explain his method, let W be the graph on vertices $0, 1, 1', 2, 2', 3, 3'$ and edges $\{0,1\}, \{0,2\}, \{0,3\}, \{1',2'\}, \{2',3'\}, \{3',1'\}, \{1,2'\}, \{2,3'\}, \{3,1'\}$. We will refer to the signed graph $(W, E(W))$ as a *whirlpool* with *central edges* $\{0,1\}, \{0,2\}, \{0,3\}$ – see Figure 1. Observe that a whirlpool has an odd-K_4 minor using its central edges, obtained after contracting the cut $\delta(\{0,1,2,3\})$.

Lemma 10.8 (Schrijver 2002)**.** *Take a graph $G = (V, E)$. Suppose that there are disjoint stable sets S_1, S_2, S_3 and distinct vertices $0, 1, 2, 3$ such that*

- $0 \in V - (S_1 \cup S_2 \cup S_3)$ *and* $i \in S_i$ *for each* $i \in [3]$,
- $\{0, i\} \in E$ *for each* $i \in [3]$,

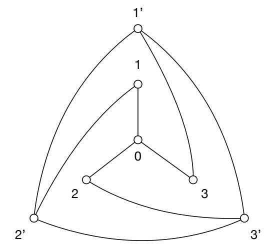

Figure 1: The whirlpool with central edges $\{0,1\},\{0,2\},\{0,3\}$. Every edge is odd.

- *for distinct $i,j\in[3]$, there is an ij-path contained in $G[S_i\cup S_j]$.*

Then $(G,E(G))$ has an odd-K_4 minor using the three edges $\{0,1\},\{0,2\},\{0,3\}$.

Proof. We prove this by induction on $|V|+|E|\geq 10$. The base case $|V|+|E|=10$ is true as $(G,E(G))$ itself is an odd-K_4. For the induction step, assume that $|V|+|E|\geq 11$. For distinct $i,j\in[3]$, let $P_{ij}\subseteq E$ be an ij-path contained in $G[S_i\cup S_j]$. We may assume that $V=\{0\}\cup V(P_{12})\cup V(P_{23})\cup V(P_{31})$ and $E=\{\{0,1\},\{0,2\},\{0,3\}\}\cup P_{12}\cup P_{23}\cup P_{31}$. If G has a vertex v of degree two, then the graph $G/\delta(v)$ still satisfies the conditions of the lemma for the same vertices $0,1,2,3$ and appropriate stable sets, so by the induction hypothesis, $(G/\delta(v),E(G/\delta(v)))=(G,E(G))/\delta(v)$ has an odd-K_4 lemma using edges $\{0,1\},\{0,2\},\{0,3\}$, implying in turn that $(G,E(G))$ has an odd-K_4 lemma using edges $\{0,1\},\{0,2\},\{0,3\}$. We may therefore assume that G does not have a vertex of degree two. This implies in turn that

> $(\star)$ for every permutation i,j,k of $1,2,3$ we have $S_i=V(P_{ij})\cap V(P_{ik})$, and that $|S_1|=|S_2|=|S_3|\geq 2$,

as $|V|+|E|\geq 11$. Let $2'\in S_2$ be the second vertex of the 12-path P_{12}, $3'\in S_3$ the second vertex of the 23-path P_{23}, and $1'\in S_1$ the second vertex of the 31-path P_{31}. Notice that $1'\neq 1, 2'\neq 2, 3'\neq 3$. Let $H:=G/\delta(0)$, and let $0'$ be the vertex corresponding to $0,1,2,3$. Notice that $\{0',1'\},\{0',2'\},\{0',3'\}\in E(H)$. For each $i\in[3]$, let $S'_i:=S_i-\{i\}$. Then for each $i\in[3]$, S'_i is stable in H and $i'\in S'_i$. Moreover, for distinct $i,j\in[3]$, the vertices i',j' lie on the path P_{ij} by $(\star)$, so $H[S'_i\cup S'_j]$ contains an $i'j'$-path. It therefore follows from the induction hypothesis that $(G,E(G))/\delta(0)=(H,E(H))$ has an odd-K_4 minor using $\{0',1'\},\{0',2'\},\{0',3'\}$. After decontracting $\delta(0)$, we get that $(G,E(G))$ has a whirlpool minor with central edges $\{0,1\},\{0,2\},\{0,3\}$, which has an odd-K_4 minor using the central edges. Consequently, $(G,E(G))$ has an odd-K_4 minor using the edges $\{0,1\},\{0,2\},\{0,3\}$, thereby completing the induction step. □

This lemma is also helpful for finding an odd-K_5 minor. A *pseudo-odd-K_5* is a signed graph $(G,E(G))$ for which the following statements hold: there exist a partition of $V(G)$ into parts S_0,S_1,S_2,S_3 and distinct vertices $x,y\in S_0$ such that

- there is an edge $e\in E$ whose ends are x,y, and for each $i\in\{0,1,2,3\}$, S_i is stable in $G\setminus e$,
- $G\setminus e$ has internally vertex-disjoint xy-paths P_1,P_2,P_3, where for each $i\in[3]$, $V(P_i)\subseteq S_0\cup S_i$,

- for distinct $i, j \in [3]$, $G[S_i \cup S_j]$ has a path with one end in $V(P_i)$ and the other in $V(P_j)$.

As a consequence of the Whirlpool Lemma 10.8, we get that,

Theorem 10.9. *A pseudo-odd-K_5 has an odd-K_5 minor.*

Proof. If $f \in E - (\{e\} \cup P_1 \cup P_2 \cup P_3)$ is an edge with an end in S_0, then $(G \setminus f, E(G \setminus f)) = (G, E(G)) \setminus f$ is still a pseudo-odd-K_5. We may therefore assume that each edge of $E - (\{e\} \cup P_1 \cup P_2 \cup P_3)$ has both ends in $S_1 \cup S_2 \cup S_3$. If $u \in S_0$ is an internal vertex of one of P_1, P_2, P_3, then as S_0 is stable, $(G/\delta(u), E(G/\delta(u))) = (G, E(G))/\delta(u)$ is still a pseudo-odd-K_5. We may therefore assume that P_1, P_2, P_3 do not have any internal vertices in S_0. Subsequently, as S_1, S_2, S_3 are stable, it follows that for each $i \in [3]$, $V(P_i) = \{x, y, v_i\}$ for some vertex $v_i \in S_i$. Let $(H, E(H)) := (G, E(G)) \setminus \delta(y)$. Then by the Whirlpool Lemma 10.8, $(H, E(H))$ has an odd-K_4 minor using edges $\{x, v_1\}, \{x, v_2\}, \{x, v_3\}$. Adding vertex y and its incident edges back in, we see that $(G, E(G))$ has an odd-K_5 minor, as required. □

10.3 A signed graph without an odd-K_5 minor is weakly bipartite.

Let $(G = (V, E), \Sigma)$ be a signed graph. Let $U, U' \subseteq V$ be different components of G, if any, and let H be the graph obtained from G by identifying a vertex of U with a vertex of U'. Notice that G, H have the same edge sets, and that the odd circuits of (G, Σ) are precisely the odd circuits of (H, Σ). Thus, (G, Σ) is weakly bipartite if, and only if, (H, Σ) is weakly bipartite. Moreover, because K_5 does not have a cut-vertex, if (H, Σ) has an odd-K_5 minor, then so does (G, Σ). We will use these observations in the proof below, due to Schrijver (2002).

Theorem 10.10 (Guenin 2001)**.** *A signed graph without an odd-K_5 minor is weakly bipartite.*

Proof. Let $(G = (V, E), \Sigma)$ be a signed graph that is not weakly bipartite. We will show that (G, Σ) has an odd-K_5 minor. To this end, we may assume that G is connected, and that every proper minor of (G, Σ) is weakly bipartite. Let $\mathcal{C}$ be the clutter of odd circuits of (G, Σ). It then follows from Proposition 10.6 that $\mathcal{C}$ is a minimally non-ideal clutter. Take an edge $e \in E$. Using Lehman's Theorem 9.12, we prove the following:

Claim 1. *There are minimum odd circuits C_1, C_2, C_3 and minimum signatures B_1, B_2, B_3 such that for distinct $i, j \in [3]$,*

(C1) $|C_i \cap B_i| \geq 3$ *and* $C_i \cap B_j = \{e\}$,

(C2) $C_i \cap C_j = \{e\} = B_i \cap B_j$,

(C3) the only odd cycles contained in $C_i \cup C_j$ *are* C_i, C_j,

(C4) the only signatures contained in $B_i \cup B_j$ *are* B_i, B_j.

Proof of Claim. Let $n := |E|$ and let $\mathcal{B}$ be the clutter of minimal signatures. By Theorem 10.5, we have $\mathcal{B} = b(\mathcal{C})$. Let M (resp. N) be the row submatrix of $M(\mathcal{C})$ (resp. $M(\mathcal{B})$) corresponding to the minimum odd circuits (resp. minimum signatures). By Theorem 9.12, M (resp. N) is a square and non-singular matrix that is r-regular (resp. s-regular) for some integers $r,s \geq 2$ such that $rs \geq n+1$. Moreover, for some labeling $C_1,\ldots,C_n$ of the minimum odd circuits and labeling $B_1,\ldots,B_n$ of the minimum signatures, we have that for all $i,j \in [n]$,

$$|C_i \cap B_j| = \begin{cases} rs-n+1 & \text{if } i=j \\ 1 & \text{if } i \neq j, \end{cases}$$

and for all elements $g,h \in E$,

$$\big|\{i \in [n] : g \in C_i, h \in B_i\}\big| = \begin{cases} rs-n+1 & \text{if } g=h \\ 1 & \text{if } g \neq h. \end{cases}$$

As signatures and odd circuits intersect in an odd number of edges, and $rs-n+1 \geq 2$, it follows that $rs-n+1 \geq 3$. By the previous equation, after possibly re-indexing the C_i and B_i's, we have that

$$e \in C_i \cap B_i \qquad i = 1,\ldots,rs-n+1.$$

Consider C_1,C_2,C_3 and B_1,B_2,B_3. We will show that these are the desired sets. **(C1)** clearly holds. **(C2)** If $f \in (C_1 \cap C_2) - \{e\}$, then $\{i \in [n] : f \in C_i, e \in B_i\} \supseteq \{1,2\}$, which is not the case. This shows that $C_1 \cap C_2 = \{e\}$ and similarly, $C_2 \cap C_3 = C_3 \cap C_1 = \{e\}$. Moreover, if $g \in (B_1 \cap B_2) - \{e\}$, then $\{i \in [n] : e \in C_i, g \in B_i\} \supseteq \{1,2\}$, which is not the case. Thus, $B_1 \cap B_2 = \{e\}$ and similarly, $B_2 \cap B_3 = B_3 \cap B_1 = \{e\}$. **(C3)** Let C be an odd cycle contained in $C_i \cup C_j$. Then $C' := C_i \triangle C_j \triangle C$ is an odd cycle. As $C \cup C' \subseteq C_i \cup C_j$ and $C \cap C' \subseteq C_i \cap C_j$, it follows that

$$2r = |C_i| + |C_j| = |C_i \cup C_j| + |C_i \cap C_j| \geq |C \cup C'| + |C \cap C'| = |C| + |C'| \geq 2r,$$

so equality holds throughout. That is, C,C' are minimum odd circuits and $\chi_{C_i} + \chi_{C_j} = \chi_C + \chi_{C'}$. As M is non-singular, it follows that $\{C,C'\} = \{C_i,C_j\}$, as required. **(C4)** Let B be a signature contained in $B_i \cup B_j$. Pick $W_i,W_j,W \subseteq V$ such that $B_i = \Sigma \triangle \delta(W_i), B_j = \Sigma \triangle \delta(W_j)$ and $B = \Sigma \triangle \delta(W)$. Then for $W' := W_i \triangle W_j \triangle W$ we have

$$B' := B_i \triangle B_j \triangle B = \Sigma \triangle \delta(W_i) \triangle \Sigma \triangle \delta(W_j) \triangle \Sigma \triangle \delta(W) = \Sigma \triangle \delta(W'),$$

so B' is also a signature. As $B \cup B' \subseteq B_i \cup B_j$ and $B \cap B' \subseteq B_i \cap B_j$, it follows that

$$2s = |B_i| + |B_j| = |B_i \cup B_j| + |B_i \cap B_j| \geq |B \cup B'| + |B \cap B'| = |B| + |B'| \geq 2s,$$

so equality holds throughout. That is, B,B' are minimum signatures and $\chi_{B_i} + \chi_{B_j} = \chi_B + \chi_{B'}$. As N is non-singular, it follows that $\{B,B'\} = \{B_i,B_j\}$, as required. $\Diamond$

We will not be using Lehman's Theorem 9.12 anymore. TO BE CONTINUED $\square$