CO 750 Packing and Covering: Lecture 24

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11.2 The sums of circuits property

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Recall that S is a binary space if $\mathbf{0} \in S$, and if $a, b \in S$ then $a \triangle b \in S$. When is a binary space cube-ideal? The following gives a partial characterization of the cube-ideal binary spaces:

Theorem 11.5 (Abdi, Cornuéjols, Guričanová, Lee 2018+)**.** *Take an integer $n \geq 1$ and a binary space $S \subseteq \{0,1\}^n$. Then S is cube-ideal if, and only if,*

$$\mathrm{conv}(S) = \big\{x \in [0,1]^n : x(F) - x(D-F) \leq |F| - 1 \quad \forall \textit{ cocycles } D \textit{ and odd subsets } F \subseteq D\big\}.$$

Proof. ($\Leftarrow$) Notice that each inequality $x(F) - x(D-F) \leq |F| - 1$ can be rewritten as

$$\sum_{i \in D-F} x_i + \sum_{j \in F} (1 - x_j) \geq 1,$$

which is a generalized set covering inequality. Thus S is cube-ideal. ($\Rightarrow$) Suppose coversely that S is cube-ideal. We first prove that

$$\mathrm{conv}(S) \subseteq \big\{x \in [0,1]^n : x(F) - x(D-F) \leq |F| - 1 \quad \forall \text{ cocycles } D \text{ and odd subsets } F \subseteq D\big\}.$$

Denote by P the polytope on the right. To prove this inclusion, it suffices to show that for every cycle C, χ_C belongs to P. Well, for every cocycle D and odd subset $F \subseteq D$, we have $C \cap D \neq F$ because $|C \cap D|$ is even, implying in turn that

$$\chi_C(F) - \chi_C(D-F) \leq 1.$$

Thus, $\chi_C \in P$. To prove the reverse inclusion, it suffices to prove that every inequality defining $\mathrm{conv}(S)$ is valid for P. Since S is cube-ideal, $\mathrm{conv}(S)$ is described by hypercube inequalities – which are valid for P – and by generalized set covering inequalities. Take disjoint subsets $I, J \subseteq [n]$ such that $\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1$ is a defining inequality of $\mathrm{conv}(S)$.

Claim. *There is a cocycle D such that $D \subseteq I \cup J$ and $|D \cap J|$ is odd.*

Proof of Claim. To see this, write

$$S = \{x \in \{0,1\}^{[n]} : Ax \equiv \mathbf{0} \pmod 2\}$$

for some $0-1$ matrix A. Let d be the sum of the columns in J of A, and let B be the submatrix of A obtained after dropping columns $I \cup J$. Since $\sum_{i\in I} x_i + \sum_{j\in J}(1-x_j) \geq 1$ is valid for every point of S, the binary system

$$By \equiv d \pmod 2$$

has no $0-1$ solution. (For if y is a solution, then by setting the coordinates in I to 0 and the coordinates in J to 1, we can extend y to a cycle x for which $\sum_{i\in I} x_i + \sum_{j\in J}(1-x_j) = 0$, which is not the case.) By Farkas' lemma for binary spaces, there is a $0-1$ vector c such that $c^\top B \equiv \mathbf{0} \pmod 2$ and $c^\top d \equiv 1 \pmod 2$. Consider the cocycle $\chi_D := c^\top A$. Then the first equation implies that $D \subseteq I \cup J$, while the second equation implies that $|D \cap J|$ is odd, as required. ◊

Let $F := D \cap J$. Then F is an odd subset of the cocycle D. Observe that the inequality

$$\sum_{i\in I} x_i + \sum_{j\in J}(1-x_j) \geq 1$$

is dominated by the inequality

$$\sum_{i\in D-F} x_i + \sum_{j\in F}(1-x_j) \geq 1 \qquad \text{which is equivalent to} \qquad x(F) - x(D-F) \leq |F| - 1,$$

because $D - F \subseteq I$ and $F \subseteq J$. As a result, every inequality defining $\mathrm{conv}(S)$ is valid for P, so $\mathrm{conv}(S) \supseteq P$. Hence, $\mathrm{conv}(S) = P$, thereby finishing the proof. □

Consider the polyhedral cone generated by S:

$$\mathrm{cone}(S) = \left\{\sum_{x\in S} \alpha_x x : \alpha \in \mathbb{R}^S_+\right\} \subseteq \{0,1\}^n.$$

Barahona and Grötschel (1986) showed that due to the transitivity of S, to describe the facets of $\mathrm{conv}(S)$, it suffices to have a facet description of $\mathrm{cone}(S)$.

Theorem 11.6 (Barahona and Grötschel 1986)**.** *Take an integer $n \geq 1$ and a binary space $S \subseteq \{0,1\}^n$. Then*

$$(1) \qquad \mathrm{conv}(S) = \{x \in [0,1]^n : x(F) - x(D-F) \leq |F| - 1 \quad \forall \textit{ cocycles } D \textit{ and odd subsets } F \subseteq D\}$$

if, and only if,

$$(2) \qquad \mathrm{cone}(S) = \{x \in \mathbb{R}^n_+ : x_f - x(D - \{f\}) \leq 0 \quad \forall \textit{ cocycles } D \textit{ and } f \in D\}.$$

Proof. ($\Rightarrow$) Suppose that (1) holds. As $\mathbf{0} \in S$, the facets of $\mathrm{conv}(S)$ tight at $\mathbf{0}$ describe the conic hull of S. Since the cocycle inequality

$$x(F) - x(D-F) \leq |F| - 1 \quad \text{cocycle } D \text{ and odd subset } F \subseteq D$$

is tight at **0** if and only if $|F| = 1$, it follows that (2) holds. ($\Leftarrow$) Conversely, suppose that (2) holds. To prove that (1) holds, let

$$(\diamond) \qquad \sum_{i\in I} a_i x_i + \sum_{j\in[n]-I} a_j(1-x_j) \geq b \qquad a\in\mathbb{R}^n_+, b\in\mathbb{R}$$

be a facet-defining inequality for $\mathrm{conv}(S)$. It suffices to show that ($\diamond$) is equivalent to a cocycle inequality. To this end, take a point $u\in S$ that lies on this facet. Consider the change of variables $x_i \mapsto 1-x_i$ for the indices in $\{i\in[n]: u_i = 1\}$; this mapping sends the above inequality to the inequality

$$(\star) \qquad \sum_{i\in I: u_i=0} a_i x_i + \sum_{i\in I:u_i=1} a_i(1-x_i) + \sum_{j\in[n]-I:u_j=0} a_j(1-x_j) + \sum_{j\in[n]-I:u_j=1} a_j x_j \geq b,$$

and the set S to the set $S\triangle u := \{x\triangle u : x\in S\}$. Then ($\star$) is a facet-defining inequality for $S\triangle u$ and the facet contains the point $\mathbf{0} = u\triangle u \in S\triangle u$. Hence, ($\star$) also defines a facet for $\mathrm{cone}(S\triangle u)$. However, since S is a binary space, $S\triangle u$ is just the original set S, so ($\star$) defines a facet of $\mathrm{cone}(S)$. By (2), there is a cocycle $D\subseteq[n]$ and an element $f\in D$ such that ($\star$) is equivalent to the inequality

$$x_f - x(D-\{f\}) \leq 0.$$

Take the cycle $C\subseteq[n]$ such that $u=\chi_C$. Reverting back the change of variables, we see that ($\diamond$) is equivalent to

$$x(F) - x(D-F) \leq |F| - 1$$

where $F = (C\cap D)\triangle\{f\}$. Since $|C\cap D|$ is even, it follows that $|F|$ is odd, so ($\diamond$) is equivalent to a cocycle inequality, as required. □

Take a finite set E and a binary space $S\subseteq\{0,1\}^E$. A *circuit* is a non-empty cycle that does not properly contain another non-empty cycle. Notice that $\{\chi_C : C \text{ is a circuit}\}$ are precisely the non-zero points in S of minimal support.

Remark 11.7. *Take a finite set E and a binary space $S\subseteq\{0,1\}^E$, and take a non-empty subset $C\subseteq E$. Then C is a cycle if, and only if, C is a disjoint union of circuits.*

Proof. ($\Leftarrow$) If C is a disjoint union of circuits, then it is also a symmetric difference of circuits, so C is a cycle. ($\Rightarrow$) We prove the converse by induction on $|C|$. As cycles of length 1 are already circuits, the base case $|C| = 1$ holds. For the induction step, assume that $|C|\geq 2$. If C does not properly contain a non-empty cycle, then it is already a circuit, and we are done. Otherwise, there is a non-empty cycle C' such that $C'\subsetneq C$. Let $C'' := C\triangle C'$. Notice that C is the disjoint union of the non-empty cycles C', C''. By the induction hypothesis, each of C', C'' is the disjoint union of circuits, implying in turn that C is a disjoint union of circuits, thereby completing the induction step. □

It follows from Remark 11.7 that $\mathrm{cone}(S)$ is the polyhedral cone generated by the circuits, that is,

$$\mathrm{cone}(S) = \left\{ \sum_{C \text{ is a circuit}} y_C\cdot\chi_C : y\in\mathbb{R}^{\text{circuits}}_+ \right\}.$$

The binary space S has the *sums of circuits property* if

$$\left\{\sum_{C \text{ is a circuit}} y_C \cdot \chi_C : y \in \mathbb{R}_+^{\text{circuits}}\right\} = \left\{x \in \mathbb{R}_+^n : x_f - x(D - \{f\}) \leq 0 \quad \forall \text{ cocycles } D \text{ and } f \in D\right\}.$$

Equivalently, a binary space S has the sums of circuits property if for each $w \in \mathbb{R}_+^E$ satisfying

$$w(D - \{f\}) \geq w_f \qquad \forall \text{ cocycles } D \text{ and } f \in D,$$

there is a vector $y \in \mathbb{R}_+^{\text{circuits}}$ such that $w = \sum (y_C \cdot \chi_C : C \text{ is a circuit})$. Combining Theorems 11.5 and 11.6, we get the following:

Corollary 11.8. *A binary space is cube-ideal if, and only if, it has the sums of circuits property.*

The deep concept of the sums of circuits property was introduced by Seymour (1979), where he showed that the binary space corresponding to the cycles of a graph has the sums of circuits property. (This result is in fact equivalent to Corollary 8.12.) That is, given a graph $G = (V, E)$, if a vector $w \in \mathbb{Z}_+^E$ satisfies

$$w(D - \{f\}) \geq w_f \qquad \forall \text{ cuts } D \text{ and } f \in D,$$

then there is an assignment $y \in \mathbb{R}_+^{\text{circuits}}$ such that

$$w = \sum (y_C \cdot \chi_C : C \text{ is a circuit}).$$

In particular, if $G = (V, E)$ is a bridgeless graph, then there is an assignment $y \in \mathbb{R}_+^{\text{circuits}}$ such that

$$\mathbf{1} = \sum (y_C \cdot \chi_C : C \text{ is a circuit}).$$

Szekeres (1973), and independently Seymour (1979), conjecture that y can be chosen to be half-integral; this is known as the notoriously difficult *cycle double cover conjecture*.