CO 750 Packing and Covering: Lecture 3

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4 Balanced matrices

Let A, B be $0-1$ matrices, where B has no column of all zeros. Why is

$$\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$$

called the set covering polyhedron and

$$\{x \geq \mathbf{0} : Bx \leq \mathbf{1}\}$$

the set packing polytope? There is a neat way to look at these polyhedra that explains the terminology and gives us good intuition about what is coming. Take a loopless graph $G = (V, E)$. Let A be the edge-vertex incidence matrix of G, that is, the columns are labeled by V and the rows are the incidence vectors of the edges. Then the $0-1$ points of

$$\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$$

correspond to the vertex covers of G, hence the "set covering polyhedron". (A *vertex cover* of a graph is a set of vertices whose deletion makes the graph stable.) Let B be the vertex-edge incidence matrix of G, i.e. $B = A^\top$. Then the $0-1$ points of

$$\{x \geq \mathbf{0} : Bx \leq \mathbf{1}\}$$

correspond to the matchings of G, hence the "set packing polytope".

It follows from various well-known theorems of Kőnig (1931) that if G is bipartite, then the set covering and the set packing systems associated to the (edge-vertex or vertex-edge) incidence matrix are totally dual integral. Well, in general, we can think of any $0-1$ matrix as the (vertex-edge or edge-vertex) incidence matrix of a "hypergraph". How can we generalize the notion of bipartite-ness to hypergraphs? However way we do this, we want the definition to be invariant of taking matrix transpose.

An odd square matrix of the form

$$\begin{pmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ 1 & & & & & 1 \end{pmatrix}$$

is called an *odd cycle matrix*. A $0-1$ matrix is *balanced* if it has no odd cycle submatrix (even after rearranging its rows and columns). Observe that if a matrix is balanced, then so is its transpose. Notice that an odd cycle matrix is the incidence matrix of a graph odd cycle. As a result, the incidence matrix of a bipartite graph is always balanced. We may therefore think of balanced matrices as generalizations of bipartite graphs.

4.1 A bicoloring characterization of balanced matrices

A *bicoloring* of a $0-1$ matrix is a partition of the columns into two color classes, where every row with at least two ones gets both colors. For instance, $R = \{1, 2\}$ and $B = \{3, 4\}$ yields a bicoloring of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

whose columns are labeled $1, 2, 3, 4$ from left to right.

Theorem 4.1 (Berge 1970)**.** *A $0-1$ matrix is balanced if, and only if, every submatrix has a bicoloring.*

Proof. Let A be a $0-1$ matrix. ($\Leftarrow$) Since an odd cycle is not bipartite, an odd cycle matrix is not bicolorable. So, if every submatrix of A is bicolorable, A must be balanced. ($\Rightarrow$) Suppose otherwise. We may assume that A is a balanced matrix that is not bicolorable, but every proper submatrix is bicolorable. In particular, every row of A has at least two ones. Let V collect the column labels of A.

Claim. *For every $v \in V$, there exist rows of the form $\{v, u\}, \{v, w\}$ for some distinct $u, w \in V - \{v\}$.*

Proof of Claim. For if not, bicolor the column submatrix of A corresponding to the columns $V - \{v\}$. Our contrary assumption allows us to extend this bicoloring to a bicoloring of A, a contradiction. $\diamondsuit$

Let G be the graph on vertices V whose edges correspond to the rows in A with exactly two ones. Since A is balanced, and the edge-vertex incidence matrix of G is a submatrix of A, it follows that G is bipartite. By Claim 1, every vertex of G has degree at least 2. In particular, G has a vertex v_0 that is not a cut-vertex. Now bicolor the column submatrix of A corresponding to the columns $V - \{v_0\}$, and extend this bicoloring uniquely to a bicoloring of A, determined by the path in $G \setminus v_0$ between two neighbors of v_0, a contradiction. This finishes the proof of Theorem 4.1. $\square$

A *hypergraph* is a pair $G = (V, E)$ where V is a finite set of *vertices*, and each element of E is a non-empty subset of V, called an *edge*. A hypergraph is *balanced* if its incidence matrix is balanced.

Corollary 4.2 (Berge 1972)**.** *Let $G = (V, E)$ be a balanced hypergraph, and let $k \geq 2$ be the minimum cardinality of an edge. Then there exists a partition of V into k color classes where every edge gets at least one vertex of each color.*

Proof. For $k = 2$, the result follows immediately from Theorem 4.1. We may therefore assume that $k \geq 3$. Let $(S_1, \ldots, S_k)$ be an arbitrary partition of V. For each edge e, let

$$k_e := |\{i \in [k] : e \cap S_i \neq \emptyset\}| \in \{1, \ldots, k\}.$$

If each k_e is k, then we have a k-coloring. Otherwise, assume that $k_g < k$ for some edge g. Since $|g| \geq k$, we may assume that

$$|g \cap S_{k-1}| \geq 2 \quad \text{and} \quad g \cap S_k = \emptyset.$$

Let A be the edge-vertex incidence matrix of G. Since A is balanced, by Theorem 4.1, we may bicolor the column submatrix of A corresponding to $S_{k-1} \cup S_k$ and get a bicoloring $S'_{k-1} \cup S'_k$. Consider now the partition $(S_1, \cdots, S_{k-2}, S'_{k-1}, S'_k)$. Notice that g intersects $k_g + 1$ many of these parts, and every other edge e intersects at least k_e many of these parts. By applying this argument recursively, we will achieve the desired k-coloring. $\square$

For an integer $k \geq 2$, a hypergraph is *k-partite* if its vertices can be partitioned into k parts such that every edge intersects each part at most once. As an immediate consequence of the preceding result, we have the following:

Corollary 4.3. *Take an integer $k \geq 2$ and a hypergraph where every edge has cardinality k. If G is balanced, then it is k-partite.*

4.2 Integral polyhedra associated with balanced matrices

Take a $0 - 1$ matrix A with column labels E, and consider the polytope

$$P(A) := \{\mathbf{1} \geq x \geq \mathbf{0} : Ax = \mathbf{1}\}.$$

Notice that for each $e \in E$,

$$P(A) \cap \{x : x_e = 0\} = P(A') \quad \text{and} \quad P(A) \cap \{x : x_e = 1\} = P(A'')$$

where A', A'' are appropriate submatrices of A. (Equality holds above after extending $P(A'), P(A'')$ to $\mathbb{R}^E$ by setting new coordinates to either 0 or 1.)