CO 750 Packing and Covering: Lecture 4

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May 11, 2017

4.2 Integral polyhedra associated with balanced matrices

Take a $0-1$ matrix A and the polytope

$$P(A) = \{\mathbf{1} \geq x \geq \mathbf{0} : Ax = \mathbf{1}\}.$$

Recall from the last lecture that for each column label e,

$$P(A) \cap \{x : x_e = 0\} = P(A') \quad \text{and} \quad P(A) \cap \{x : x_e = 1\} = P(A'')$$

where A', A'' are appropriate submatrices of A.

Proposition 4.4. *Let A be a balanced matrix. Then the polytope $P(A)$ is integral.*

Proof. Suppose otherwise. Let E be the column labels of A. We may assume that $P(A)$ is not integral, but for every proper submatrix A' of A, $P(A')$ is integral. In particular, for every $e \in E$, the two polytopes

$$P(A) \cap \{x : x_e = 0\} \quad \text{and} \quad P(A) \cap \{x : x_e = 1\}$$

are integral. Let $x^\star$ be a fractional extreme point of $P(A)$. Since the polytopes above are integral, it follows that $\mathbf{1} > x^\star > \mathbf{0}$. Our minimality assumption implies that A is a square non-singular matrix.

Claim. *Every row of A has exactly two ones.*

Proof of Claim. By our minimal choice, every row of A has at least two ones. Let A' be the matrix obtained from A after removing the first row. Since $P(A')$ is integral and $x^\star \in P(A')$, it follows that $x^\star$ lies on an edge of $P(A')$. So for some vertices $\chi_S, \chi_T \in P(A')$ and $\lambda \in (0,1)$,

$$x^\star = \lambda \chi_S + (1-\lambda)\chi_T.$$

Since $\mathbf{1} > x^\star > \mathbf{0}$, it follows that $S \cap T = \emptyset$ and $S \cup T = E$. Since $A'\chi_S = \mathbf{1} = A'\chi_T$, every row of A other than the first row has exactly two ones. A similar argument applied to the second row implies that even the first row has exactly two ones. $\Diamond$

Since A is balanced, it is the incidence edge-vertex incidence matrix of a bipartite graph G. As A is a square matrix, G has an even cycle, which in turn contradicts the non-singularity of A. This finishes the proof of Proposition 4.4. □

Theorem 4.5 (Fulkerson, Hoffman, Oppenheim 1974)**.** *Let* $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ *be a balanced matrix. Then the polyhedron*

$$P = \{x \geq \mathbf{0} : Ax \geq \mathbf{1}, Bx \leq \mathbf{1}, Cx = \mathbf{1}\}$$

is integral. In particular, the set packing polytope and the set covering polyhedron corresponding to a balanced matrix are both integral.

Proof. Let $x^\star$ be an extreme point of P. Observe that $x^\star \leq \mathbf{1}$, and that $x^\star$ is also an extreme point of the polytope $\{\mathbf{1} \geq x \geq \mathbf{0} : Dx = \mathbf{1}\}$, where D is the row submatrix of $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ corresponding to the constraints of $Ax \geq \mathbf{1}, Bx \leq \mathbf{1}, Cx = \mathbf{1}$ that are tight at $x^\star$. Since $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ is balanced, so is D, so by Proposition 4.4, $x^\star$ is integral, as required. □

In fact, the linear system above is totally dual integral. We will prove a similar result in the next section.

4.3 Hall's theorem for balanced hypergraphs

Let $G = (V, E)$ be a hypergraph. A *matching* is a packing of pairwise disjoint edges. A *perfect matching* is a matching that uses every vertex. Recall Hall's condition for the existence of perfect matchings in bipartite graphs:

Theorem 4.6 (Hall 1935)**.** *Let G be a bipartite graph. Then the following statements are equivalent:*

- *G has no perfect matching,*
- *there exist disjoint vertex sets R, B such that $|R| > |B|$ and every edge with an end in R has an end in B.*

We will see a generalization of this to balanced hypergraphs. We will need two lemmas.

Lemma 4.7. *Let A be an $m \times n$ balanced matrix. Then the polyhedron*

$$P = \{x, s, t \geq \mathbf{0} : Ax + Is - It = \mathbf{1}\}$$

is integral.

Proof. Denote by a_i the ith row of A, for each $i \in [m]$. Take an extreme point $(x^\star, s^\star, t^\star)$ of P. Since the corresponding columns are linearly dependent, we see that $s_i^\star t_i^\star = 0$ for each $i \in [m]$. As a result, $x^\star$ is also an extreme point of the polyhedron

$$\left\{ x \geq \mathbf{0} : \begin{array}{lll} a_i^\top x & \leq 1 & \forall i \in [m] \text{ s.t. } s_i^\star > 0 \\ a_i^\top x & \geq 1 & \forall i \in [m] \text{ s.t. } t_i^\star > 0 \\ a_i^\top x & = 1 & \text{otherwise.} \end{array} \right\}$$

By Theorem 4.5, this polyhedron is integral, implying in turn that $x^\star$ is integral. This easily implies that $(x^\star, s^\star, t^\star)$ is also integral, thereby finishing the proof. □

Lemma 4.8. *Let A be a balanced matrix. Then the linear system $x, s, t \geq \mathbf{0}, Ax + Is - It = \mathbf{1}$ is totally dual integral.*

Proof. We prove this by induction on the number of rows of A. The base case is obvious. For the induction step, consider for integral weights b, c, d the primal program

$$(P) \qquad \begin{array}{ll} \max & b^\top x + c^\top s + d^\top t \\ \text{s.t.} & Ax + Is - It = \mathbf{1} \\ & x, s, t \geq \mathbf{0} \end{array}$$

and the dual

$$(D) \qquad \begin{array}{lll} \min & \mathbf{1}^\top y & \\ \text{s.t.} & A^\top y & \geq b \\ & y & \geq c \\ & -y & \geq d. \end{array}$$

We will construct an integral optimal solution to (D). To this end, take an optimal solution $\bar{y}$ to (D). If $\bar{y}$ is integral, we are done. Otherwise, we may assume that $\bar{y}_1$ is fractional. Write $\bar{y} = (\bar{y}_1, \bar{z})$. Let a be the first row of A, and let A' (resp. c', d') be the matrix (resp. vector) obtained from A (resp. c, d) after removing the first row. Consider the program

$$(D') \qquad \begin{array}{lll} \min & \mathbf{1}^\top z & \\ \text{s.t.} & A'^\top z & \geq b - \lceil \bar{y}_1 \rceil a \\ & z & \geq c' \\ & -z & \geq d'. \end{array}$$

Since $\bar{y} = (\bar{y}_1, \bar{z})$ is feasible for (D), we get that $\bar{z}$ is feasible for (D'). Our induction hypothesis implies that (D') has an integral optimal solution $z^\star$. In particular,

$$\mathbf{1}^\top \bar{z} \geq \mathbf{1}^\top z^\star.$$

As $z^\star$ is feasible for (D'), and c, d are integral, it follows that $(\lceil \bar{y}_1 \rceil, z^\star)$ is feasible for (D), so

$$\lceil \bar{y}_1 \rceil + \mathbf{1}^\top z^\star \geq \mathbf{1}^\top \bar{y} = \bar{y}_1 + \mathbf{1}^\top \bar{z}.$$

Combining the preceding two inequalities yields

$$\lceil \bar{y}_1 \rceil + \mathbf{1}^\top z^\star \geq \mathbf{1}^\top \bar{y} \geq \bar{y}_1 + \mathbf{1}^\top z^\star.$$

By Lemma 4.7, (P) has an integral optimal solution, so as b, c, d are integral, (P) has an integer optimal value. Thus, by LP Strong Duality, $\mathbf{1}^\top \bar{y}$ is an integer. Hence, the inequalities above imply that $\lceil \bar{y}_1 \rceil + \mathbf{1}^\top z^\star = \mathbf{1}^\top \bar{y}$, so $(\lceil \bar{y}_1 \rceil, z^\star)$ is an integral optimal solution for (D), as required. This completes the induction step. $\square$