CO 750 Packing and Covering: Lecture 5

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4.3 Hall's theorem for balanced hypergraphs

Recall the following two lemmas from the last lecture:

Lemma 4.7. *Let A be an $m \times n$ balanced matrix. Then the polyhedron*

$$P = \{x, s, t \geq \mathbf{0} : Ax + Is - It = \mathbf{1}\}$$

is integral.

Lemma 4.8. *Let A be a balanced matrix. Then the linear system $x, s, t \geq \mathbf{0}, Ax + Is - It = \mathbf{1}$ is totally dual integral.*

We are now ready to prove the following generalization of Hall's Theorem 4.6 for balanced hypegraphs:

Theorem 4.9 (Conforti, Cornuéjols, Kapoor, Vušković 1996)**.** *Let $G = (V, E)$ be a balanced hypergraph. Then the following statements are equivalent:*

- *G has no perfect matching,*
- *there are disjoint vertex sets R, B such that $|R| > |B|$ and for every edge e, $|e \cap B| \geq |e \cap R|$.*

Proof. ($\Leftarrow$) Suppose for a contradiction that G has a perfect matching $e_1, \dots, e_k$. Then

$$|R| = \sum_{i=1}^{k} |e_i \cap R| \leq \sum_{i=1}^{k} |e_i \cap B| = |B| < |R|,$$

a contradiction. ($\Rightarrow$) Suppose G has no perfect matching. Let A be the vertex-edge incidence matrix of G. Notice that A is a balanced matrix. Consider the linear program

$$(P) \qquad \begin{array}{ll} \max & \mathbf{0}^\top x - \mathbf{1}^\top s - \mathbf{1}^\top t \\ \text{s.t.} & Ax + Is - It = \mathbf{1} \\ & x, s, t \geq \mathbf{0} \end{array}$$

Since G has no perfect matching, (P) has no integer feasible solution of value ≥ 0. It therefore follows from Lemma 4.7 that the optimal value of (P) is < 0. As a result, by Lemma 4.8, the dual program has an integral feasible solution of negative value, that is, there is an integral point $\bar{y}$ such that

$$\begin{aligned}\mathbf{1}^\top y &< 0\\ A^\top y &\geq \mathbf{0}\\ y &\leq \mathbf{1}\\ y &\geq -\mathbf{1}\end{aligned}$$

Let $B := \{v \in V : \bar{y}_v = 1\}$ and $R := \{v \in V : \bar{y}_v = -1\}$. Clearly, $B \cap R = \emptyset$. The first inequality implies that $|R| > |B|$ while the second inequality implies that, for each edge e, $|e \cap B| \geq |e \cap R|$, as required. □

This result has a nice Kőnig-type consequence. Given a hypergraph, the *degree* of a vertex is the number of edges containing that vertex. For an integer $d \geq 1$, a hypergraph is *d-regular* if every vertex has degree d.

Corollary 4.10. *The edges of a balanced hypergraph with maximum degree d can be partitioned into d matchings.*

Proof. Let $G = (V, E)$ be a balanced hypergraph with maximum degree $d \geq 1$. Let us first prove the result for d-regular hypergraphs:

Claim 1. *If G is d-regular, then its edges can be partitioned into d perfect matchings.*

Proof of Claim. We prove this by induction on $d \geq 1$. The base case $d = 1$ is obvious. Assume that $d \geq 2$. Let us use Theorem 4.9 to find a perfect matching in G. Take disjoint vertex subsets R, B of V such that for every edge e, $|e \cap B| \geq |e \cap R|$. Then

$$d \cdot |B| = \sum_{e \in E} |e \cap B| \geq \sum_{e \in E} |e \cap R| = d \cdot |R|,$$

implying in turn that $|B| \geq |R|$. It therefore follows from Theorem 4.9 that G has a perfect matching $M_d \subseteq E$. Notice that $G \setminus M_d$ is $(d-1)$-regular, so by the induction hypothesis, the edges of $G \setminus M_d$ can be partitioned into $d - 1$ perfect matchings $M_1, \ldots, M_{d-1}$. Together with M_d, we get a partition of the edges of G into d perfect matchings, thereby completing the induction step. ◊

Claim 2. *There is a d-regular balanced hypergraph $H = (V, E')$ such that $E \subseteq E'$.*

Proof of Claim. To obtain H, for every vertex v of G, add $d - \deg(v)$ edges of the form $\{v\}$. It is clear that H is a d-regular hypergraph. It is easy to see that H is a balanced hypergraph. ◊

By Claim 1, the edges of H can be partitioned into d perfect matchings. It is easy to see that this corresponds to a partition of the edges of G into d matchings, thereby finishing the proof. □

In particular,

Theorem 4.11 (Kőnig 1931)**.** *Let G be a loopless bipartite graph of maximum degree d. Then the edges of G can be partitioned into d matchings, that is, G can be d-edge-colored.*

5 Perfect graphs

Let $G = (V, E)$ be a simple graph. Denote by $\chi(G)$ the minimum number of stable sets needed to cover V. Notice that $\chi(G)$ records the *chromatic number of* G, i.e. the minimum number of colors needed for a vertex-coloring. Denote by $\omega(G)$ the maximum cardinality of a clique. Since the vertices of a clique get different colors in any vertex-coloring, it follows that

$$\chi(G) \geq \omega(G).$$

Denote by $\overline{G}$ the *complement* of G, that is, $\overline{G}$ has vertex set V where distinct vertices u, v are adjacent in $\overline{G}$ if they are non-adjacent in G. Notice that the cliques and stable sets of $\overline{G}$ are precisely the stable sets and cliques of $\overline{G}$.

Remark 5.1. *Let $G = (V, E)$ be a simple graph. Then*

$$\theta(G) := \chi(\overline{G})$$

is the minimum number of cliques of G needed to cover V, and

$$\alpha(G) := \omega(\overline{G})$$

is the maximum cardinality of a stable set. In particular, $\theta(G) \geq \alpha(G)$.

Recall the following two theorems from Assignment 1:

Theorem 5.2 (Kőnig 1931)**.** *In a loopless bipartite graph, the minimum cardinality of a vertex cover is equal to the maximum cardinality of a matching.*

Theorem 5.3. *In a partially ordered set, the minimum number of antichains needed to cover the ground set is equal to the maximum cardinality of a chain.*

We will need this result moving forward, as well as a few notions. The *line graph* of a simple graph G is the graph on vertex set $E(G)$ where distinct $e, f \in E(G)$ are adjacent if e, f share a vertex of G. Given a partially ordered set $(V, \leq)$, its *comparability graph* is the graph on vertex set V where distinct $u, v \in V$ are adjacent if they are comparable.

The main theme of this section is, when does equal hold in $\chi \geq \omega$?

Theorem 5.4. *$\chi(G) = \omega(G)$ if G is any of the following graphs:*

(1) G or $\overline{G}$ is bipartite,

(2) G or $\overline{G}$ is the line graph of a bipartite graph,

(3) G or $\overline{G}$ is a comparability graph.

Proof. **(1)** Let G be a bipartite graph. Then $\chi(G) = 2 = \omega(G)$. We need to show that $\theta(G) = \alpha(G)$. Clearly,

$$\alpha(G) = |V| - k$$

where k is the minimum cardinality of a vertex cover. Since G is bipartite,

$$\theta(G) = |V| - m$$

where m is the maximum cardinality of a matching. By Theorem 5.2, $m = k$, implying in turn that $\theta(G) = \alpha(G)$, as required. **(2)** Let G be the line graph of a bipartite graph H. Observe that the stable sets and cliques of G are in correspondence with the matchings and stars of H, respectively. Thus $\chi(G)$ is equal to the minimum number of colors needed in an edge-coloring of H, while $\omega(G)$ is equal to the maximum degree of a vertex of H. It therefore follows from Theorem 4.11 that $\chi(G) = \omega(G)$. Moreover, $\theta(G)$ is equal to the minimum cardinality of a vertex cover, while $\alpha(G)$ is equal to the maximum cardinality of a matching. So by Theorem 5.2, $\theta(G) = \alpha(G)$. **(3)** Let $G = (V, E)$ be the comparability graph of a partially ordered set $(V, \leq)$. Then the cliques and stable sets of G are in correspondence with the chains and antichains of $(V, \leq)$. It therefore follows from Theorem 1.2 that $\theta(G) = \alpha(G)$, and it follows from Theorem 5.3 that $\chi(G) = \omega(G)$. $\square$