CO 750 Packing and Covering: Lecture 7

Ahmad Abdi

May 25, 2017

5.2 Odd holes and odd antiholes

We say that a simple graph is *minimally imperfect* if it is not perfect, but every proper induced subgraph is perfect. Equivalently, a simple graph G is minimally imperfect if $\chi(G) > \omega(G)$, but for every proper induced subgraph G', $\chi(G') = \omega(G')$. The latter implies that a minimally imperfect graph is always connected.

Remark 5.8. *A graph is perfect if, and only if, it has no minimally imperfect induced subgraph.*

Let H be an odd circuit with at least 5 vertices. Then $3 = \chi(H) > \omega(H) = 2$, so G is imperfect. Since every proper induced subgraph of H is bipartite, and therefore perfect, it follows that H is minimally imperfect. Notice that the Weak Perfect Graph Theorem 5.7 equivalently states that,

Corollary 5.9. *The complement of a minimally imperfect graph is also minimally imperfect.*

Thus, the complement of an odd circuit with at least 5 vertices is also minimally imperfect. Let G be a simple graph. We say that G has an *odd hole* if it has as an induced subgraph an odd circuit with at least 5 vertices, and we say that G has an *odd antihole* if $\overline{G}$ has an odd hole. It follows from the preceding remark that,

Remark 5.10. *A perfect graph has no odd hole and no odd antihole.*

In 1961, Claude Berge conjectured that the converse of this statement is also true. In 2006, this conjecture was proved by Chudnovsky, Robertson, Seymour and Thomas, and their theorem is referred to as the strong perfect graph theorem. We will see some of the milestones and highlights leading to the proof, as well as a sketch of the proof.

5.3 Star cutsets and antitwins

Let $G = (V, E)$ be a simple graph. A *star cutset* is a non-empty $X \subseteq V$ such that

- a vertex of X is adjacent to all the other vertices in X, and
- $G \setminus X$ is not connected.

Lemma 5.11 (Chvátal 1985)**.** *A minimally imperfect graph does not have a star cutset.*

Proof. Let $G = (V, E)$ be a minimally imperfect graph, and let $\omega := \omega(G)$. Then

$$\omega(G \setminus S) = \omega \quad \text{for every stable set } S \subseteq V.$$

Suppose for a contradiction that G has a star cutset $X \subseteq V$. Then the vertices of $G \setminus X$ can be partitioned into non-empty parts V_1, V_2 such that G has no edge between V_1 and V_2. Since every proper induced subgraph of G is perfect, for each $i \in [2]$, there is a vertex-coloring $f_i : X \cup V_i \to [\omega]$ of the induced subgraph $G[X \cup V_i]$. Since X is a star cutset, it has a vertex v that is adjacent to all other vertices of X. For $i \in [2]$, let $S_i := \{w \in X \cup V_i : f_i(w) = f_i(v)\}$. Clearly, each S_i is stable and $S_i \cap X = \{v\}$. Moreover, since there are no edges between V_1 and V_2, it follows that $S := S_1 \cup S_2$ is also stable. In particular, $\omega(G \setminus S) = \omega$, so $G \setminus S$ has a clique C of cardinality ω. However, either $C \subseteq X \cup V_1$ or $C \subseteq X \cup V_2$, implying in turn that C is an ω-clique of some $G[X \cup V_i] \setminus S_i$, which has an $(\omega - 1)$-vertex-coloring, a contradiction. □

This lemma was a key milestone for what led to the proof of the strong perfect graph theorem. To demonstrate the power of this lemma, let us see some applications of it. Let G_1 be a perfect graph, and take a vertex $v \in V(G_1)$. To *duplicate* v is to introduce a new vertex $\bar{v}$, join it to all the neighbors of v, and then join it to $\bar{v}$. More generally, given another perfect graph G_2 over a disjoint vertex set, to *substitute* G_2 *for* v is to remove v, and join every vertex of G_2 to all the neighbors of v in $G_1 \setminus v$.

Theorem 5.12 (Lovász 1972)**.** *Let G_1, G_2 be perfect graphs over disjoint vertex sets. If G is obtained by substituting G_2 for a vertex v of G_1, then G is perfect. In particular, duplication preserves perfection.*

Proof. Suppose otherwise. Since every induced subgraph of G is either an induced subgraph of G_1, or of G_2, or arises from induced subgraphs of G_1, G_2 by substitution, we may assume that G is minimally imperfect. Clearly, G_2 has at least two vertices, and $G_1 \setminus v$ has at least one vertex. Take an arbitrary vertex u of G_2, and denote by N its neighbors of G in $V(G_1 \setminus v)$. Notice that for each vertex in $V(G_2)$, its neighbors of G in $V(G_1 \setminus v)$ is precisely N. As G is minimally imperfect, $\overline{G}$ is minimally imperfect by Corollary 5.9, so $\overline{G}$ is connected, implying in turn that $V(G_1 \setminus v) - N \neq \emptyset$. Let $X := \{u\} \cup N$. Then X is a star cutset as u is adjacent to all the vertices in N, and in $G \setminus X$, there are no edges between $V(G_2) - \{u\}$ and $V(G_1 \setminus v) - N$. This contradicts the Star Cutset Lemma 5.11. □

Let $G = (V, E)$ be a simple graph. A *skew partition* is a partition of V into a pair (A, B) such that $G[A]$ is not connected and $\overline{G}[B]$ is not connected. Notice that if (A, B) is a skew partition for G, then it is a skew partition for $\overline{G}$. Notice further that if X is a star cutset and $|X| \geq 2$, then $(V - X, X)$ is a skew partition. In an attempt to generalize Lemma 5.11, Chvátal (1985) conjectured that a minimally imperfect graph does not have a skew partition. The *length* of a path is the number of edges in it. A path of $\overline{G}$ is called an *antipath of* G. We say that a skew partition (A, B) is *balanced* if

- there is no induced odd path between non-adjacent vertices in B with interior in A,
- there is no induced odd antipath between adjacent vertices in A with interior in B.

Theorem 5.13 (Chudnovsky, Robertson, Seymour, Thomas 2006)**.** *A minimally imperfect graph does not have a balanced skew partition.*

Let $G = (V, E)$ be a simple graph. Distinct vertices u, v are *antitwins* if every other vertex is adjacent to precisely one of u, v. Notice that if u, v are antitwins in G, then they are also antitwins in $\overline{G}$. We will see in the next lecture that,

Lemma 5.14 (Oraliu 1988)**.** *A minimally imperfect graph does not have antitwins.*