# CO 750 Packing and Covering: Lecture 9

Ahmad Abdi

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## 6 Perfect matrices

Let $G = (V, E)$ be a perfect graph. Let $A$ be the $0-1$ matrix whose columns are labeled by $V$ and whose rows are the incidence vectors of the stable sets of $G$. Take weights $c \in \mathbb{Z}_+^V$. Consider the set packing primal-dual pair

$$(P) \quad \begin{array}{ll} \max & c^\top x \\ \text{s.t.} & Ax \leq \mathbf{1} \\ & x \geq \mathbf{0} \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{ll} \min & \mathbf{1}^\top y \\ \text{s.t.} & A^\top y \geq c \\ & y \geq \mathbf{0}. \end{array}$$

We can rewrite the primal as

$$(P) \quad \begin{array}{ll} \max & \sum (c_v x_v : v \in V) \\ \text{s.t.} & \sum (x_v : v \in S) \leq 1 \quad \forall \text{ stable sets } S \\ & x_v \geq 0 \quad \forall v \in V. \end{array}$$

Observe that a clique gives a feasible solution to this program. So the maximum weight of a clique is a lower-bound on the optimal value of (P). To make this precise, let $G_c$ be the graph obtained from $G$ after replacing each vertex $v$ by $c_v$ duplicates. (If $c_v = 0$ then delete $v$.) Notice that by Theorem 5.12, $G_c$ is also a perfect graph. Observe that the maximum weight of a clique of $G$ is equal to the maximum cardinality $\omega(G_c)$ of a clique of $G_c$. Thus, $\omega(G_c)$ is a lower-bound on the optimal value of (P). Let us next rewrite the dual as

$$(D) \quad \begin{array}{ll} \min & \sum (y_S : \text{ stable sets } S) \\ \text{s.t.} & \sum (y_S : \text{ stable sets } S \text{ such that } v \in S) \geq c_v \quad \forall v \in V \\ & y_S \geq 0 \quad \forall \text{ stable sets } S. \end{array}$$

Observe that a covering of $V(G_c)$ using stable sets gives a feasible solution to (D). Thus, the minimum number of stable sets needed to cover $V(G_c)$, which is $\chi(G_c)$, is an upper-bound on the optimal value of (D). Since $G_c$ is perfect, we have $\chi(G_c) = \omega(G_c)$, implying in turn that,

**Corollary 6.1.** *Let $G$ be a perfect graph. Then the set packing system corresponding to the stable sets of $G$ is totally dual integral. In particular, the set packing polytope*

$$\left\{ x \in \mathbb{R}_+^V : \sum (x_v : v \in S) \leq 1 \quad \forall \textit{ stable sets } S \right\}$$

*is integral.*

In fact, we will see that these are essentially the *only* examples of integral set packing polytopes and totally dual integral set packing systems! To this end, let $A$ be a $0-1$ matrix without a column of all zeros. We say that $A$ is *perfect* if the set packing polytope $\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$ is integral.

## 6.1 Perfection implies total dual integrality

From the discussion in the previous section, it seems more natural to call a matrix perfect when the corresponding set packing system is totally dual integral. The following amazing result justifies our choice of terminology:

**Theorem 6.2** (Fulkerson 1972)**.** *Let $A$ be a perfect matrix. Then the linear system $x \geq \mathbf{0}, Ax \leq \mathbf{1}$ is totally dual integral.*

*Proof.* Denote by $E$ the column labels of $A$. Consider the set packing primal-dual pair

$$(P)\quad \begin{array}{ll} \max & c^\top x \\ \text{s.t.} & Ax \leq \mathbf{1} \\ & x \geq \mathbf{0} \end{array} \qquad \text{and} \qquad (D)\quad \begin{array}{ll} \min & \mathbf{1}^\top y \\ \text{s.t.} & A^\top y \geq c \\ & y \geq \mathbf{0} \end{array} \qquad c \in \mathbb{Z}^E.$$

As $A$ is perfect, (P) has an integral optimal solution for all $c \in \mathbb{Z}^E$. We will prove by induction on the optimal value $\omega \in \mathbb{Z}_+$ of (P) that (D) has an integral dual solution for all $c \in \mathbb{Z}^E$. If $\omega = 0$ for some $c \in \mathbb{Z}^E$, then as $A$ has no column of all zeros, it follows that $c \leq \mathbf{0}$, implying in turn that $\mathbf{0}$ is an optimal solution for (D). For the induction step, assume that $\omega \geq 1$ for some $c \in \mathbb{Z}^E$. Take an arbitrary row $a$ of $A$ such that

$$a^\top x^\star = 1 \qquad \text{for all optimal solutions } x^\star \text{ of (P).}$$

(To find this row, take an optimal dual solution $y^\star$, and pick $a$ so that $y^\star_a > 0$; apply the complementary slackness conditions.) We may assume that $a$ is the first row of $A$. Consider the set packing primal-dual pair

$$(P')\quad \begin{array}{ll} \max & (c-a)^\top x \\ \text{s.t.} & Ax \leq \mathbf{1} \\ & x \geq \mathbf{0} \end{array} \qquad \text{and} \qquad (D')\quad \begin{array}{ll} \min & \mathbf{1}^\top y \\ \text{s.t.} & A^\top y \geq c - a \\ & y \geq \mathbf{0} \end{array}$$

Clearly, the optimal value of (P') is at most $\omega$, and our choice of $a$ implies that it is exactly $\omega - 1$. Thus, by the induction hypothesis, (D') has an integral optimal solution $\bar{y} = (\bar{y}_1, \bar{z})$ of value $\omega - 1$. Let $y^\star := (\bar{y}_1 + 1, \bar{z})$. Then $y^\star$ is an integral feasible solution for (D) and has value $\omega$, so it is optimal. This completes the induction step. □

## 6.2 The pluperfect graph theorem

In an attempt to prove it, Ray Fulkerson proposed and proved a polyhedral analogue of the weak perfect graph theorem, and he called it the *pluperfect graph theorem*. To prove his theorem, we will need two ingredients. Let $A$ be a non-negative matrix without a column of all zeros. Let

$$P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}.$$

The *antiblocker of* $P$ is the set

$$a(P) := \{y \geq \mathbf{0} : x^\top y \leq 1 \ \ \forall x \in P\}.$$

**Proposition 6.3.** *Let $A$ be a non-negative matrix without a column of all zeros. Let $B$ be the matrix whose rows are the extreme points of $P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$. Then $B$ is non-negative, has no column of all zeros, and*

$$\begin{aligned} a(P) &= \{y \geq \mathbf{0} : By \leq \mathbf{1}\} \\ a(a(P)) &= P. \end{aligned}$$

*Proof.* Clearly, $B$ is a non-negative matrix. Since $A$ has no column of all zeros, $P$ is a polytope, so every point of $P$ can be written as a convex combination of the rows of $B$ – this has two consequences. Firstly, as $\epsilon\mathbf{1} \in P$ for a sufficiently small $\epsilon > 0$, $B$ cannot have a column of all zeros. Secondly, $\{y \geq \mathbf{0} : By \leq \mathbf{1}\} \subseteq a(P)$. As the reverse inclusion holds trivially, we see that $a(P) = \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$. For the next equation, by definition

$$a(a(P)) = \{x \geq \mathbf{0} : y^\top x \leq 1 \ \ \forall y \in a(P)\}.$$

So clearly, $P \subseteq a(a(P))$. To prove the reverse inclusion, it suffices to show that every row $a$ of $A$ belongs to $a(P)$. Since $a \geq \mathbf{0}$ and $Ba \leq \mathbf{1}$, the result follows. ☐

Next we study the extreme points of the antiblocker. Let's see an example first. Consider the matrix

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then the extreme points of $P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$ are the rows of the matrix

$$B := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Proposition 6.3, the antiblocker of $P$ is the polytope $a(P) = \{x \geq \mathbf{0} : Bx \leq \mathbf{1}\}$. Aside from the three rows of $A$, the extreme points of $a(P)$ are $(1\ 0\ 0), (0\ 1\ 0), (0\ 0\ 1), (0\ 0\ 0)$, which are all orthogonal projections of the rows of $A$. In the next lecture, we will show that this is true in general.