Signed graphs

- graph $G = (V, E)$
- $\Sigma \subseteq E$
- (G, Σ) is a signed graph
- a circuit/cycle C is odd if

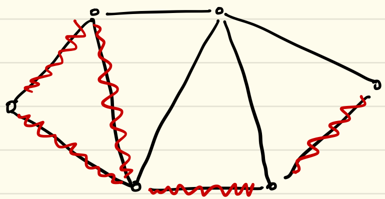
$$|C \cap \Sigma| \equiv 1 \pmod 2,$$

otherwise it's even

- (G, Σ) is weakly bipartite if

$$\{ C : C \text{ an odd circuit} \}$$

is an ideal clutter



Σ

> What are the weakly bipartite signed graphs?

Signed graph minors

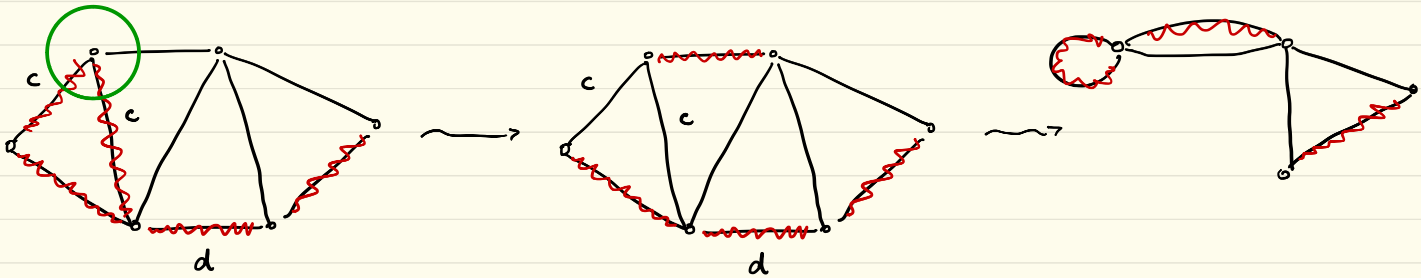
→ For disjoint $I, J \subseteq E$, define $(G, \Sigma) \setminus I / J$ as

- if J contains an odd circuit, set

$$(G, \Sigma) \setminus I / J := (G \setminus I / J, \emptyset)$$

- otherwise, pick a signature B disjoint from J, set

$$(G, \Sigma) \setminus I / J := (G \setminus I / J, B - I)$$

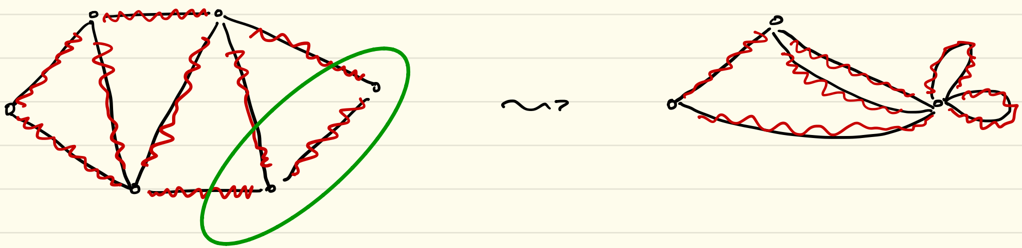


Signed graph minors

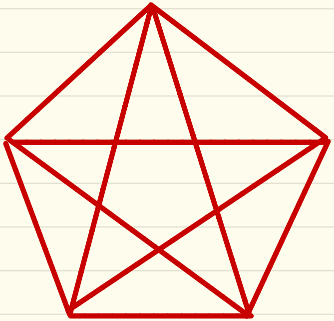
Note: Consider a signed graph $(G, E(G))$ where every edge is odd. Then for every cut $\delta(U)$,

$$(G, E(G)) / \delta(U) = (H, E(H))$$

where $H := G / \delta(U)$.



Remark 10.7: A weakly bipartite signed graph has no odd-K_5 minor:



every edge is odd

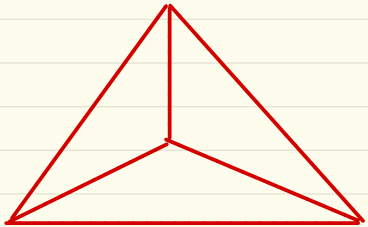
$(K_5, E(K_5))$

Goal: Prove the converse.

$$\boxed{\text{We need tools for finding odd-}K_5\text{ minors.}}$$

Odd-K_4s and Whirlpools

→ The signed graph $(K_4, E(K_4))$ is called an odd-K_4.

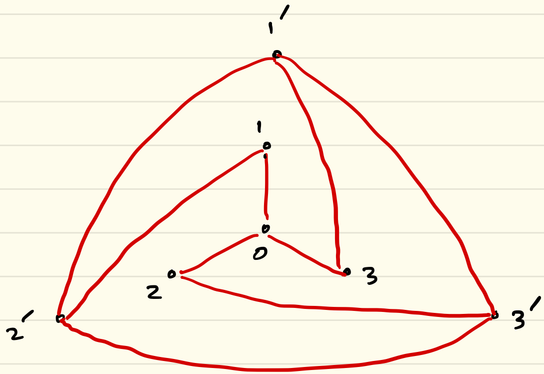


every edge is odd

→ Schrijver (2002) gave a tool for finding odd-K_4 minors.

Odd-K_4s and Whirlpools

→ Consider the signed graph:



every edge is odd.

→ Let's call this a whirlpool with central edges 01, 02, 03.

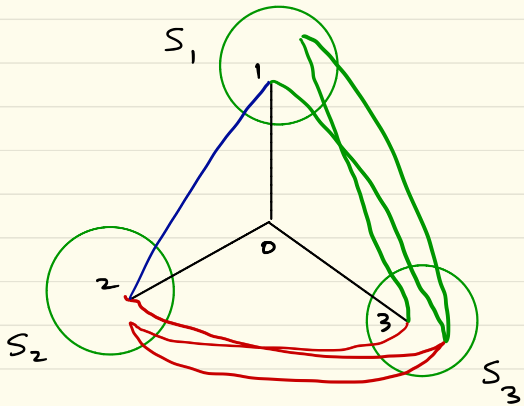
→ The whirlpool has an odd-K_4 minor using the central edges.

Lemma 10.8 (Schrijver 2002)

→ given a graph $G=(V,E)$, disjoint stable sets S_1, S_2, S_3, distinct vxs $0,1,2,3$ s.t.

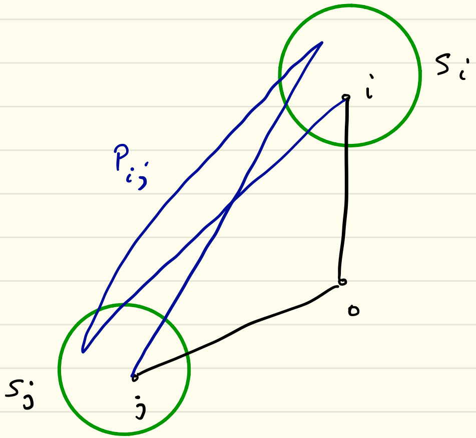
- $0 \in V-(S_1 \cup S_2 \cup S_3)$, $i \in S_i$ and $\{0,i\} \in E$ for $i \in [3]$,
- $G[S_i \cup S_j]$ has an ij-path for distinct $i,j \in [3]$.

→ Then (G,E) has an odd-K_4 minor using $01, 02, 03$



Proof: Proceed by induction on $|V| + |E|$.

→ let P_{ij} be an ij-path contained in $G[S_i \cup S_j]$

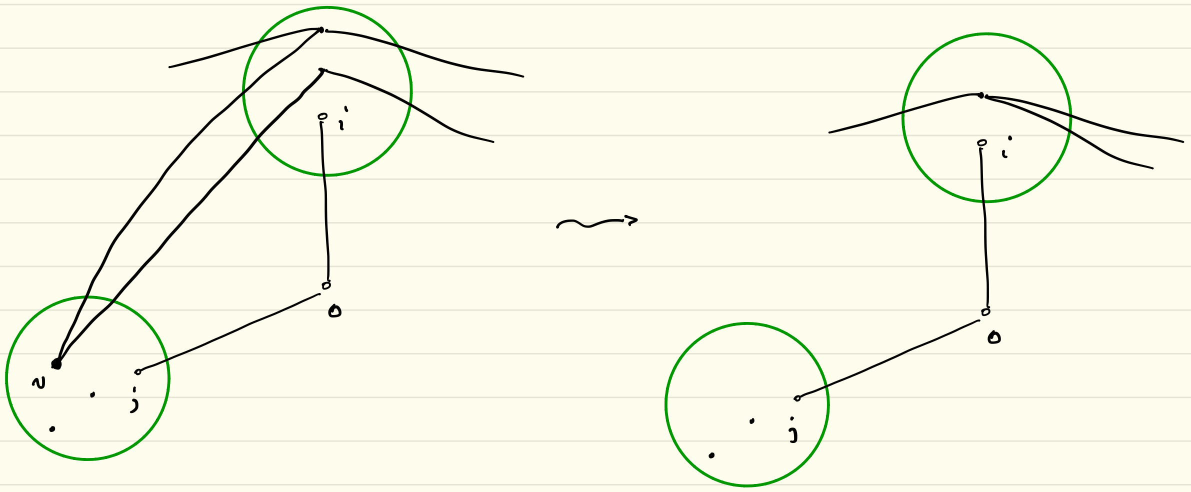


→ we may assume that

$$V = \{0\} \cup V(P_{12}) \cup V(P_{23}) \cup V(P_{31})$$

$$E = \{01, 02, 03\} \cup P_{12} \cup P_{23} \cup P_{31}$$

→ if $\deg(v)=2$, then $(G, E(G))/\delta(v)$ satisfies the conditions of the lemma for the same vertices $0,1,2,3$



→ so we may assume that every vertex has degree ≥ 3

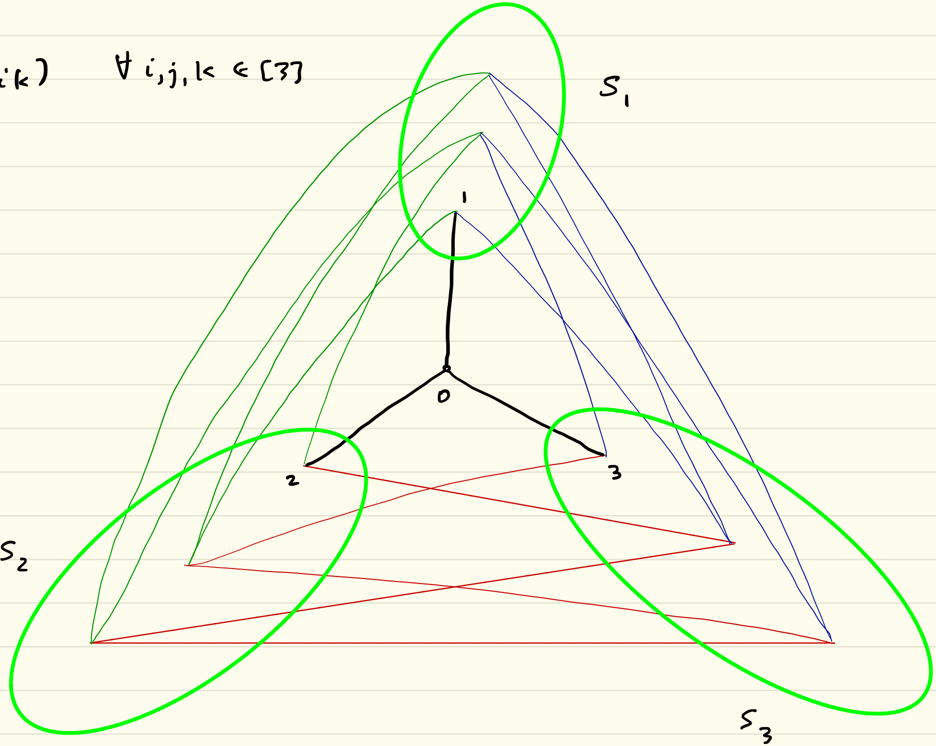
→ thus

$$S_i = V(P_{ij}) \cap V(P_{ik}) \quad \forall\, i,j,k \in [3]$$

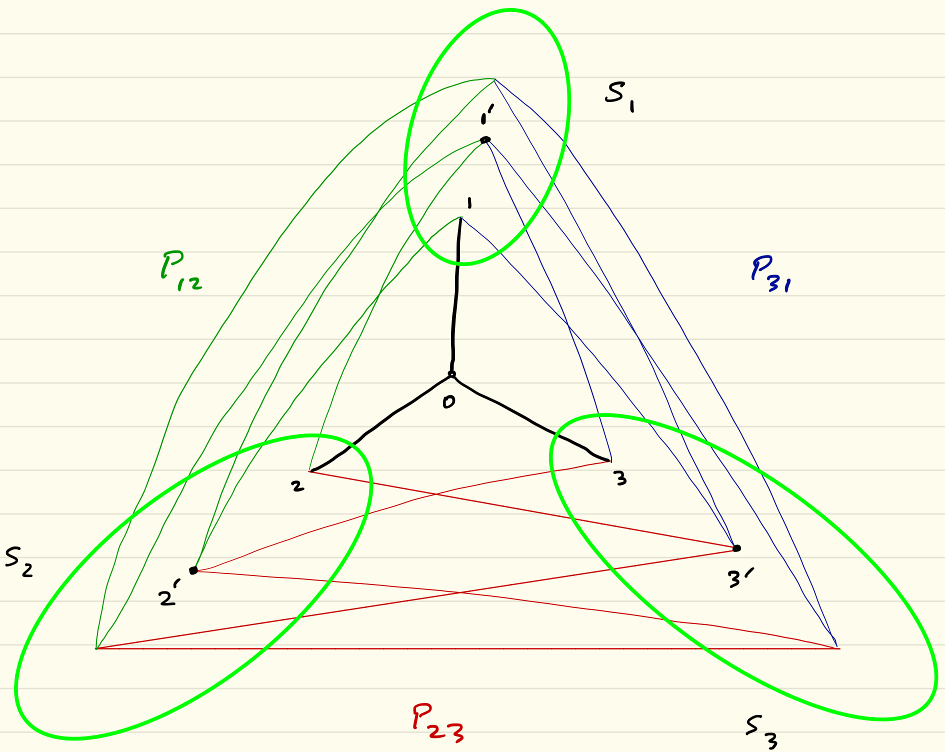
$$|S_1| = |S_2| = |S_3| \geq 1$$

→ if $|S_1| = 1$ then

$$(G, E) = (K_4, E(K_4))$$

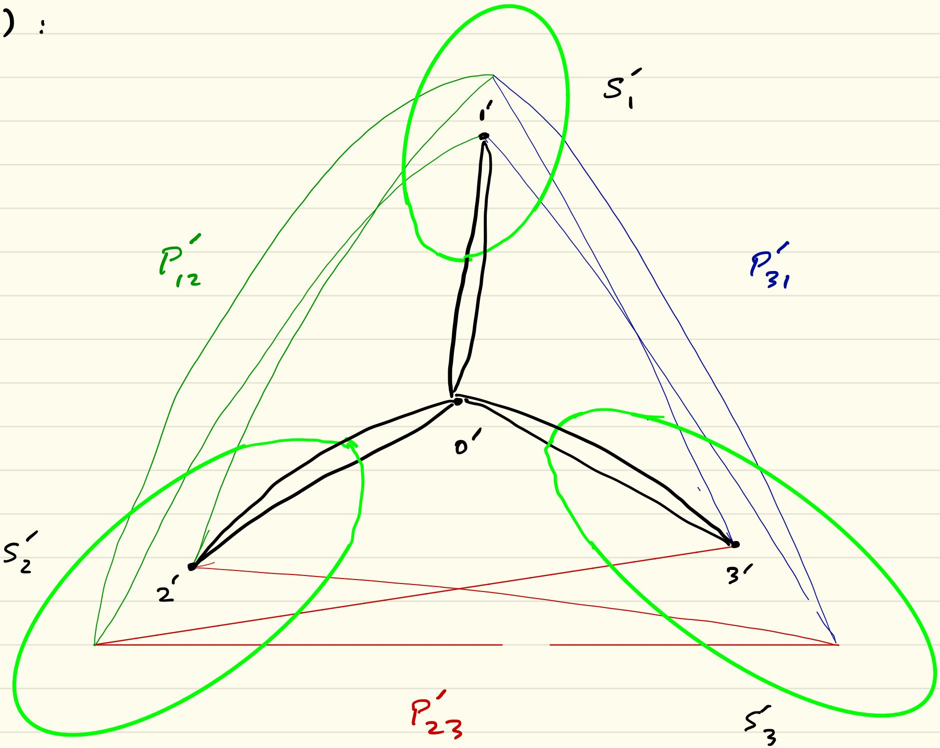


→ otherwise, define $1', 2', 3'$:

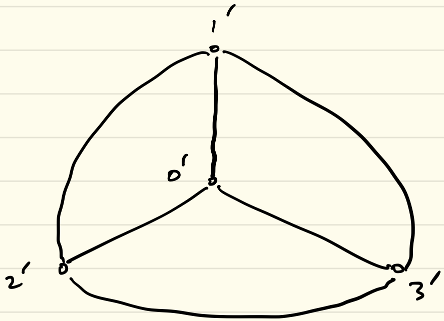


→ otherwise, define 1', 2', 3' :

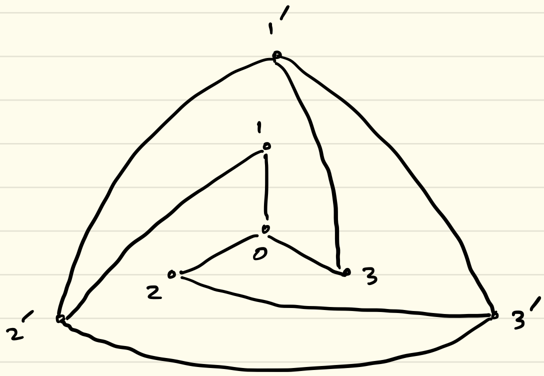
→ Contract $\delta(0)$:



→ by IH there's an odd-K_4 minor using $0'1'$, $0'2'$, $0'3'$



→ deconstract 0:



→ this is a whirlpool with central edges 01, 02, 03

→ so (G, Σ) has an odd-K_4 minor using 01, 02, 03

∎