Lecture 2: Perfect Graphs

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1 Perfect graphs

Let $G = (V, E)$ be a simple graph. Denote by $\chi(G)$ the minimum number of stable sets needed to cover V. Notice that $\chi(G)$ records the *chromatic number of* G, i.e. the minimum number of colours needed to colour the vertices such that neighbouring vertices receive different colours. In general, computing $\chi(G)$ is an NP-hard problem, but for a certain class of graphs called *perfect*, this problem belongs to P. By and large, this is the case for two main reasons: the class is "hereditary", and there is a powerful duality relation involving $\chi(G)$. Let us elaborate.

Denote by $\omega(G)$ the maximum cardinality of a clique, called the *clique number of* G. Since the vertices of a clique all get different colours in any proper vertex-colouring, it follows that $\chi(G) \geq \omega(G)$.

By swapping the roles of cliques and stable sets, we get two other parameters. The *theta number of* G, denoted $\theta(G)$, is the minimum number of cliques of G needed to cover V. The *stability number of* G, denoted $\alpha(G)$, is the maximum cardinality of a stable set.

Denote by $\overline{G}$ the *complement* of G, that is, $\overline{G}$ has vertex set V where distinct vertices u, v are adjacent in $\overline{G}$ if they are non-adjacent in G. Notice that the cliques and stable sets of $\overline{G}$ are precisely the stable sets and cliques of G, respectively. Subsequently,

Remark 1.1. Let $G = (V, E)$ be a simple graph. Then $\theta(G) = \chi(\overline{G})$ and $\alpha(G) = \omega(\overline{G})$. In particular, $\theta(G) \geq \alpha(G)$.

Equality does not always hold in $\chi \geq \omega$. For instance, for the odd circuit C_5 on five vertices, $\chi(C_5) = 3 > 2 = \omega(C_5)$. Can we then characterise when equality does hold? Is this even a well-posed question? Let H be an arbitrary graph, and let $k := \chi(H) - \omega(H) \geq 0$. Let $K \subseteq V(H)$ be a maximum clique of H. Let G be the graph obtained from H after adding k vertices and just enough edges so as to grow K into a clique of cardinality $\omega(H) + k$. Notice now that $\chi(G) = \chi(H) = \omega(H) + k = \omega(G)$. Starting from an arbitrary graph, we just constructed a graph for which $\chi = \omega$. This construction tells us that asking when equality holds in $\chi \geq \omega$ is an ill-posed question. To make sure this construction is ruled out, we must come up with a stronger notion.

For $X \subseteq V$, the subgraph of G induced on vertices X is called an *induced subgraph* and is denoted $G[X]$. The graph G is *perfect* if, for every induced subgraph G' of G, $\chi(G') = \omega(G')$. (Notice that G' may be G.) In words, a simple graph is perfect if in each induced subgraph, the chromatic number equals the clique number.

In this lecture, we first see some classical examples of perfect graphs that originate from Dilworth's theorem on partially ordered sets, and Kőnig's theorem on bipartite graphs. We then prove the *Perfect Graph Theorem* stating that perfection is closed under complementation. After that, we shall exhibit the power of *Kempe switching* in vertex-colouring to prove that *Meyniel graphs* are perfect. We then prove a structure theorem for a special class of perfect graphs, called *cographs*, which are important in *Ramsey Theory*. Finally, we present fast algorithms for another special class of perfect graphs, called *chordal* graphs, which are historically important as they led to the advent of *lexicographic breadth-first search*.

2 Classical examples of perfect graphs

Take a partially ordered set $(V, \leq)$, that is, the following statements hold for all $a, b, c \in V$:

- $a \leq a$,
- if $a \leq b$ and $b \leq a$, then $a = b$,
- if $a \leq b$ and $b \leq c$, then $a \leq c$.

We say that a, b are *comparable* if $a \geq b$ or $b \geq a$; otherwise they are *incomparable*. A *chain* is a set of pairwise comparable elements. An *antichain* is a set of pairwise incomparable elements. Notice that every antichain intersects every chain at most once.

What is the minimum number of (not necessarily disjoint) chains whose union is V? That is, what is the least number of chains needed to *cover* the ground set?

Theorem 2.1 (Dilworth's Theorem)**.** Let $(V, \leq)$ be a partially ordered set. Then the minimum number of chains needed to cover V is equal to the maximum cardinality of an antichain.

Proof. Since every chain intersects every antichain at most once, the minimum number of chains needed to cover V is greater than or equal to the maximum cardinality of an antichain. We will prove the other inequality by induction on $|V|$. The base case $|V| = 1$ is obvious. For the induction step, assume that $|V| \geq 2$. Let α be the maximum cardinality of an antichain. We will find α chains covering V. If $\alpha = |V|$, then we are clearly done. Otherwise, $\alpha < |V|$, implying in turn that there is a chain $\{a, b\}$ where a is a minimal element and b is a maximal element. Let $V' := V - \{a, b\}$.

Claim. If the maximum cardinality of an antichain of $(V', \leq)$ is $\alpha - 1$, then there are α chains covering V.

Proof of Claim. By the induction hypothesis, there are $\alpha - 1$ chains of V' covering $V - \{a, b\}$. Together with $\{a, b\}$, we get a covering of V using α chains. $\lozenge$

We may therefore assume that V' has an antichain A such that $|A| = \alpha$. Let

$$V^+ := A \cup \{x \in V - A : x \geq z \text{ for some } z \in A\}$$
$$V^- := A \cup \{y \in V - A : y \leq z \text{ for some } z \in A\}.$$

Since A is an antichain, $V^+ \cap V^- = A$, and since it is a maximum antichain, $V^+ \cup V^- = V$. As a is minimal and $a \notin A$, it follows that $a \notin V^+$. As b is maximal and $b \notin A$, we get that $b \notin V^-$. In particular, $|V^+|, |V^-| < |V|$. Thus, by the induction hypothesis, V^+ has α chains covering it, and V^- has α chains covering it. Gluing these chains together, we get α chains covering $V^+ \cup V^- = V$, thereby completing the induction step. $\square$

Given a partially ordered set $(V, \leq)$, its *comparability graph* is the graph on vertex set V where distinct $u, v \in V$ are adjacent if they are comparable. In Exercise 1, we see that every comparability graph is perfect. Moreover, Dilworth's Theorem implies that,

Corollary 2.2. The complement of a comparability graph is perfect.

Proof. Let $\overline{G}$ be the complement of a comparability graph G, corresponding to a partially ordered set $(V, \leq)$. Since every induced subgraph of G is a comparability graph, every induced subgraph of $\overline{G}$ is the complement of a comparability graph. Thus, it suffices to prove that $\chi(\overline{G}) = \omega(\overline{G})$. The cliques and stable sets of $\overline{G}$ correspond to the antichains and chains of $(V, \leq)$, respectively. Thus, $\chi(\overline{G})$ is equal to the minimum number of chains needed to cover V, while $\omega(\overline{G})$ is equal to the maximum cardinality of an antichain. Thus, by Dilworth's Theorem, $\chi(\overline{G}) = \omega(\overline{G})$, as required. $\square$

Let us exhibit another classical example of a perfect graph. Let $G = (V, E)$ be a graph $G = (V, E)$. Recall from Lecture 1 that a vertex cover is a subset of vertices incident with every edge of the graph, that is, a vertex cover is the complement of a stable set. In Lecture 1, we proved the following theorem:

Theorem 2.3 (Kőnig's Theorem)**.** Let $G = (V, E)$ be a bipartite graph. Then the minimum size of a vertex cover is equal to the maximum size of a matching.

We use this theorem to present two other important classes of perfect graphs. It can be readily checked by the reader that every bipartite graph is perfect. As a consequence of Kőnig's Theorem, we get that the complements of such graphs are also perfect:

Corollary 2.4. The complement of every bipartite graph is perfect.

Proof. Since the class is hereditary, it suffices to show that the chromatic number is equal to the maximum size of a clique. Let $G = (V, E)$ be a bipartite graph. We need to show that $\theta(G) = \alpha(G)$. Clearly,

$$\alpha(G) = |V| - k$$

where k is the minimum cardinality of a vertex cover. Since G is bipartite,

$$\theta(G) = |V| - m$$

where m is the maximum cardinality of a matching. By Theorem 2.3, $m = k$, implying in turn that $\theta(G) = \alpha(G)$, as required. □

The *line graph* of a simple graph G is the graph on vertex set $E(G)$ where distinct $e, f \in E(G)$ are adjacent if e, f share a vertex of G. In Exercise 2, we see that the line graph of a bipartite graph is perfect. Kőnig's Theorem implies that the complement of such a graph is also perfect:

Corollary 2.5. The complement of the line graph of a bipartite graph is perfect.

Proof. Let G be the line graph of a bipartite graph H. Observe that the stable sets and cliques of G are in correspondence with the matchings and stars of H, respectively. Thus, $\theta(G)$ is equal to the minimum cardinality of a vertex cover of H, while $\alpha(G)$ is equal to the maximum cardinality of a matching of H. Thus, by Kőnig's Theorem, $\theta(G) = \alpha(G)$. □

3 The Perfect Graph Theorem

The results of the previous section lead to a natural question: Does complementation preserve perfection? Although this may seem too good to be true, the answer is yes.

Theorem 3.1. Let G be a simple graph. The following statements are equivalent:

(i) G is perfect,

(ii) $\omega(H) \cdot \alpha(H) \geq |V(H)|$ for every induced subgraph H.

Proof. **(i) ⇒ (ii)**: Let H be an induced subgraph. By definition, $\chi(H) = \omega(H)$, that is, $V(H)$ can be covered by $\omega(H)$ stable sets. Since each stable set has cardinality at most $\alpha(H)$, it follows that

$$|V(H)| \leq \omega(H) \cdot \alpha(H).$$

(ii) ⇒ (i): Suppose for a contradiction that G is not perfect. Let H be an induced subgraph of G that is not perfect, but every proper induced subgraph of H is perfect. Let $\omega := \omega(H)$, $\alpha := \alpha(H)$ and $n := |V(H)|$. Note that $n > 1$. Clearly,

$$\omega \geq \omega(H \setminus S) \geq \omega - 1 \qquad \text{for every nonempty stable set } S \subseteq V(H);$$

since $H \setminus S$ is perfect and H is not, it follows that

$$\omega(H \setminus S) = \omega \qquad \text{for every nonempty stable set } S \subseteq V(H).$$

Let S_0 be a maximum stable set of H. Then for every vertex $v \in S_0$, $H \setminus v$ is perfect, so its vertices can be partitioned into $\omega(H \setminus v) = \omega$ nonempty stable sets. As S_0 has α vertices, we get $\alpha\omega$ stable sets $S_1, \ldots, S_{\alpha\omega}$.

Claim. Every maximum clique of H intersects all but one of $S_0, S_1, \ldots, S_{\alpha\omega}$ exactly once.

Proof of Claim. Let C be a maximum clique of H. Clearly C intersects each of $S_0, S_1, \ldots, S_{\alpha\omega}$ at most once. For a vertex $v \in S_0$, if

- $v \in C$: then C intersects all but one stable set in every partition of $V(H \setminus v)$ into ω stable sets,
- $v \notin C$: then C intersects all stable sets in every partition of $V(H \setminus v)$ into ω stable sets.

This observation immediately implies the claim. $\Diamond$

For each $i \in \{0, 1, \ldots, \alpha\omega\}$, let C_i be a maximum clique of $H \setminus S_i$; notice that $|C_i| = \omega$. Let A be the $0-1$ matrix whose columns are labeled by $V(H)$, and whose rows are the incidence vectors of $S_0, S_1, \ldots, S_{\alpha\omega}$. Let B be the $0-1$ matrix whose columns are labeled by $V(H)$, and whose rows are the incidence vectors of $C_0, C_1, \ldots, C_{\alpha\omega}$. It then follows from the claim above that $AB^\top = J - I$, where J is the all-ones matrix and I the identity matrix of appropriate dimensions. Since $J - I$ is a nonsingular $(\alpha\omega + 1) \times (\alpha\omega + 1)$ matrix, it follows that both A and B have full row rank, implying in turn that

$$|V(H)| = n \geq \alpha\omega + 1 = \alpha(H) \cdot \omega(H) + 1 > |V(H)|,$$

a contradiction. $\square$

As a consequence,

Theorem 3.2 (Perfect Graph Theorem)**.** If a graph is perfect, then so is its complement.

Proof. Suppose that G is perfect. Then by Theorem 3.1, for every induced subgraph H of G,

$$\omega(H) \cdot \alpha(H) \geq |V(H)|,$$

implying in turn that for every induced subgraph $\overline{H}$ of $\overline{G}$,

$$\alpha(\overline{H}) \cdot \omega(\overline{H}) \geq |V(\overline{H})|,$$

so by Theorem 3.1, $\overline{G}$ is perfect, as required. $\square$

3.1 The Strong Perfect Graph Theorem

We say that a simple graph is *minimally imperfect* if it is not perfect, but every proper induced subgraph is perfect. Equivalently, a simple graph G is minimally imperfect if $\chi(G) > \omega(G)$, but for every proper induced subgraph G', $\chi(G') = \omega(G')$. The latter implies that a minimally imperfect graph is always connected.

Remark 3.3. A graph is perfect if, and only if, it has no minimally imperfect induced subgraph.

Let H be an odd circuit with at least five vertices. Then $3 = \chi(H) > \omega(H) = 2$, so G is imperfect. Since every proper induced subgraph of H is bipartite, and therefore perfect, it follows that H is minimally imperfect. Notice that the Perfect Graph Theorem equivalently states that,

Corollary 3.4. The complement of a minimally imperfect graph is also minimally imperfect.

Thus, the complement of an odd circuit with at least five vertices is also minimally imperfect. Let G be a simple graph. We say that G has an *odd hole* if it has as an induced subgraph an odd circuit with at least 5 vertices, and we say that G has an *odd antihole* if $\overline{G}$ has an odd hole. It follows from the preceding remark that,

Remark 3.5. A perfect graph has no odd hole and no odd antihole.

In fact, the converse of this statement is also true!

Theorem 3.6 (Strong Perfect Graph Theorem)**.** A graph is perfect if, and only if, it has no odd hole and no odd antihole.

The original proof of this theorem spans over 150 pages, so we will not cover it here (parts of the proof have been simplified over the years, but the proof is still too long to be covered here).

An important consequence of the Strong Perfect Graph Theorem is the following:

Corollary 3.7. Every simple graph G satisfies at least one of the following statements:

- $\chi(G) = \omega(G)$, or
- G has an odd hole or an odd antihole.

4 Kempe switches and Meyniel graphs

In this section, we introduce a powerful tool in graph colouring, called a *Kempe switch*, to prove that a large class of so-called *Meyniel graphs* are perfect. We then see several consequences.

4.1 Kempe switches

Let $G = (V, E)$ be a simple graph, and let $f : V \to [k]$ be a proper vertex-colouring for some integer $k \geq 2$. Given distinct colours $r, b \in [k]$, an *rb-coloured component* is any connected component of the induced subgraph $G[\{v \in V : f(v) = r \text{ or } b\}]$; the rb-coloured component is *trivial* if it consists of exactly one vertex, otherwise it is *nontrivial*. Since f is a proper vertex-colouring, every rb-coloured component is bipartite, and an rb-coloured component is trivial if and only if it involves only one colour.

Consider one of the following operations applied to f:

- Take a nontrivial rb-coloured component. Swap the two colours on its vertex set. Keep the colour of all the other vertices unchanged.
- Take all the trivial rb-coloured components whose vertices are coloured the same. Swap the colour of every vertex to the other colour. Keep the colour of all the other vertices unchanged.

What is obtained is another proper vertex-colouring $g : V \to [k]$, one whose image could potentially omit colour r or b. We say that g is obtained from f by a *Kempe switch*. Kempe switches are historically important due to their connection to the *Four-Colour Theorem*. In Exercise 3, we see how to use this tool to prove the *Five-Colour Theorem*. In this section we use it to prove that a certain class of graphs is perfect.

4.2 Meyniel graphs are perfect.

A simple graph is *Meyniel* if every odd circuit of length at least five has at least two chords (a *chord* is an edge connecting two non-consecutive vertices of the circuit). We shall use Kempe switching to prove that every Meyniel graph is perfect. Let us start with the following important lemma:

Lemma 4.1. Let $G = (V, E)$ be a Meyniel graph. Pick a vertex $v \in V$, let $N \subseteq V$ be its neighbourhood, and let $G_N := G[N]$. Let $f : V - \{v\} \to [k]$ be a proper vertex-colouring of $G \setminus v$ for some integer $k \geq 2$, and let f_N be the restriction of f to N. Then the following statements hold:

(i) Let H be an rb-coloured component of $G \setminus v$. Then $H \cap G_N$ is either: a nontrivial rb-coloured component of G_N, or a union of trivial rb-coloured components of G_N whose vertices are coloured the same.

(ii) If g_N is obtained from f_N by a series of Kempe switches in G_N, then g_N can be extended to a proper vertex-colouring g that is obtained from f by a series of Kempe switches in $G \setminus v$.

Proof. **(i)** If $H \cap G_N$ does not contain vertices of different colour, then it must be a union of trivial rb-coloured components of G_N. Otherwise, $H \cap G_N$ contains vertices of different colour. Let u, w be vertices of $H \cap G_N$ coloured r, b, respectively. It suffices to show that $H \cap G_N$ contains a uw-path.

By definition, there is a uw-path contained in H; among all such paths pick one of minimum length and call it P. Consider the circuit $C := v, P, v$. As the ends of P are coloured differently, P has odd length, so C has odd length.

We claim that P is contained in G_N. To this end, partition P into pairwise edge-disjoint subpaths, each of which has its ends in G_N and its interior vertices outside G_N. Suppose for a contradiction that P is not contained in G_N. Then some subpath must have at least one interior vertex. However, for each such subpath P', the circuit v, P', v is chordless of length at least four, so the circuit, and therefore the path P', must be of even length since G is Meyniel. Moreover, if P'' appears next to P', then the circuit v, P', P'', v has only one chord, so the circuit, and therefore the path P', P'', must be of even length since G is Meyniel. Subsequently, the path P'' must also be of even length. This argument propagates and tells us that every subpath in the partition of P has even length, implying in turn that P has even length, a contradiction.

(ii) It suffices to prove this for a g_N obtained by a single Kempe switch. It follows easily from (i) that g_N can be extended to a proper vertex-colouring g of $G \setminus v$ that is obtained from f by either: a single Kempe switch if the switch in G_N involves a nontrivial rb-coloured component, or a series of Kempe switches otherwise. $\square$

Theorem 4.2. Let $G = (V, E)$ be a Meyniel graph, and let $f : V \to [k]$ be a proper vertex-colouring for some integer $k \geq 1$. Then f can be turned into an optimal proper vertex-colouring by a sequence of Kempe switches.

Proof. Let us proceed by induction on $|V| \geq 1$. The base case is trivial. For the induction step, assume that $|V| \geq 2$.

Clearly, $k \geq \chi(G) =: \chi$. If $\{u \in V : f(u) \in [\chi]\} = V$, then f is already optimal, so we are done. Otherwise, we shall find a proper vertex-colouring $h : V \to [k]$ obtained from f by a series of Kempe switches such that $\{u \in V : f(u) \in [\chi]\} \subsetneq \{u \in V : h(u) \in [\chi]\}$; repeating this procedure shall eventually result in an optimal proper vertex-colouring, thereby completing the induction step. To this end, take a vertex $v \in V$ such that $f(v) \notin [\chi]$, and let $r := f(v)$.

Claim. There exists a proper vertex-colouring g that is obtained from f after a series of Kempe switches such that

(1) $\{u \in V : f(u) \in [\chi]\} = \{u \in V : g(u) \in [\chi]\}$, and

(2) there exists a colour $b \in [\chi]$ such that v has no neighbour u such that $g(u) = b$.

Proof of Claim. Define $V' := \{u \in V : f(u) \in [\chi]\}$, $G' := G[V' \cup \{v\}]$, and $G'_N := G'[N]$ where N is the neighbourhood of v in G'. Then the vertices of G'_N are coloured with at most χ colours. However, any optimal proper vertex-colouring of G'_N uses at most $\chi - 1$ colours, as $\chi(G'_N) \leq \chi(G') - 1 \leq \chi - 1$. This is the key idea for finding the desired g.

Let $f_N, f_{V'}$ be the restrictions of f to N, V', respectively. By the induction hypothesis, f_N can be turned into an optimal proper vertex-colouring g_N by a series of Kempe switches in G'_N. By Lemma 4.1 (ii), g_N can be extended to a proper vertex-colouring $g_{V'}$ that is obtained from $f_{V'}$ by a series of Kempe switches in $G' \setminus v$. Let g be the extension of $g_{V'}$ to V defined as $g(u) := f(u)$ for all $u \in V - V'$. We claim that g is the desired vertex-colouring.

It can be readily checked that g is a proper vertex-colouring obtained from f by a series of Kempe switches (in fact, these are the same Kempe switches used to obtain $g_{V'}$ from $f_{V'}$). By construction, $\{u \in V : g(u) \in [\chi]\} = V'$, so g satisfies (1). Moreover, since $\chi(G'_N) \leq \chi - 1$ and g_N is an optimal vertex-colouring of G'_N, there exists a colour $b \in [\chi]$ such that $N \cap \{u \in V : g(u) = b\} = \emptyset$. Subsequently, g satisfies (2) for the colour b. $\Diamond$

Consider now the proper vertex-colouring g. By (2), there exists a colour $b \in [\chi]$ such that v has no b-coloured neighbour. Then v forms a trivial rb-coloured component of G. Let h be obtained from g after applying a single Kempe switch to all the trivial rb-coloured components whose vertices are r-coloured. Then

$$\{u \in V : h(u) \in [\chi]\} \supseteq \{u \in V : g(u) \in [\chi]\} \cup \{v\} = \{u \in V : f(u) \in [\chi]\} \cup \{v\}$$

where the last equality follows from (1). Consequently, h is the promised vertex-colouring. This finishes the proof. $\square$

Corollary 4.3. Let $G = (V, E)$ be a Meyniel graph, let v be a vertex, and let N be its neighbourhood. Then $G \setminus v$ has an optimal proper vertex-colouring that induces an optimal proper vertex-colouring of $G[N]$.

Proof. Let f be an optimal proper vertex-colouring of $G \setminus v$, and let f_N be its restriction to N. By Theorem 4.2, f_N can be turned into an optimal proper vertex-colouring g_N by a series of Kempe switches in $G[N]$. By Lemma 4.1 (ii), g_N can be extended to a proper vertex-colouring g that is obtained from f by a series of Kempe switches in $G \setminus v$. Since f is optimal, and since Kempe switches do not increase the number of colours used, it follows that g is optimal also. Thus, g is the desired proper vertex-colouring of $G \setminus v$. □

Theorem 4.4. Every Meyniel graph is perfect.

Proof. Let $G = (V, E)$ be a Meyniel graph. It suffices to show that $\chi(G) \leq \omega(G) =: \omega$. We proceed by induction on $|V| \geq 1$. The base case holds trivially. For the induction step, assume that $|V| \geq 2$. Pick an arbitrary vertex v and let N be its neighbourhood. By Corollary 4.3, $G \setminus v$ has an optimal proper vertex-colouring f that induces a proper vertex-colouring of $G[N]$. By the induction hypothesis, $\chi(G[N]) \leq \omega(G[N])$ and $\chi(G \setminus v) \leq \omega(G \setminus v)$. Clearly, $\omega(G[N]) \leq \omega - 1$ and $\omega(G \setminus v) \leq \omega$. Thus, f uses at most ω colours, while its restriction to N uses at most $\omega - 1$ colours, so f can easily be extended to a proper vertex-colouring of G, one which uses at most ω colours. Thus, $\chi(G) \leq \omega$, thereby completing the induction step. □

Consequently,

Theorem 4.5. The complement of every Meyniel graph is perfect.

Proof. Theorem 4.4, combined with the Perfect Graph Theorem, implies that the complement of every Meyniel graph is perfect. □

Corollary 4.6. Let G be a graph where every odd circuit has at least two non-crossing chords. Then G is perfect.

Corollary 4.7. Let G be a graph where every odd circuit has at least two crossing chords. Then G is perfect.

5 Cographs and a structure theorem

Let us start with the following rather immediate result:

Theorem 5.1. Every n-vertex perfect graph has a clique or stable set of cardinality at least $n^{\frac{1}{2}}$.

Proof. This follows, for instance, from Theorem 3.1. □

An important problem in *Ramsey Theory* is the following conjecture of Erdős and Hajnal [4]:

(?) *For every simple graph H there exists an $\epsilon > 0$ such that, for all sufficiently large n, every n-vertex simple graph without an induced H has a clique or stable set of cardinality n^{ϵ}.* (?)

The best result towards this conjecture was shown in the same paper where the conjecture was posed: *For every simple graph H, and for all sufficiently large n, every n-vertex simple graph without an induced H has a clique or stable set of cardinality $2^{\Omega(\sqrt{\log n})}$*. This result was proved by showing that every n-vertex simple graph without an induced H has a large perfect graph as an induced subgraph, and then using Theorem 5.1! In this section, we study this class of perfect graphs, and prove a structure theorem that makes them useful in the context of the conjecture above.

Denote by P_4 the path on four vertices. Observe that $P_4 \cong \overline{P_4}$. A *cograph* is a simple graph without an induced P_4. Observe that the class of cographs is closed under complementation and taking induced subgraphs.

Theorem 5.2. Every cograph is perfect.

Proof. Every odd circuit of length at least five with at most one chord has an induced P_4, therefore does not appear as an induced subgraph of any cograph. In particular, given a cograph G, every odd circuit of length at least five has at least two chords, implying in particular that G is Meyniel. Thus, G is perfect by Theorem 4.4, as required. □

Let $G = (V, E)$ be a simple graph. Given disjoint sets $A, B \subseteq V$, we say that A *covers* B if every vertex in B has a neighbour in A.

Lemma 5.3. Let $G = (V, E)$ be a cograph with at least two vertices. Then either G or $\overline{G}$ is not connected.

Proof. Suppose otherwise. As G has no chordless path of length at least three, every pair of vertices are at distance at most two. Similarly, every pair of vertices in $\overline{G}$ are at distance at most two. Fix a vertex $v \in V$. Let $A := N_G(v)$, and let $B := N_{\overline{G}}(v)$. Observe that $A, B, \{v\}$ partition V.

Claim 1. A covers B in G, and B covers A in $\overline{G}$.

Proof of Claim. For each $u \in B$, $\mathrm{dist}_G(v, u) > 1$, so $\mathrm{dist}_G(v, u) = 2$, implying that u has a neighbour in A. This implies that A covers B in G. By symmetry, B covers A in $\overline{G}$. ♢

In particular, A, B each have at least two vertices. An *alternating circuit* is a sequence $(a_1, b_1, a_2, b_2, \ldots, a_k, b_k)$ where $k \geq 2$ is an integer, the vertices $a_1, \ldots, a_k$ belong to A and are distinct, the vertices $b_1, \ldots, b_k$ belong to B and are distinct, $\{a_i, b_i\}, i \in [k]$ are edges of G, and $\{a_{i+1}, b_i\}, i \in [k]$ are edges of $\overline{G}$. (Here, $k + 1 = 1$.)

Claim 2. There exists an alternating circuit (a_1, b_1, a_2, b_2) with four vertices.

Proof of Claim. It follows from Claim 1 that an alternating circuit $(a_1, b_1, a_2, b_2, \ldots, a_k, b_k)$ does exist. If $\{a_1, b_2\}$ is an edge of $\overline{G}$, then (a_1, b_1, a_2, b_2) is the desired alternating circuit. Otherwise, $\{a_1, b_2\}$ is an edge of G, implying that $(a_1, b_2, a_3, b_3, \ldots, a_k, b_k)$ is another alternating circuit. Repeating the argument on this alternating circuit, which has fewer vertices than before, will eventually yield an alternating circuit with exactly four vertices. ♢

If $\{a_1, a_2\}$ is an edge of $\overline{G}$, then (b_1, a_1, v, a_2) is an induced P_4 in G, which is not possible. Thus, $\{a_1, a_2\}$ is an edge of G. If $\{b_1, b_2\}$ is an edge of G, then (v, a_2, b_2, b_1) is an induced P_4 in G, which again is not possible. Thus, $\{b_1, b_2\}$ is an edge of $\overline{G}$. But then (b_1, a_1, a_2, b_2) is an induced P_4 in G, a contradiction. □

As a consequence,

Theorem 5.4. A simple graph is a cograph if, and only if, it can be constructed from the single vertex graph K_1 by complementation and disjoint union.

Proof. $(\Rightarrow)$ follows immediately from Lemma 5.3. $(\Leftarrow)$ If G is constructed from the single vertex graph K_1 by complementation and disjoint union, then so is every induced subgraph of G. Thus, if H is an induced subgraph with at least two vertices, then either H or $\overline{H}$ is not connected, implying in turn that $H \ncong P_4$. Hence, G is a cograph. □

6 Chordal graphs and fast algorithms

A simple graph is *chordal* if every circuit of length at least four has a chord. Observe that if a graph is chordal, then so is every induced subgraph of it.

Theorem 6.1. Every chordal graph, as well as its complement, is perfect.

Proof. Let G be a chordal graph. It can be readily verified that G is Meyniel, that is, every odd circuit of length at least five has at least two chords. It therefore follows from Theorem 4.4 that G is perfect, and from Theorem 4.5 that $\overline{G}$ is perfect. □

It follows from Theorem 6.1 that in a chordal graph, the chromatic number equals the clique number, while the clique covering number equals the stability number. In this section, we provide five fast algorithms for chordal graphs: recognition of such graphs, maximum clique, maximum stable set, minimum proper vertex-colouring, and minimum clique covering.

Let $G = (V, E)$ be a connected simple graph. A *cutset* is a subset $X \subseteq V$ where $G \setminus X$ is not connected. A *clique cutset* is a cutset X such that $G[X]$ is a clique.

Lemma 6.2. Let $G = (V, E)$ be a connected chordal graph on at least three vertices. Then every minimal cutset, if any, is a clique cutset. In particular, if G is not a clique, then it has a clique cutset.

Proof. Let X be a minimal cutset. Suppose for a contradiction that X is not a clique cutset. Then there exists distinct vertices $x, y \in X$ that are non-adjacent. Take disjoint $V_1, V_2 \subseteq V - X$ such that $G[V_1], G[V_2]$ are connected components of $G \setminus X$. Observe that $G[V_1 \cup X], G[V_2 \cup X]$ are connected.

Claim. For each $i \in \{1, 2\}$, there exists an xy-path in $G[V_i \cup X]$ that is internally vertex-disjoint from X.

Proof of Claim. Pick $v_1 \in V_1$ and $v_2 \in V_2$. By minimality of X, $X - \{x\}$ is not a cutset, so there exists a v_1v_2-path P in $G[V_1 \cup V_2 \cup X]$ such that $V(P) \cap X = \{x\}$. Similarly, $X - \{y\}$ is not a cutset, so there exists a v_1v_2-path P' in $G[V_1 \cup V_2 \cup X]$ such that $V(P') \cap X = \{y\}$. Observe that $P[v_i, x] \cup P'[v_i, y]$ is an xy-trail in $G[V_i \cup X]$ that is internally vertex-disjoint from X, so it contains an xy-path in $G[V_i \cup X]$ that is internally vertex-disjoint from X, as required. ◊

For each $i \in \{1, 2\}$, amongst all xy-paths in $G[V_i \cup X]$ internally vertex-disjoint from X, pick one P_i of minimum length. Then P_1, P_2 must be chordless paths. As there is no edge between V_1, V_2, there is no edge between an internal node of P_1 and an internal node of P_2. Thus, as x, y are non-adjacent, $P_1 \cup P_2$ is a chordless circuit of length at least four, a contradiction as G is chordal. $\square$

A vertex $v \in V$ is *simplicial* if its neighbourhood induces a clique.

Lemma 6.3. Let $G = (V, E)$ be a connected chordal graph that is not a clique and has at least three vertices. Then G has two non-adjacent simplicial vertices. In particular, every chordal graph has a simplicial vertex.

Proof. We shall proceed by induction on $|V| \geq 3$. For the base case $|V| = 3$, notice that G is a path of length two, so the ends of the path are the desired simplicial vertices. For the induction step, assume that $|V| \geq 4$. It follows from Lemma 6.2 that G has a clique cutset X. Choose disjoint $V_1, V_2 \subseteq V - X$ such that $G[V_1], G[V_2]$ are connected components of $G \backslash X$. Observe that $G[V_1 \cup X], G[V_2 \cup X]$ are connected chordal graphs, each with fewer than $|V|$ vertices. It therefore follows from the induction hypothesis that for each $i \in \{1, 2\}$, $G[V_i \cup X]$ is either a clique or has two non-adjacent simplicial vertices; in particular, V_i contains a vertex v_i that is simplicial in $G[V_i \cup X]$, because $G[X]$ is a clique so it does not contain non-adjacent vertices. Since every neighbour of v_i is contained in $V_i \cup X$, it follows that v_i is simplicial in G, for each $i \in \{1, 2\}$. Since v_1, v_2 are non-adjacent, we have successfully completed the induction step. $\square$

Let G be a simple graph with n vertices. A *perfect vertex elimination scheme* is an ordering $\sigma(1), \sigma(2), \ldots, \sigma(n)$ of the vertices such that for each $i \in \{1, \ldots, n-1\}$, the neighbours of $\sigma(i)$ in $\{\sigma(i+1), \ldots, \sigma(n)\}$ form a clique in G. That is, $\sigma(1), \sigma(2), \ldots, \sigma(n)$ is a perfect vertex elimination scheme if for each $i \in \{1, \ldots, n-1\}$, $\sigma(i)$ is a simplicial vertex of $G[\{, \sigma(i), \ldots, \sigma(n)\}]$.

Theorem 6.4. A simple graph is chordal if, and only if, it has a perfect vertex elimination scheme.

Proof. $(\Rightarrow)$ follows immediately by recursively finding a simplicial vertex, whose existence is guaranteed by Lemma 6.3, and then deleting the vertex. $(\Leftarrow)$ Let $\sigma(1), \sigma(2), \ldots, \sigma(n)$ be a perfect vertex elimination scheme of a simple graph G. Pick a circuit of length at least four. Pick the smallest index $i \in [n]$ such that $\sigma(i)$ is a vertex of the circuit. Then the two neighbours of $\sigma(i)$ in the circuit belong to $\{\sigma(i+1), \ldots, \sigma(n)\}$ and are therefore adjacent, leading to a chord in the circuit, as required. $\square$

Recognising chordal graphs: Let $G = (V, E)$ be a simple graph. By Theorem 6.4, G is chordal if, and only if, it has a perfect vertex elimination scheme. Since a simplicial vertex, if any, can be found in time $O(|V|^2)$, a perfect vertex elimination scheme, if any, can be constructed in time $O(|V|^3)$, implying in turn that testing whether or not G is chordal can be accomplished in time $O(|V|^3)$. In fact, by using an appropriate data structure representing G, one can find a perfect vertex elimination scheme in time $O(|V| + |E|)$ ([11]).

Minimum proper vertex-colourings and maximum cliques: Given a chordal graph G and a perfect vertex elimination scheme $\sigma(1), \ldots, \sigma(n)$, we have the following algorithm outputting a minimum proper vertex-colouring $f : \{\sigma(1), \ldots, \sigma(n)\} \to \Phi$ and a maximum clique K:

$\Phi \leftarrow \{1\}$ and $f(\sigma(n)) := 1$
$K \leftarrow \{\sigma(n)\}$
$i \leftarrow n - 1$
if $i \geq 1$ **then**
  Let K' be the set of neighbours of $\sigma(i)$ in $\{\sigma(i+1), \ldots, \sigma(n)\}$.
  Let $\Phi' \subseteq \Phi$ be the set of colours used to colour the vertices in K'.
  if $\Phi' \subsetneq \Phi$ **then**
    Pick a colour $j \in \Phi - \Phi'$.
    $\Phi \leftarrow \Phi$ and $f(\sigma(i)) := j$
    $K \leftarrow K$
    $i \leftarrow i - 1$
  else
    $\Phi \leftarrow \Phi \cup \{|K'| + 1\}$ and $f(\sigma(i)) := |K'| + 1$.
    $K \leftarrow K' \cup \{\sigma(i)\}$
    $i \leftarrow i - 1$
  end if
else
  Return $f : \{\sigma(1), \ldots, \sigma(n)\} \to \Phi$ and K.
end if

Observe that at every iteration of the algorithm, f remains a proper partial vertex-colouring with colour set Φ, and K remains a clique of cardinality $|\Phi|$. Notice further that at the end, f becomes a proper vertex-colouring, implying in turn that f is a minimum proper vertex-colouring while K is a maximum clique. The running time of this algorithm is clearly $O(|V| + |E|)$.

Minimum clique coverings and maximum stable sets: Given a chordal graph G and a perfect vertex elimination scheme $\sigma(1), \ldots, \sigma(n)$, we have the following algorithm outputting a minimum clique covering $Y_1, \ldots, Y_\alpha$ and a maximum stable set $\{y_1, \ldots, y_\alpha\}$:

Let $y_1 := \sigma(1)$.
Let Y_1 be the clique consisting of y_1 and its neighbours.
$\alpha \leftarrow 1$
if $Y_1 \cup \cdots \cup Y_\alpha \neq \{\sigma(1), \ldots, \sigma(n)\}$ **then**
  Let i be the smallest index such that $\sigma(i) \notin Y_1 \cup \cdots \cup Y_\alpha$.

Let $y_{\alpha+1} := \sigma(i)$.

`Note:` $y_{\alpha+1}$ `is not adjacent to` $y_1, \ldots, y_\alpha$.

Let $Y_{\alpha+1}$ be the clique consisting of $\sigma(i)$ and its neighbours in $\{\sigma(i+1), \ldots, \sigma(n)\} - (Y_1 \cup \cdots \cup Y_\alpha)$.

$\alpha \leftarrow \alpha + 1$

else

Return the clique covering $Y_1, \ldots, Y_\alpha$ and the stable set $\{y_1, \ldots, y_\alpha\}$.

end if

Since the clique covering and the stable set have the same cardinality α, it follows that $Y_1, \ldots, Y_\alpha$ is a minimum clique covering while $\{y_1, \ldots, y_\alpha\}$ is a maximum stable set. The running time of this algorithm is clearly $O(|V| + |E|)$.

Exercises

1. Let $(V, \leq)$ be a partially ordered set. Prove that the minimum number of antichains needed to cover V is equal to the maximum cardinality of a chain. Then, prove that every comparability graph is perfect.

2. (a) Let $G = (V, E)$ be a bipartite graph. A *proper edge-colouring* is an assignment of colours to the edges such that every pair of edges sharing a vertex receive different colours. Prove that the minimum number of colours needed in a proper edge-colouring is equal to the maximum degree of G.

   (b) Prove that the line graph of a bipartite graph is perfect.

3. Let $G = (V, E)$ be a planar graph. Prove the following statements:

   (a) Use *Euler's formula ($v - e + f = 2$)* to prove that G has a vertex of degree at most five.

   (b) Use Kempe switches to prove that $\chi(G) \leq 5$.

4. Prove that every cograph is a comparability graph.

5. A *stable transversal* is a stable set that intersects every maximal clique once. A graph is *strongly perfect* if every induced subgraph has a stable transversal. Prove that every strongly perfect graph is perfect.

6. Prove that every cograph is strongly perfect.

7. An *odd hole* is an induced odd circuit of length at least five. Let G_0 be the graph with vertices a, b, c, d and edges $\{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}$. Prove that a graph without a G_0 induced subgraph is perfect if, and only if, the graph has no odd hole.

8. Given an ordering $v_1 < v_2 < \cdots < v_n$ of the vertices of a simple graph, a natural way of colouring them by positive integers is to scan the sequence from v_1 to v_n, and assign to each v_j the smallest positive integer

$f(v_j)$ assigned to none of its neighbours in $\{v_i : i < j\}$. Observe that this was the algorithm used to colour the vertices of a chordal graph.

We refer to the graph with the linear order on the set of its vertices as an *ordered graph* and to the largest integer appearing as some $f(v_j)$ as the *Grundy number* of this ordered graph. Clearly, the Grundy number is at least as large as the chromatic number.

Given a simple graph, a *perfect ordering* is an ordering of the vertices such that for each induced ordered subgraph, the Grundy number is equal to the chromatic number. A graph is *perfectly orderable* if it admits a perfect ordering.

(a) Show that chordal graphs, as well as their complements, are perfectly orderable.

(b) Prove that an ordering is perfect if, and only if, there exists no four vertices a, b, c, d such that $a < b, d < c$, and $G[\{a, b, c, d\}]$ has edges $\{a, b\}, \{b, c\}, \{c, d\}$.

(c) Show that comparability graphs are perfectly orderable.

9. Prove that every perfectly orderable graph is strongly perfect.

10. A simple graph $G = (V, E)$ is an *interval graph* if there exists a set of intervals $\{I_v : v \in V\}$ on the real line such that $uv \in E$ iff $I_u \cap I_v \neq \emptyset$.

    (a) Prove that every interval graph is chordal.

    (b) Prove that the complement of every interval graph is a comparability graph.

11. Let $G = (V, E)$ be a simple graph. Prove that the following statements are equivalent:

    (i) $\chi(G) \leq k$,

    (ii) G has an orientation without directed paths of length $k - 1$,

    (iii) G has an acyclic orientation without directed paths of length $k - 1$.

12. Let $G = (V, E)$ be a simple graph. Create a new copy V' of V, and a new vertex w. For every edge $uv \in E$, introduce two new edges $uv', u'v$. Also, for every vertex $v' \in V'$, introduce the edge $v'w$. Denote by G' the new graph (which includes G as an induced subgraph). Prove the following statements:

    (a) $\chi(G') = \chi(G) + 1$

    (b) every triangle of G', if any, is contained in G.

13. For any integer k, give a construction of a graph where $\chi(G) \geq k$ and $\omega(G) = 2$.

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The fact that Meyniel graphs are perfect is due to Meyniel himself [10]. Corollary 4.6 is due to [6], while Corollary 4.7 is from [12]. Theorem 5.2 and Theorem 5.4 are from [13]. Theorem 6.1 is due to [1, 8]. Lemma 6.3 is due to [3], while Theorem 6.4 is from [5].

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