Lecture 3: Multicommodity flows and weakly bipartite signed graphs

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1 Menger's theorem

Packing is a basic problem in Combinatorial Optimisation. In its most general form, we are given a family $\mathcal{C}$ of subsets of a finite ground set V, and we are asked to find the maximum number of pairwise disjoint members of $\mathcal{C}$, that is, we are asked to find a maximum *packing*. In an important variant of the problem, we are allowed to pick members of $\mathcal{C}$ *fractionally*, where the disjointness property is replaced by the requirement that in our fractional packing, the *congestion* of every element in the ground set is at most 1.

The earliest example of packing comes from the classical theorem of Menger. Let $G = (V, E)$ be a graph, and take distinct vertices $s, t \in V$.[1] An *st-path* is an inclusionwise minimal edge subset connecting s and t. An *st-cut* is an edge subset of the form $\delta(U) := \{e \in E : |e \cap U| = 1\}$ where $U \subseteq V$ satisfies $U \cap \{s, t\} = \{s\}$. We will refer to U and $V - U$ as the *shores* of G. Notice that every st-path intersects every st-cut.

What is the maximum number of (pairwise) disjoint st-paths? In other words, how many st-paths can we *pack*? This is answered (in a min-max sense) by the following beautiful theorem.

Theorem 1.1 (Menger's Theorem)**.** Let $G = (V, E)$ be a graph, and take distinct vertices $s, t \in V$. Then the maximum number of disjoint st-paths is equal to the minimum cardinality of an st-cut.

Proof. Every st-path intersects an st-cut, so the maximum number of disjoint st-paths is at most the minimum cardinality of an st-cut. We prove the other inequality by induction on $|V| + |E| \geq 3$. The result is obvious for $|V| + |E| = 3$. For the induction step, assume that $|V| + |E| \geq 4$. Let τ be the minimum cardinality of an st-cut. We may assume that $\tau \geq 1$. We will find τ disjoint st-paths.

Claim 1. If an edge e does not appear in a minimum st-cut, then G has τ disjoint st-paths.

Proof of Claim. Notice that the cardinality of a minimum st-cut in $G \setminus e$ is still τ. As a result, the induction hypothesis implies the existence of τ disjoint st-paths in $G \setminus e$, and therefore in G. $\Diamond$

[1] We allow parallel edges but disallow loops, until further notice.

We may therefore assume that every edge appears in a minimum st-cut. An st-cut $\delta(U)$ is *trivial* if either $|U| = 1$ or $|V - U| = 1$.

Claim 2. If there is a minimum st-cut that is not trivial, then G has τ disjoint st-paths.

Proof of Claim. Let $\delta(U), s \in U \subseteq V - \{t\}$ be a minimum st-cut that is non-trivial. Let G_1 be the graph obtained from G by shrinking U to a single vertex s', and let G_2 be the graph obtained from G after shrinking $V - U$ to a single vertex t'. Since $\delta(U)$ is non-trivial, it follows that $|V(G_i)| + |E(G_i)| < |V| + |E|$, for each $i \in [2]$. We may therefore apply the induction hypothesis to G_1 and G_2. Notice that τ is still the minimum cardinality of an $s't$-cut in G_1 and of an st'-cut in G_2. Thus, by the induction hypothesis, G_1 has τ disjoint $s't$-paths and G_2 has disjoint st'-paths. Gluing these paths along the edges of $\delta(U)$ gives us τ disjoint st-paths in G. $\diamondsuit$

We may therefore assume that every minimum st-cut is trivial. Since every edge appears in a minimum st-cut, it follows that every edge has either s or t as an end. In this case, G has a special form and it is clear that $\tau = \nu$ for this graph, thereby completing the induction step. $\square$

2 Mutlicommodity flows

Menger's Theorem is a classical example of an *integer single-commodity flow problem*, where we have one source s, and one sink t, and we would like to send as much integer flow as possible from the source to the sink. One can extend this problem to a *multicommodity flow problem* where we have multiple pairs of terminals with possibly different demands of different commodities, all of which need to be satisfied simultaneously over the same graph, via not just integer flows but fractional flows.

To elaborate, let $G = (V, E)$ be a graph, $F \subseteq E$, and $p \in \mathbb{Z}_+^E$. We shall think of each edge $f \in F$ as a *dummy* edge with a *demand* of p_f units of commodity between its ends. We would like to send the commodities from one end of each dummy edge to the other end through the edges in $E - F$. As such, we shall think of each edge $e \in E - F$ as a *supply* edge with a *capacity* of p_e units. Note that each supply edge can be used to carry multiple commodities corresponding to different dummy edges.

The commodity corresponding to dummy edge $f \in F$ is to be sent along paths in $E - F$ connecting the ends of f. Let $\mathcal{C}$ be the set of circuits $C \subseteq E$ of G such that $|C \cap F| = 1$. Observe that the commodities must be sent along the paths in $\{C - F : C \in \mathcal{C}\}$.

The *multicommodity flow problem* can now be formulated as the problem of assigning a fraction $y_C \geq 0$ to each circuit $C \in \mathcal{C}$, indicating the amount of flow to be sent along $C - F$ from one end of the dummy edge in $C \cap F$ to the other end, satisfying the following conditions:

(*Capacity Constraints*) the total amount of flow (i.e. the traffic/congestion) through each supply edge must not exceed its capacity:

$$\sum (y_C : C \in \mathcal{C}, e \in C) \leq p_e \quad \forall e \in E - F$$

(*Demand Constraints*) the total amount of flow sent from one end of dummy edge to the other end must meet its demand:

$$\sum (y_C : C \in \mathcal{C}, f \in C) \geq p_f \quad \forall f \in F$$

Such an assignment $(y_C : C \in \mathcal{C})$ is called a *multicommodity flow*. If, in addition, each fraction y_C assigned is an integer, then $(y_C : C \in \mathcal{C})$ is an *integer multicommodity flow*.

Note that the multicommodity flow problem and its integer variant are *feasibility* problems, asking whether certain assignments exist. From the phrasing of the problem, it is implicit that such assignments may not necessarily exist. Let us make this more explicit. For a multicommodity flow to exist, a certain *cut condition* has to be satisfied.

Take a cut $\delta(U)$ of the graph, and let f be a dummy edge crossing the cut, if any. The edge has a demand of p_f units of commodity between its ends. The single commodity is carried along paths in $E - F$ connecting the ends of f, each of which must cross the cut $\delta(U)$ at least once. Thus, for the flow to exist, the cut $\delta(U)$ should have a total capacity of at least p_f so as to accommodate for the demand p_f: $\sum (p_e : e \text{ is a supply edge in } \delta(U)) \geq p_f$. Now, the more demand edges we have crossing $\delta(U)$, the larger the capacity of the cut must be. Thinking along these lines, we get the following proposition:

Proposition 2.1 (Cut Condition)**.** If a multicommodity flow y exists, then for every cut $\delta(U)$ of G, we have

$$\sum (p_e : e \text{ is a supply edge in } \delta(U)) \geq \sum (p_f : f \text{ is a dummy edge in } \delta(U)) .$$

Proof. We have

$$\begin{aligned}
\sum_{f \in \delta(U) \cap F} p_f &\leq \sum_{f \in \delta(U) \cap F} \sum (y_C : f \in C \in \mathcal{C}) \quad \text{demand constraints} \\
&\leq \sum_{e \in \delta(U) \cap (E-F)} \sum (y_C : e \in C \in \mathcal{C}) \\
&\leq \sum_{e \in \delta(U) \cap (E-F)} p_e \quad \text{capacity constraints}
\end{aligned}$$

where the second inequality follows from the fact that if $f \in C \in \mathcal{C}$ for some $f \in \delta(U) \cap F$, then $(C - \{f\}) \cap \delta(U)$ is a nonempty subset of $\delta(U) \cap (E - F)$, thereby proving the cut condition. ☐

Thus, the cut condition is a necessary condition for the existence of a multicommodity flow. A natural question is whether the cut condition is sufficient? That is, does the cut condition necessarily imply the existence of a multicommodity flow? While this is true for $|F| = 1$ (the single-commodity case – see exercise 2), and $|F| = 2$ (the two-commodity case [6]), the answer is unfortunately negative in general. In exercise 5, we see an instance with $|F| = 3$ satisfying the cut condition, but no multicommodity flow exists. The graph illustrated there, however, is not planar. We prove that this must be the case in any instance where the cut condition is not sufficient!

3 Signed graphs

Let us take a detour and provide a primer on signed graphs. This study will pay off in the next section when we get back to multicommodity flows.

Let $G = (V, E)$ be a graph. A *cycle* is an edge subset $C \subseteq E$ where every vertex of G is incident with an even number of edges from C. Note that $\emptyset$ is a cycle. A *circuit* is a nonempty cycle that is inclusionwise minimal with respect to this property. Observe that an edge subset is a circuit if, and only if, it induces a connected subgraph where every vertex has degree two. Observe further that a nonempty cycle is a union of (edge-)disjoint circuits.

Remark 3.1. Every cycle and cut have an even intersection.

Proof. Let $\delta(U)$ be a cut for some $U \subseteq V$. Then for any $C \subseteq E$, we have

$$\sum_{v \in U} |\delta(v) \cap C| = 2|\{e \in C : e \subseteq U, \text{i.e. both ends of } e \text{ are in } U\}| + |C \cap \delta(U)|,$$

so

$$\sum_{v \in U} |\delta(v) \cap C| \equiv |C \cap \delta(U)| \pmod 2.$$

If C is a cycle, then the left-hand side is even, implying in turn that $|C \cap \delta(U)|$ is even. $\square$

Let $\Sigma \subseteq E$. The pair (G, Σ) is called a *signed graph*. A circuit $C \subseteq E$ is *odd* if $|C \cap \Sigma|$ is odd. One can similarly define *even circuits*, *odd cycles*, and *even cycles*. We will use the following observation throughout this lecture without reference.

Remark 3.2. Let $(G = (V, E), \Sigma)$ be a signed graph, and take a subset C. The following statements are equivalent:

- C is an even (resp. odd) cycle,
- C is a union of disjoint circuits, an even (resp. odd) number of which are odd circuits.

Proof. Exercise. $\square$

Given two sets A, B, denote by $A \triangle B$ their symmetric difference $(A \cup B) - (A \cap B)$. It can be readily checked that $C \cap (A \triangle B) = (C \cap A) \triangle (C \cap B)$ for any set C. Also, $|A \triangle B| \equiv |A| + |B| \pmod 2$. We will use these facts below.

To *re-sign*, or simply *resign* (G, Σ) is to replace it by the signed graph $(G, \Sigma \triangle \delta(U))$ for some $U \subseteq V$.

Remark 3.3. Resigning preserves the parity of a cycle.

Proof. Let (G, Σ) be a signed graph, and $(G, \Sigma \triangle \delta(U))$ a resigning of it. Let C be a cycle. Then

$$|C \cap (\Sigma \triangle \delta(U))| = |(C \cap \Sigma) \triangle (C \cap \delta(U))| \equiv |C \cap \Sigma| + |C \cap \delta(U)| \equiv |C \cap \Sigma| \pmod 2$$

where the last equality follows from the fact that $|C \cap \delta(U)|$ is even. Thus, C has the same parity in both (G, Σ) and $(G, \Sigma \triangle \delta(U))$, thereby finishing the proof. $\square$

A *signature* of (G, Σ) is any set of the form $\Sigma \triangle \delta(U), U \subseteq V$.

Proposition 3.4. Let $(G = (V, E), \Sigma)$ be a signed graph. Then the following statements hold:

1. If there is no odd cycle, then $\emptyset$ is a signature of (G, Σ).
2. If $B \subseteq E$ intersects every odd circuit at least once, then B contains a signature of (G, Σ).

Proof. **(1)** Let A be the $0, 1$ $(|V| + 1)$-by-$|E|$ matrix whose columns are labeled by the edges, and whose first $|V|$ rows are the incidence vectors of $\delta(v), v \in V$ and whose last row is the incidence vector of Σ. Let b be the column vector whose first $|V|$ coordinates are 0 and whose last entry is 1. As (G, Σ) has no odd cycle, the system $Ax \equiv b \pmod 2$ has no solution over $GF(2)$. By exercise 8, there is a certificate $c \in \{0, 1\}^{|V|+1}$ such that

$$c^\top A \equiv \mathbf{0} \quad \text{and} \quad c^\top b \equiv 1 \pmod 2.$$

The second equation implies that the last entry of c is 1. Let $U := \{u \in V : c_u = 1\}$. Then the first equation implies that $\Sigma = \delta(U)$. Thus, $\emptyset = \Sigma \triangle \delta(U)$ is a signature, as required.

(2) Since B intersects every odd circuit of (G, Σ), it follows that the signed graph $(G \setminus B, \Sigma - B)$ has no odd circuit, implying in turn that it has no odd cycle. It therefore follows from (1) that $\Sigma - B = \delta_{G \setminus B}(U)$ for some $U \subseteq V$. Then $\Sigma \triangle \delta_G(U) \subseteq B$, so B contains a signature of (G, Σ), as required. $\square$

In fact, the converse of Proposition 3.4 holds as well – see exercise 7.

4 Weakly bipartite signed graphs

Consider a multicommodity flow problem with graph $G = (V, E)$, dummy edges $F \subseteq E$, and $p \in \mathbb{Z}_+^E$. The connection between this problem and signed graphs becomes a bit clearer when we analyse the cut condition.

Proposition 4.1. The cut condition is satisfied if, and only if, F is a minimum weight signature of the signed graph (G, F).

Proof. By definition, the cut condition is satisfied if, and only if, for every cut $\delta(U)$,

$$p(\delta(U) - F) = \sum_{e \in \delta(U) - F} p_e \geq \sum_{f \in \delta(U) \cap F} p_f = p(\delta(U) \cap F).$$

The latter is equivalent to asking, for every cut $\delta(U)$, that

$$p(F \triangle \delta(U)) \geq p(F),$$

which is equivalent to asking that F is a minimum weight signature of the signed graph (G, F). $\square$

Denote by $\mathcal{C}(G,F)$ the family of odd circuits of the signed graph (G,F). Consider the following linear program:[2]

$$\begin{array}{cclcll} & \min & p^\top x & & & \\ (P) & \text{s.t.} & x(C) & \geq & 1 & C \in \mathcal{C}(G,F) \\ & & x_e & \geq & 0 & e \in E \end{array}$$

Every signature and odd circuit intersect (see exercise 7), so the incidence vector of every signature of (G,F) is a feasible solution to (P). In fact, by Proposition 3.4 (2), the minimum weight of a signature is the value of an optimal *integer* solution to (P).

Now, consider the dual linear program

$$\begin{array}{cclcll} & \max & \sum(y_C : C \in \mathcal{C}(G,F)) & & & \\ (D) & \text{s.t.} & \sum(y_C : e \in C \in \mathcal{C}(G,F)) & \leq & p_e & e \in E \\ & & y_C & \geq & 0 & C \in \mathcal{C}(G,F) \end{array}$$

Proposition 4.2. The following statements hold:

1. (D) has optimal value at most $p(F)$,
2. if (D) has optimal value $p(F)$, then any optimal solution to (D) yields a multicommodity flow.

Proof. **(1)** As a signature of (G,F), F intersects every odd circuit at least once, so its incidence vector, $\chi_F \in \{0,1\}^E$, is a feasible solution to the primal LP (P), and has value $p(F)$. Thus, by Weak LP Duality, (D) must have optimal value at most $p(F)$. **(2)** Assume that (D) has optimal value $p(F)$, and let $y^\star$ be an optimal solution to (D). It then follows that χ_F is an optimal solution to the primal (P). As a result, $(\chi_F, y^\star)$ is an optimal pair for the primal-dual pair, so the *Complementary Slackness conditions* must hold:

i. If $y^\star_C > 0$, then $\chi_F(C) = 1$. This means that $y^\star$ assigns a nonzero fraction to only those odd circuits C for which $|C \cap F| = 1$.

ii. If $(\chi_F)_f > 0$, then $\sum(y^\star_C : f \in C \in \mathcal{C}(G,F)) = p_f$. This means that whenever $f \in F$, then

$$\sum(y^\star_C : f \in C \in \mathcal{C}(G,F)) = p_f.$$

Let $\mathcal{C}$ be the family of circuits C of G such that $|C \cap F| = 1$. Observe that each $C \in \mathcal{C}$ is an odd circuit. Now define $\bar{y} \in \mathbb{R}^{\mathcal{C}}_+$ as follows: for each $C \in \mathcal{C}$, let $\bar{y}_C = y^\star_C$. We claim that $\bar{y}$ is a multicommodity flow. By (i), $\bar{y}$ and $y^\star$ have the same nonzero entries. By (ii), the demand constraints are satisfied at equality:

$$\sum_{f \in C \in \mathcal{C}} \bar{y}_C = p_f \quad \forall f \in F$$

Moreover, the congestions are bounded above by p in $y^\star$, so we have that the capacity constraints are satisfied:

$$\sum_{e \in C \in \mathcal{C}} \bar{y}_C \leq p_e \quad \forall e \in E - F.$$

Thus, $\bar{y}$ is indeed a multicommodity flow, as claimed. □

[2] For a vector x and a subset I of indices, $x(I)$ denotes $\sum_{i \in I} x_i$.

Let $(G = (V, E), \Sigma)$ be a signed graph. We say that (G, Σ) is *weakly bipartite* if the polyhedron

$$\left\{x \in \mathbb{R}_+^E : x(C) \geq 1 \text{ for every odd circuit } C \text{ of } (G, \Sigma)\right\}$$

is integral, that is, if the linear program

$$\begin{array}{lrcll} \min & p^\top x & & & \\ \text{s.t.} & x(C) & \geq & 1 & \text{for every odd circuit } C \text{ of } (G,\Sigma) \\ & x_e & \geq & 0 & e \in E \end{array}$$

has an integral optimal solution, for all $p \in \mathbb{Z}_+^E$.

Theorem 4.3. Consider a multicommodity flow problem with graph $G = (V, E)$, dummy edges $F \subseteq E$, and arbitrary $p \in \mathbb{Z}_+^E$. If (G, F) is weakly bipartite, then the cut condition is sufficient for the existence of a multicommodity flow.

Proof. Suppose the cut condition holds. Then by Proposition 4.1, F is a minimum weight signature of (G, F). In particular, among all integer solutions to (P), χ_F is optimal. Since (G, F) is weakly bipartite, χ_F must in fact be an optimal solution to (P), so the optimal value of (P) is $p(F)$. It therefore follows from Strong LP Duality that (D) has optimal value $p(F)$, so by Proposition 4.2, there exists a multicommodity flow, as required. □

5 The cut condition is sufficient for planar graphs.

We will need two ingredients.

Proposition 5.1. Let $(G = (V, E), \Sigma)$ be a signed graph, and let G' be the graph obtained from G after subdividing every edge in $E - \Sigma$ once. Then (G, Σ) is weakly bipartite, if and only if, $(G', E(G'))$ is weakly bipartite.

Proof. Exercise. □

Let $G = (V, E)$ be an arbitrary graph, and let T be a nonempty even cardinality subset of V. Recall from Lecture 1 that a T-cut is a cut of the form $\delta(U)$ where $|U \cap T|$ is odd, while a T-join is a subset $J \subseteq E$ whose odd-degree vertices coincide with T. Recall the following result from Lecture 1:

Theorem 5.2 (Edmonds-Johnson Theorem)**.** Let $G = (V, E)$ be a graph, and let $T \subseteq V$ be nonempty and of even cardinality. Then $P(G, T) := \{x \in \mathbb{R}_+^E : x(B) \geq 1, B \text{ is a } T\text{-cut}\}$ is an integral polyhedron. In particular, every vertex of $P(G, T)$ is the incidence vector of an inclusionwise minimal T-join of G.

We are now ready to prove the following:

Theorem 5.3. Let $(G = (V, E), \Sigma)$ be a signed graph, where G is planar. Then (G, Σ) is weakly bipartite.

Proof. Since subdividing preserves planarity, we may assume by Proposition 5.1 that $\Sigma = E$. Thus, the odd circuits of (G, Σ) are exactly the odd circuits of G. Let us further assume that G is a plane graph, i.e. it has already been drawn on the plane without any pair of edges crossing. Notice that

($\star$) every circuit has an inside and an outside; the circuit can be written as the symmetric difference of the facial circuits that are inside (resp. outside); the circuit is odd if and only if the number of facial odd circuits used in the sum is odd.

Consider now the plane dual $G^\star = (V^\star, E)$, and let $T \subseteq V^\star$ denote the odd-degree vertices. Observe that T corresponds to the facial odd circuits of G. Notice that the cycles of G are the cuts of $G^\star$, and so the circuits of G are the inclusionwise minimal nonempty cuts of $G^\star$. Thus, by ($\star$), the odd circuits of G are the inclusionwise minimal T-cuts of $G^\star$.

By the Edmonds-Johnson Theorem, $P(G^\star, T) := \{x \in \mathbb{R}_+^E : x(B) \geq 1 \text{ for every } T\text{-cut } B \text{ of } G^\star\}$ is an integral polyhedron. Observe that we may only focus on the nonnegativity contraints as well as those corresponding to the inclusionwise minimal T-cuts of $G^\star$, as the other constraints are redundant for $P(G^\star, T)$. As the inclusionwise minimal T-cuts of $G^\star$ are in correspondence with the odd circuits of G, it follows that

$$P(G^\star, T) = \left\{x \in \mathbb{R}_+^E : x(C) \geq 1 \text{ for every odd circuit } C \text{ of } G\right\}.$$

As $P(G^\star, T)$ is integral, the equality above implies that (G, Σ) is a weakly bipartite signed graph, as required.

□

Consequently, by Theorem 4.3, we get the following.

Corollary 5.4. In planar graphs, the cut condition is sufficient for the existence of a multicommodity flow.

6 An excluded minor characterisation of weakly bipartite signed graphs

Take disjoint edge subsets I, J of (G, Σ). It follows from Proposition 3.4 that J does not contain an odd cycle if, and only if, there is a signature disjoint from J. Let

$$(G, \Sigma) \setminus I/J := \begin{cases} (G \setminus I/J, \emptyset) & \text{if } J \text{ contains an odd cycle,} \\ (G \setminus I/J, B - I) & B \text{ is a signature disjoint from } J. \end{cases}$$

We refer to $(G, \Sigma) \setminus I/J$ as a *minor* of (G, Σ) obtained after deleting I and contracting J.[3] We note that $(G, \Sigma) \setminus I/J$ is defined up to resigning.

Proposition 6.1. Let (G, Σ) be a signed graph, and let $I, J \subseteq E$ be disjoint.

1. If C is an odd circuit of (G, Σ) disjoint from I, then $C - J$ is an odd cycle of $(G, \Sigma) \setminus I/J$.

2. If C is an odd circuit of $(G, \Sigma) \setminus I/J$, then there is an odd circuit of (G, Σ) containing C and contained in $C \cup J$.

3. If (G, Σ) is weakly bipartite, then so is $(G, \Sigma) \setminus I/J$.

[3] In this setting, to contract a loop is to delete it.

Proof. **(1)** and **(2)** are left as exercises (see exercise 10). **(3)** Suppose (G, Σ) is weakly bipartite, that is, the linear program

$$\begin{array}{llll} \min & p^\top x & & \\ \text{s.t.} & x(C) & \geq & 1 \quad \text{for every odd circuit } C \text{ of } (G,\Sigma) \\ & x_e & \geq & 0 \quad e \in E \end{array}$$

has an integral optimal solution, for all $p \in \mathbb{Z}_+^E$. Given disjoint $I, J \subseteq E$, we would like to prove that $(G, \Sigma) \backslash I/J$ is also weakly bipartite. To see this, consider $p \in \mathbb{Z}_+^E$ where $p_e = 0\ \forall e \in I$ and $p_f = \infty\ \forall f \in J$. (Technically, $\infty \notin \mathbb{Z}$, but for all intents and purposes, we may treat ∞ as a sufficiently large integer here.) By assumption, the linear program has an integral optimal solution. Let us look at the linear program more closely however. Since $p_e = 0\ \forall e \in I$, we may set $x_e = 1\ \forall e \in I$ without loss of generality; in doing so, we satisfy constraint involving e without incurring any cost. Since $p_e = \infty\ \forall e \in J$, we must set $x_e = 0\ \forall e \in J$; in doing so, we basically drop $x_e, e \in J$ from every constraint as well as the objective function. Thus, the linear program for $p \in \mathbb{Z}_+^E$ where $p_e = 0\ \forall e \in I$ and $p_f = \infty\ \forall f \in J$ is equivalent to

$$\begin{array}{llll} \min & \sum_{e \in E-(I\cup J)} p_e x_e & & \\ \text{s.t.} & x(C-J) & \geq & 1 \quad \text{for every odd circuit } C \text{ of } (G,\Sigma) \text{ disjoint from } I \\ & x_e & \geq & 0 \quad e \in E-(I \cup J). \end{array}$$

It follows from parts (1) and (2) that this linear program is equivalent to

$$\begin{array}{llll} \min & \sum_{e \in E-(I\cup J)} p_e x_e & & \\ \text{s.t.} & x(C') & \geq & 1 \quad \text{for every odd circuit } C' \text{ of } (G,\Sigma) \setminus I/J \\ & x_e & \geq & 0 \quad e \in E-(I \cup J). \end{array}$$

Thus, the linear program above has an integral optimal solution for all $p \in \mathbb{Z}_+^{E-(I\cup J)}$, so by definition $(G, \Sigma) \setminus I/J$ is weakly bipartite. $\square$

Consider the signed graph $(K_5, E(K_5))$, we will refer to it as *odd*-K_5. We see below that odd-K_5 is not weakly bipartite.

Proposition 6.2. If a signed graph is weakly bipartite, then it has no odd-K_5 minor.

Proof. By Proposition 6.1 (3) it suffices to show that odd-K_5 is not weakly bipartite. One way to see this is to note that the linear program

$$\begin{array}{llll} \min & \mathbf{1}^\top x & & \\ \text{s.t.} & x(C) & \geq & 1 \quad \text{for every odd circuit } C \text{ of } (K_5,E(K_5)) \\ & x_e & \geq & 0 \quad e \in E(K_5) \end{array}$$

does not have an integral optimal solution. The minimum size of a signature of odd-K_5 is 4, so the minimum objective value of an integer solution to the linear program is 4. However, the fractional vector $x_e = \frac{1}{3}\ \forall e \in E(K_5)$ is a feasible solution whose objective value is $\frac{10}{3} < 4$, proving our claim. $\square$

In 1977, Paul Seymour [9] conjectured that the converse of this remark also holds. Observe that Theorem 5.3 supports this conjecture, because planar graphs have no K_5 minor. In the late 1990s, in his PhD thesis, Bertrand Guenin proved the conjecture [4].

Theorem 6.3 (Characterisation of Weakly Bipartite Signed Graphs)**.** A signed graph is weakly bipartite if, and only if, it has no odd-K_5 minor.

His proof made a spectacular use of a powerful result of Alfred Lehman on *minimally non-ideal matrices* [7]. Guenin's original proof was over 50 pages, but was condensed significantly to its essence in only a few pages by Schrijver [8]. A few years later, building on Schrijver's ideas, Geelen and Guenin gave a proof that avoided the use of Lehman's theorem altogether, and showed an even stronger result (characterising the class of *evenly bipartite signed graphs*) [2].

Exercises

1. Let $(G = (V, E), \Sigma)$ be a signed graph, and let $p \in \mathbb{Z}_+^E$. Recall that the *weight* of a signature Γ is $p(\Gamma) = \sum_{e\in\Gamma} p_e$. A *weighted packing of odd circuits* is an assignment of a nonnegative integer y_C to every odd circuit C such that the congestion at each edge e is at most p_e:
$$\sum (y_C : e \in C) \leq p_e \quad \forall e \in E.$$
The *value* of the weighted packing is $\mathbf{1}^\top y$. Prove that the minimum weight of a signature is greater than or equal to the maximum value of a weighted packing of odd circuits.

2. (a) Let $(G = (V, E), \{e\})$ be a signed graph, and let $p \in \mathbb{Z}_+^E$. Prove that the minimum weight of a signature is equal to the maximum value of a weighted packing of odd circuits.
   (b) Conclude that for the single-commodity problem, the cut condition necessarily implies the existence of an integer multicommodity flow.

3. Consider the signed graph $(K_4, \{e, f\})$, where e, f is any perfect matching. Show that the minimum cardinality of a signature is strictly larger than the maximum number of pairwise disjoint odd circuits.

4. Let $(G = (V, E), \Sigma)$ be a signed graph, and choose $p \in \mathbb{Z}_+^E$ such that every two signatures have the same weight parity. Let τ be the minimum weight of a signature.
   (a) Prove that if $\tau = 2$, then there exists a weighted packing of odd circuits of value two.
   (b) How does the previous part compare with exercise 3?

5. Consider the multicommodity flow problem displayed in Figure 1. In this problem, you will show this instance satisfies the cut condition, but has no multicommodity flow. Let (G, Σ) be the signed graph whose underlying graph is the same, where Σ consists of the three dummy edges.
   (a) Show that the minimum p-weight of a signature is 4.
   (b) Conclude that the cut condition is satisfied.
   (c) Show that there exists no odd circuit that intersects every minimum weight signature exactly once.

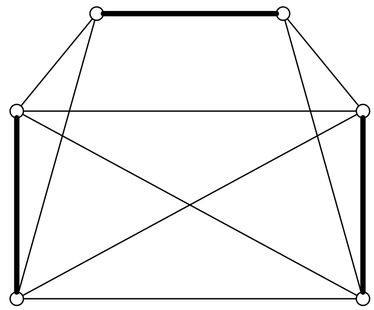

Figure 1: The dummy edges are the bold edges. $p_e = 1$ for every edge e except for the horizontal dummy edge f, for which $p_f = 2$.

(d) Conclude that there is no multicommodity flow.

6. Prove Remark 3.2.

7. In this problem we prove the converse of Proposition 3.4. Let (G, Σ) be a signed graph.

   (a) Prove that if $\emptyset$ is a signature, then (G, Σ) has no odd cycle.

   (b) Prove that if B contains a signature, then B intersects every odd circuit.

8. Consider a linear system $Ax \equiv b \pmod 2$ that has no solution over $GF(2)$, where A, b have $0, 1$ entries. Prove that there exists a $c \in \{0,1\}^m$ such that $c^\top A \equiv \mathbf{0}$ and $c^\top b \equiv 1 \pmod 2$.

9. Prove Proposition 5.1.

10. Prove Proposition 6.1 parts (1) and (2).

11. Use the theory of multicommodity flows to give another proof that the signed graph $(K_5, E(K_5))$ is not weakly bipartite.

Acknowledgements

Proposition 3.4 is originally due to Zaslavsky [10]. The notion of weakly bipartite signed graphs is due to Grötschel and Pulleyblank [3]. Theorem 5.3 is due to Hadlock [5] and Barahona [1].

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