MA431 Assignment 2

Due Wednesday, February 16 at 10:00am GMT

Collect as many points as you can. The threshold for 100% is 35 points.

Please do not look up the solutions. You can collaborate with other students, and if you do, you should acknowledge it in writing.

Email your solutions by the due date to `a.abdi1@lse.ac.uk`.

Q1. (Recommended, 5 points)

Let G be an n-vertex graph, and let A be an $n \times n$ real symmetric matrix such that $|A| \leq A(G)$. Prove that $\Delta(G) \geq \rho(A)$, where $\Delta(G)$ denotes the maximum degree of G.

Q2. (Recommended, 10 points)

Let G be a graph on n vertices with at least one edge whose spectrum is $\theta_1 \geq \cdots \geq \theta_n$, and let x be an arbitrary eigenvector of $A := A(G)$. Let $X_1, \ldots, X_k$ be a partition of the vertex set into k nonempty stable sets, and let B be the $k \times k$ matrix where

$$B_{ij} = \frac{1}{\sum_{u \in X_i} x_u^2} \cdot \sum \left(x_u x_v : u \in X_i, v \in X_j,\ u, v \text{ are adjacent} \right).$$

(a) Prove that the spectrum of B interlaces the spectrum of A.

(b) Prove that $k \geq 1 - \frac{\theta_1}{\theta_n}$.

(c) Conclude that G has chromatic number at least $1 - \frac{\theta_1}{\theta_n}$.

Q3. (Recommended, 5 points)

Let G be a graph, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be its Laplacian spectrum. Prove that the number of spanning trees of G is equal to $\frac{1}{n} \prod_{i=2}^{n} \lambda_i$.

Q4. (Recommended, 5 points)

Let G be an n-vertex simple graph, and let $\overline{G}$ be its complement. Prove the following statements:

(a) $\lambda_i(\overline{G}) = n - \lambda_{n-i+2}(G)$ for $2 \leq i \leq n$,

(b) $\lambda_n(G) \leq n$,

(c) if $\overline{G}$ has $\bar{c}$ connected components, and $\bar{c} \geq 2$, then $\lambda_n(G) = n$ and its multiplicity is $\bar{c} - 1$.

Q5. (Optional, 5 points)

Let A be an $n \times n$ matrix. Recall that $p_A(x) = \det(xI - A)$. Choose $\sigma_0(A), \sigma_1(A), \ldots, \sigma_n(A)$ such that

$$p_A(x) = \sum_{k=0}^{n} (-1)^k \sigma_k(A) x^{n-k}.$$

Prove the following statements for each k:

(a) $\sigma_k(A)$ is the sum of the product of any k eigenvalues, counted according to their algebraic multiplicity. That is, if $\lambda_1, \ldots, \lambda_n$ are the n eigenvalues of A, repeated according to the algebraic multiplicity of the eigenvalues, then

$$\sigma_k(A) = \sum_{S \subseteq [n], |S| = k} \prod_{i \in S} \lambda_i.$$

(b) $\sigma_k(A)$ is the sum of the determinants of all principal $k \times k$ submatrices. That is,

$$\sigma_k(A) = \sum \left(\det(B) : B \text{ is a } k \times k \text{ principal submatrix of } A\right).$$

Q6. (Optional, 5 points)

Let $G = (V, E)$ be a k-regular graph with spectrum $k \geq \theta_2 \geq \cdots \geq \theta_n$. Prove that $\alpha(G) \leq n \cdot \frac{-\theta_n}{k - \theta_n}$. Moreover, prove that if S is a stable set meeting this bound, then every vertex outside of S has exactly $-\theta_n$ neighbours inside S.

Q7. (Optional, 5 points)

Let M be an $n \times n$, and let C be its cofactor matrix. Recall that C is an $n \times n$ matrix whose ij-entry is $(-1)^{i+j}$ times the determinant of the submatrix of M obtained after removing row i and column j. By a Laplace expansion along any row of M, we get the matrix equation $C^\top M = \det(M) I$. The matrix $C^\top$ is called the *adjugate of* M, and denoted $\mathrm{adj}(M)$.

Let G be a graph, and let L be its Laplacian matrix. Prove that every entry of $\mathrm{adj}(L)$ is equal to $T(G)$, the number of spanning trees of G.

Q8. (Optional, 5 points)

Let G be a connected graph on n vertices. Prove that

$$\lambda_2(G) = \min_x \frac{n \sum_{ij \in E} (x_i - x_j)^2}{\sum_{i<j} (x_i - x_j)^2}$$

where the minimum is taken over all non-constant vectors x.

Q9. (Optional, 5 points)

The *Laplacian matrix* of a graph $G = (V, E)$ is the matrix $\Delta(G) - A(G)$, where $\Delta(G)$ is the diagonal matrix whose diagonal entries correspond to vertex degrees, and $A(G)$ is the adjacency matrix. The *Laplacian spectrum* of G is the spectrum of its Laplacian matrix.

Find the Laplacian spectra of the graphs below. Show your work. (You may use a solver.)