MA431 Assignment 3

Due Wednesday, March 2 at 10:00am GMT

Collect as many points as you can. The threshold for 100% is 35 points.

Please do not look up the solutions. You can collaborate with other students, and if you do, you should acknowledge it in writing.

Email your solutions by the due date to `a.abdi1@lse.ac.uk`.

Q1. (Recommended, 5 points)

Let $G = (V, E)$ be a graph, and let $x \in \mathbb{R}^E$. Prove that for every edge e,

$$\mathrm{Kir}(G; x) = x_e \cdot \mathrm{Kir}(G/e; x^e) + \mathrm{Kir}(G \setminus e; x^e),$$

where x^e denotes the vector obtained from x after dropping the coordinate corresponding to e.

Q2. (Optional, 10 points)

Let $G = (V, E)$ be a graph, let $w \in \mathbb{R}_+^E$, and let L_w be the Laplacian of the weighted graph (G, w). Prove the following statements:

(a) $L_w = \sum_{e=\{u,v\}\in E} w_e \cdot (e_u - e_v)(e_u - e_v)^\top$,

(b) for each $x \in \mathbb{R}^V$, $x^\top L_w x = \sum_{e=\{u,v\}\in E} w_e (x_u - x_v)^2$,

(c) L_w is a positive semidefinite matrix,

(d) $\mathbf{1}$ is an eigenvector of L_w with eigenvalue 0,

(e) if every edge has nonzero weight, then L_w has rank $n - c$, and 0 as an eigenvalue has multiplicity c, where c is the number of connected components of G

Q3. (Recommended, 5 points)

Let $\overrightarrow{G} = (V, \overrightarrow{E})$ be the digraph displayed in Figure 1, and let $T \subseteq \overrightarrow{E}$ be the spanning tree displayed by the dashed arcs.

(a) Write down the incidence matrix of $\overrightarrow{G}$.

(b) What is the dimension of the cycle space $W^\diamond$? Compute a basis for $W^\diamond$ consisting of the fundamental cycles of T.

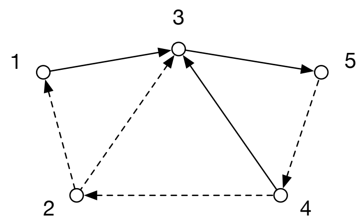

Figure 1: Oriented graph for Q1

(c) What is the dimension of the cut space $W^\star$? Compute a basis for $W^\star$ consisting of the fundamental cuts of T.

Q4. (Optional, 5 points)

Let $G = (V, E)$ be a connected graph, and let $\overrightarrow{G} = (V, \overrightarrow{E})$ be an orientation. Let $b \in \mathbb{R}^V$. Prove that the following statements are equivalent:

(a) there exists f such that $\nabla f = b$,

(b) for each $\pi \in \mathbb{R}^V$, if grad $\pi = \mathbf{0}$, then $b^\top \pi = 0$.

Q5. (Recommended, 10 points)

Let $G = (V, E)$ be a connected graph, and let $e \in E$. Prove the following statements:

(a) The effective resistance of e is at most 1.

(b) The effective resistance of e is 1 if, and only if, $G \setminus e$ is not connected.

Q6. (Recommended, 5 points)

Let $G = (V, E)$ be a connected graph, let $\overrightarrow{G} = (V, \overrightarrow{E})$ be an orientation, and let $e \in \overrightarrow{E}$. Let i be the unit electrical flow across the endpoints of e, i.e., $i = P_\star \chi^e$. Show that for any edge $e' \in \overrightarrow{E}$, $i_{e'} \leq i_e$.

Q7. (Optional, 5 points)

Let $G = (V, E)$ be a connected graph. For all $u, v \in V$, let $d(u, v)$ denote the effective resistance between u and v, for all $u, v \in V$. Show that $d(\cdot, \cdot)$ defines a metric. That is, prove the following:

(a) $d(u, v) = 0$ if, and only if, $u = v$

(b) $d(u, v) = d(v, u)$ for all $u, v \in V$, and

(c) $d(u, v) + d(v, w) \geq d(u, w)$ for all $u, v, w \in V$.

Q8. (Optional, 10 points) Let M_n be the $n \times n$ 2-dimensional grid.

(a) (5 points) Show that the effective resistance between opposite corners of the grid is $\Omega(\log n)$.

(Hint: exploit Rayleigh monotonicity.)

(b) (5 points) Show that it is $O(\log n)$.