MA431 Assignment 4

Due Wednesday March 16 at 10:00am GMT

Collect as many points as you can. The threshold for 100% is 35 points.

Please do not look up the solutions. You can collaborate with other students, and if you do, you should acknowledge it in writing.

Email your solutions by the due date to `a.abdi1@lse.ac.uk`.

Q1. (Recommended, 10 points) In this question, you are asked to complete the proof of the Transfer-Current Theorem presented in Lecture 7.

Let $G = (V, E)$ be a graph, $\overrightarrow{G} = (V, \overrightarrow{E})$ be an arbitrary orientation, and $F \subseteq \overrightarrow{E}$ an arbitrary set.

(a) Let $x \in \mathbb{R}^{\overrightarrow{E}}$ be a flow such that $\text{support}(x) \subseteq F$. Let i be the electrical flow such that $\nabla i = \nabla x$. Suppose $i_a = 0$ for all $a \in F$. Prove that $i = \mathbf{0}$ (and so x is a circulation).

Suppose now that F is a forest. For each $a = (u, v) \in \overrightarrow{E}$, let i^a be the unit electrical flow from u to v. Fix $e \in F$ and let $\hat{F} := F \setminus e$.

(b) Prove that there exists $\alpha_a \in \mathbb{R}$ for each $a \in \hat{F}$ such that for the vector

$$i := i^e - \sum_{a \in \hat{F}} \alpha_a i^a,$$

we have $i_f = 0$ for all $f \in \hat{F}$.

(c) Let $K \subseteq V$ be a connected component of the subgraph $(V, \hat{F})$. Prove that

$$\sum_{v \in K} (\nabla i)_v = \begin{cases} 0 & \text{if } K \cap \{s, t\} = \emptyset \\ -1 & \text{if } s \in K \\ 1 & \text{if } t \in K \end{cases}$$

where $e = (s, t)$.

Q2. (Optional, 5 points)

Let $\overrightarrow{G}$ be an arbitrary orientation the Petersen graph, and let $e = (s, t)$ be an arbitrary edge. Use Julia (or any other software) to address the following:

(a) Find the unit electrical flow from s to t, the associated electrical potentials, and the effective resistance of e.

(b) Increase the conductance of e to 2, and keep all the other edges of unit conductance. What is the effective resistance of e? Compare this with (a).

Q3. (Recommended, 5 points)

Let $G = (V, E)$ be a connected graph with a given orientation, and $s, t \in V$. A *loop-erased random walk* from s to t is obtained from a simple random walk $(X_0, X_1, \ldots)$ starting from s as follows. Start with $X' = (X_0, X_1, \ldots, X_r)$, where r is minimal such that $X_r = t$. Then find $j < k$ with k minimal such that $X'_j = X'_k$, and replace X' with $(X'_0, \ldots, X'_j, X'_{k+1}, \ldots, X'_r)$ (also update r to the new shorter walk length.) Repeat until no such j, k exist.

Let i be the unit electrical current from s to t. Let $(X'_i)_{i\geq 0}$ be a loop-erased random walk started from s and stopped at t. Show that i_e is equal to the probability that (X'_i) uses edge e in the forward direction, minus the probability that (X'_i) uses edge e in the backward direction.

Q4. (Optional, 5 points)

Consider simple random walk in G started at s and stopped at t. For any $v, w \in V$ with $\{v, w\} \in E$, let S_{vw} denote the number of times the walk transitions from v to w (not counting transitions from w to v). For any arc $a = (v, w) \in \vec{E}$, show that

$$|\mathbb{E}[S_{vw}] - \mathbb{E}[S_{wv}]| \leq 1.$$

Q5. (Recommended, 5 points)

Show that the commute time between any two nodes of a connected graph with n nodes is at most n^3. Find an example where the commute time between some pair of vertices is $\Omega(n^3)$.

Q6. (Recommended, 5 points)

Consider C_n, the cycle of length n. Show that one direction of Cheeger's inequality is tight for C_n up to constant factors, for all n. *(Hint: you don't need to determine the second eigenvalue of the Laplacian, a bound suffices.)*

Q7. (Optional, 10 points)

The *cover time* of a connected graph G (which we will denote by $\text{Cover}(G)$ is the maximum over all choices of starting node s, of the expected time needed for a simple random walk starting from s to visit each node of G.

Let R be the maximum over $u, v \in V$ of the effective resistance between u and v in G; also let m denote the number of edges of G. Show that

$$\mathrm{Cover}(G) = \Omega(mR) \text{ and } \mathrm{Cover}(G) = O(\log|V| \cdot mR).$$

Hint: Markov's inequality is very useful here.

Q8. (Optional, 10 points)

Consider Q_n, the n-dimensional binary hypercube; identify its vertex set with $\{-1, 1\}^n$.

Determine the second eigenvalue of the Laplacian of Q_n. *(Hint: for $a \in \{0,1\}^n$, consider vectors $x \in \mathbb{R}^{V(Q_n)}$ of the form $x_v = (-1)^{a \cdot v}$ for all $v \in \{-1,1\}^n$.)*

Hence show that one direction of Cheeger's inequality is tight (up to constant factors) for the hypercube. Also give bounds on the mixing time, both directly from the second eigenvalue, and from Cheeger's inequality.