MA431 Assignment 5

Due Wednesday March 30 at 10:00am BST

Collect as many points as you can. The threshold for 100% is 35 points.

Please do not look up the solutions. You can collaborate with other students, and if you do, you should acknowledge it in writing.

Email your solutions by the due date to `a.abdi1@lse.ac.uk`.

Q1. (Recommended, 5 points)

Let G be a d-regular graph with edge expansion $\alpha > 0$. Show that there is a constant C depending only on d and α so that if r edges are removed from G, the resulting graph has a connected component of size at least $n - Cr$.

Q2. (Optional, 10 points)

Fix $d \geq 3$ and $\beta > 0$. Let $G = (V, E)$ be a d-regular spectral β-expander of size n. For any $v \in V$, $r \leq n$, define

$$B(v, r) = \{u \in V : \text{ there is a path from } v \text{ to } u \text{ of length at most } r\}.$$

(a) Show that $|B(v, r)| \geq \min\{(1 + c)^r, n/2\}$ for some constant c depending only on d and β.

(b) Deduce that the diameter of G is $O(\log n)$. (The *distance* between two vertices in a graph is the length of a shortest path between the two vertices. The *diameter* is the maximum distance between any pair of vertices.)

Q3. (Recommended, 5 points)

Let $v_1, \ldots, v_k$ be k points in $\mathbb{R}^n$. Prove that $x^\star = \frac{1}{k}\sum_{i=1}^{k} v_i$ is the unique minimiser of the function $f(x) = \sum_{i=1}^{k} \|x - v_i\|^2$.

Q4. (Optional, 5 points)

Let G be a connected graph, and let L be its Laplacian matrix. Prove that every proper principal submatrix of L is nonsingular.

Q5. (Recommended, 5 points)

Let $G = (V, E)$ be a connected graph, and let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be its Laplacian spectrum. Prove that if G forms a path, then λ_2 has multiplicity at most 1.

Q6. (Optional, 10 points)

Let $G = (V, E)$ be a 3-connected graph that is planar. Describe an algorithm for finding a peripheral cycle of G whose running time is bounded above by a polynomial in $|V|$.

Q7. (Optional, 10 points)

A graph is *outerplanar* if it has a plane embedding where every vertex belongs to the boundary of the same face. Let $G = (V, E)$ be a 2-connected outerplanar graph, and let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be its Laplacian spectrum. Prove that λ_2 has multiplicity at most 2.

Q8. (Recommended, 5 points)

Let $G = (V, E)$ be a connected graph. A *generalised Laplacian* is a symmetric $V \times V$ matrix Q such that for all $u, v \in V$,

$$Q_{uv} \begin{cases} < \ 0 & \text{if } u, v \text{ are adjacent} \\ = \ 0 & \text{if } u, v \text{ are nonadjacent and distinct,} \end{cases}$$

Let λ be the smallest eigenvalue of Q. Prove that λ is a simple eigenvalue, and each associated eigenvector has nonzero entries of the same sign.