MA431 Spectral Graph Theory: Lecture 10

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22 The multiplicity of λ_2 for planar graphs

Let $G = (V, E)$ be a 3-connected planar graph, let L be the Laplacian matrix, and let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be its Laplacian spectrum. In this section, we prove that the second eigenvalue, λ_2, has multiplicity at most 3.

Lemma 22.1. Let $G = (V, E)$ be a connected graph, let L be its Laplacian matrix, and let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be its Laplacian spectrum. Let $f \in \mathbb{R}^V_+$ be a λ_2-eigenvector whose support is minimal amongst all λ_2-eigenvectors of L. Let $U_+ := \{u \in V : f_u > 0\}$, $U_- := \{u \in V : f_u < 0\}$, and $U := U_+ \cup U_-$. Then the following statements hold:

1. every vertex of $V - U$ with a neighbour in one of U_+, U_- has a neighbour in other set,
2. $G[U_+], G[U_-]$ are connected subgraphs.

Proof. As $\mathbf{1}^\top f = 0$, the sets U_+, U_- are nonempty; this will be a useful fact in our proof. For every vertex $u \in V$, we have $(\deg(u) - \lambda_2) \cdot f_u = \sum_{v \in N(u)} f_v$. These equalities imply **(1)** immediately. **(2)** We will show that $G[U_+]$ is connected; that $G[U_-]$ is connected follows from applying a similar argument to $-f$. Suppose for a contradiction $G[U_+]$ is not connected. Then there exists a partition of U_+ into nonempty parts I, J such that there is no edge between the two parts. Define the nonzero vector $g \in \mathbb{R}^V$ as follows:

$$g_u := \begin{cases} f_u & \text{if } u \in I \\ -\alpha \cdot f_u & \text{if } u \in J \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha \in \mathbb{R}$ is chosen so that $\mathbf{1}^\top g = 0$. We claim that g is a λ_2-eigenvector of L, thereby contradicting the minimality of the support of f.

Claim. $\frac{g^\top L g}{g^\top g} \leq \lambda_2$.

Proof of Claim. For subsets $S_1, S_2 \subseteq V$, denote by $L[S_1, S_2]$ the submatrix of L whose rows correspond to S_1 and whose columns correspond to S_2, and by v_{S_1} the subvector of $v \in \mathbb{R}^V$ restricted to the coordinates in S_1.

Then

$$\begin{aligned} g^\top L g &= g_I^\top L[I,I] g_I + g_J^\top L[J,J] g_J && \text{because } L[I,J] = \mathbf{0} \\ &= f_I^\top L[I,I] f_I + \alpha^2 f_J^\top L[J,J] f_J \\ &= f_I^\top (\lambda_2 f_I - L[I,U_-] f_{U_-}) + \alpha^2 f_J^\top (\lambda_2 f_J - L[J,U_-] f_{U_-}) && \text{because } Lf = \lambda_2 f \\ &= \lambda_2 g^\top g - f_I^\top L[I,U_-] f_{U_-} - \alpha^2 f_J^\top L[J,U_-] f_{U_-} \\ &\le \lambda_2 g^\top g \end{aligned}$$

where the last inequality follows from the inequalities $f_I, f_J > \mathbf{0}$, $f_{U_-} < \mathbf{0}$, and the fact that $L[I,U_-], L[J,U_-]$ have nonpositive entries. $\Diamond$

However, as $g \in \langle \mathbf{1} \rangle^\perp$, CFT (3) implies that $\frac{g^\top L g}{g^\top g} \ge \lambda_2$, and equality is achieved only for vectors g in the λ_2-eigenspace. The claim above implies that indeed equality is achieved, and so g must be a λ_2-eigenvector, thereby contradicting the support minimality of f. $\square$

We need the following classic result from Graph Theory:

Theorem 22.2 (Menger's Theorem)**.** Let $G = (V, E)$ be a graph, and let s, t be distinct vertices. Then the following statements are equivalent:

1. there exist k internally vertex-disjoint st-paths,
2. for all $X \subseteq V - \{s,t\}$ such that $|X| < k$, the vertices s, t belong to the same connected component of $G \setminus X$.

We are now ready for the main result of this section:

Theorem 22.3. Let $G = (V, E)$ be a 3-connected planar graph, let L be the Laplacian matrix, and let $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n$ be its Laplacian spectrum. Then λ_2 has multiplicity at most 3.

Proof. Suppose for a contradiction λ_2 has multiplicity at least 4. Embed G on the plane; let C be a facial (i.e. peripheral) cycle, and let v_1, v_2, v_3 be distinct vertices of $V(C)$. Our contrary assumption implies that there exists a λ_2-eigenvector f such that $f_{v_1} = f_{v_2} = f_{v_3} = 0$. We may assume that f is support minimal amongst all λ_2-eigenvectors. Let $U_+ := \{u \in V : f_u > 0\}, U_- := \{u \in V : f_u < 0\}$, and $U := U_+ \cup U_-$.

As G is 3-connected, we may apply Menger's Theorem and conclude that there exist vertex-disjoint paths P_1, P_2, P_3 such that for each $i \in [3]$,

- P_i is a $u_i v_i$-path in $G[V - U]$, and
- u_i has a neighbour in U.

To see this, let G' be the graph obtained from G after introducing a new vertex, t, with neighbours v_1, v_2, v_3. Observe that G' remains 3-connected. Now, pick an arbitrary vertex $s \in U$, and find three internally vertex-disjoint st-paths in G', whose existence is guaranteed by Menger's Theorem. The three paths P_1, P_2, P_3 are appropriate subpaths of these st-paths.

Moving forward, note that by placing t in the face bounded by C, we get a plane embedding of G' as well. Our contradiction will come from the fact that G' has a $K_{3,3}$ minor, which is at odds with the planarity of G' by Remark 21.3.

By Lemma 22.1, in G, each u_i has a neighbour in U_+ and a neighbour in U_-, and $G[U_+], G[U_-]$ are disjoint connected subgraphs. Thus, by contracting $G[U_+], G[U_-]$ to single vertices u_+, u_-, respectively, and by contracting P_1, P_2, P_3, we obtain a (not necessarily simple) minor of G' where each of u_+, u_-, t is a neighbour of each of v_1, v_2, v_3, implying in turn that G' has a $K_{3,3}$ minor, which is a contradiction. $\square$

Let $G = (V, E)$ be an arbitrary connected graph. A *generalised Laplacian* is a symmetric $V \times V$ matrix Q such that for all $u, v \in V$,

$$Q_{uv} \begin{cases} < & 0 \qquad \text{if } u, v \text{ are adjacent} \\ = & 0 \qquad \text{if } u, v \text{ are nonadjacent and distinct,} \end{cases}$$

Observe that there are no requirements on the diagonal entries of Q. The generalised Laplacian matrix exhibits similar behaviour as the Laplacian matrix. For instance, in Exercise 7, we see that $\lambda_1(Q)$ is simple, and in Exercises 8 and 9, we see that if G is 3-connected and planar, then $\lambda_2(Q)$ has multiplicity at most three.

Observe that for any $\lambda \in \mathbb{R}$, $Q - \lambda I$ is also a generalised Laplacian, one whose eigenvectors are the same as the eigenvectors of Q, and whose eigenvalues are obtained by subtracting α from the eigenvalues of Q. Subsequently, the multiplicity of $\lambda := \lambda_2(Q)$ can be thought of as the *corank*, i.e. dimension of the kernel, of $Q - \lambda I$.

Putting things together, we obtain that if G is 3-connected and planar, then the corank of any generalised Laplacian is at most three. In fact, when G is 3-connected and planar, one can obtain a planar drawing of G from the kernel of a generalised Laplacian of maximum corank; we refer the interested reader to [2] (§13.11).

An interesting graph invariant, known as the *Colin de Verdière number* and denoted $\mu(G)$, is defined as the maximum corank of a generalised Laplacian Q of G subject to an additional condition that

> there is no nonzero $V \times V$ matrix B such that $QB = \mathbf{0}$ and $B_{uv} = 0$ whenever u, v are equal or adjacent.

This technical condition is known as the *Strong Arnold Property*. The parameter was introduced in [1], where it was shown that $\mu(G)$ is monotone under taking minors and that planarity of G is characterized by the inequality $\mu(G) \leq 3$. Later on, it was shown that *linkless embeddability* is characterized by the inequality $\mu(G) \leq 4$ [3]. See [4] for a survey on the parameter.

Acknowledgements

The main result of §22 is due to Colin de Verdière [1], but the short proof is due to van der Holst [5].

References

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Exercises

1. Prove Theorem 21.5, Claim 1.

2. Prove Theorem 21.5, Claim 7.

3. Let $v_1, \ldots, v_k$ be k points in $\mathbb{R}^n$. Prove that $x^\star = \frac{1}{k}\sum_{i=1}^k v_i$ is the unique minimiser of the function $f(x) = \sum_{i=1}^k \|x - v_i\|^2$.

4. Let G be a connected graph, and let L be its Laplacian matrix. Prove that every proper principal submatrix of L is nonsingular.

5. Based on the results of this lecture, describe an algorithm that given a 3-connected graph $G = (V, E)$ runs in time polynomial in $|V|$ and outputs a straight-line embedding of G or certifies that G is not planar.

6. Let $G = (V, E)$ be a connected graph, and let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be its Laplacian spectrum.

   (a) Prove that if G is a path, then λ_2 has multiplicity at most 1.

   (b) G is *outerplanar* if it has a plane embedding where every vertex belongs to the boundary of the same face. Prove that if G is a 2-connected outerplanar graph, then λ_2 has multiplicity at most 2.

7. Let $G = (V, E)$ be a connected graph. Let Q be a generalised Laplacian matri. Let λ be the smallest eigenvalue of Q. Prove that λ is a simple eigenvalue, and each associated eigenvector has nonzero entries of the same sign.

8. Let G be an n-vertex connected graph, let Q be a generalised Laplacian, and let $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be the spectrum of Q. Let $f \in \mathbb{R}_+^V$ be a λ_2-eigenvector whose support is minimal amongst all λ_2-eigenvectors of Q. Let $U_+ := \{u \in V : f_u > 0\}$, $U_- := \{u \in V : f_u < 0\}$, and $U := U_+ \cup U_-$. Prove that $G[U_+], G[U_-]$ are connected subgraphs.

9. Let G be an n-vertex connected graph, let Q be a generalised Laplacian, and let $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be the spectrum of Q. Prove that if G is 3-connected and planar, then λ_2 has multiplicity at most 3.