MA431 Spectral Graph Theory: Lecture 2

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2 The Perron-Frobenius Theorem, continued

Let A be an $n \times n$ real symmetric matrix, and let $\theta \in \mathbb{R}$. Recall that a vector $x \in \mathbb{R}^n$ is θ-subharmonic if $x \geq \mathbf{0}$, $x \neq \mathbf{0}$, and $Ax \geq \theta x$. We saw the remark below last time.

Remark 2.1. Let A be a real symmetric matrix, and let x be an eigenvector with eigenvalue θ. Then $|x|$ is $|\theta|$-subharmonic for the matrix $|A|$. In particular, if A is nonnegative, then $|x|$ is $|\theta|$-subharmonic for the matrix A.

Recall that a real symmetric matrix is irreducible if its underlying graph is connected. We proved the following important lemma last time:

Lemma 2.2. Let A be a real symmetric matrix that is nonnegative and irreducible. Then there exists a maximum real number ρ such that there is a ρ-subharmonic vector for A. Moreover, every ρ-subharmonic vector x is an eigenvector with eigenvalue ρ, and satisfies $x > \mathbf{0}$. In particular, ρ is an eigenvalue.

Let A be a real symmetric matrix. The *spectral radius of* A is

$$\rho(A) := \max\{|\theta| : \theta \text{ is an eigenvalue of } A\}.$$

The spectral radius may not necessarily itself be an eigenvalue; for instance, $-I$ with the sole eigenvalue of -1 has spectral radius 1. In stark contrast to this rather trivial example, the following theorem proves that the spectral radius is an eigenvalue for real symmetric matrices that are nonnegative and irreducible:

Theorem 2.3 (Perron-Frobenius Theorem)**.** Let A be a real symmetric matrix that is nonnegative and irreducible. Then the following statements hold:

1. $\rho(A)$ is an eigenvalue of A,
2. for every eigenvector corresponding to $\rho(A)$, all entries are nonzero and have the same sign,
3. $\rho(A)$ is a simple eigenvalue, i.e. it has multiplicity one,
4. if $-\rho(A)$ is also an eigenvalue, then the underlying graph of A is bipartite (and is therefore loopless).

Proof. Let ρ be the maximum real number such that there exists a ρ-subharmonic vector for A. We have that

$$\rho(A) \leq \rho \leq \max\{\theta : \theta \text{ is an eigenvalue for } A\} \leq \rho(A)$$

where the leftmost inequality follows from Remark 2.1, the middle one from Lemma 2.2, and the rightmost one is obvious. Equality must hold throughout, implying in turn that $\rho(A)$ is indeed an eigenvalue, thereby proving **(1)**.

Let x be an eigenvector with eigenvalue ρ. We want to prove that all entries of x are nonzero and have the same sign. After negating x, if necessary, we may assume that x has an entry $x_u > 0$. By Remark 2.1, $|x|$ is a ρ-subharmonic for A. By Lemma 2.2, $|x|$ is also an eigenvector with eigenvalue ρ, and $|x| > \mathbf{0}$. Subsequently, $A(|x| - x) = \rho(|x| - x)$. Since $|x| - x \geq \mathbf{0}$ and $(|x| - x)_u = 0$, it follows from Lemma 2.2 that $|x| - x$ cannot be a ρ-subharmonic vector for A, implying in turn that $|x| - x = \mathbf{0}$. Thus, $x = |x| > \mathbf{0}$, thereby proving **(2)**.

The ρ-eigenspace must have dimension one. For if not, then it would contain a nonzero vector with a zero entry, thereby contradicting (2). As a consequence, ρ has multiplicity one. This proves **(3)**.

(4) Suppose $-\rho$ is also an eigenvalue for A. Pick a ρ-eigenvector x and a $(-\rho)$-eigenvector y. Note that x, y are orthogonal vectors. Then $A^2x = \rho^2 x$ and $A^2 y = \rho^2 y$, so ρ^2 is a non-simple eigenvalue for A^2. It is clear that $\rho(A^2) = \rho^2$. But then (3) could not possibly hold for the matrix A^2 which is real, symmetric, and nonnegative. It must therefore be the case that A^2 is not irreducible.

Let G_1 be the underlying graph of A, and let G_2 be that of A^2. Note that G_1, G_2 have the same set of vertices, and edges of G_2 correspond to length-two walks in G_1 between the same ends. As a result, G_2 has a walk between two vertices if, and only if, G_1 has an *even* length walk between the same two vertices.

That A^2 is not irreducible means that G_2 is not connected. Thus, our argument above implies the existence of a partition of the vertex set of G_1 into nonempty parts X, Y such that every walk between them has odd length. This, combined with the fact that G_1 is connected, implies that G_1 is a bipartite graph with bipartition X, Y, as required. □

By studying the spectral radius of (the adjacency matrix of) a graph, we can tell whether or not the graph is bipartite, regular, or otherwise.

Theorem 2.4. Let G be a connected graph, and let $A := A(G)$. Then the following statements are equivalent:

1. G is bipartite,
2. the spectrum of G is symmetric about the origin, that is, if θ belongs to the spectrum, then so does $-\theta$, and both eigenvalues have the same multiplicity,
3. $-\rho(A)$ is an eigenvalue.

Proof. Exercise. □

Before characterizing regular graphs in terms of the spectrum, let us state the following bounds on the spectral radius of a graph:

Theorem 2.5. Let $G = (V, E)$ be a graph, and let $A := A(G)$. Then

$$\frac{\sum_{v\in V} \deg(v)}{|V|} \leq \rho(A) \leq \max\{\deg(v) : v \in V\},$$

that is, $\rho(A)$ is sandwiched between the average degree and the maximum degree of G.

Proof. Exercise. □

We are now ready to characterize regular graphs:

Theorem 2.6. Let G be a connected graph with maximum degree Δ, and let $A = A(G)$. Then $\rho(A) := \Delta$ if, and only if, G is a Δ-regular graph.

Proof. Exercise. □

Exercises

1. Consider the wheel W_5 on vertices $\{1, 2, 3, 4, 5\}$ and edges $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}$. Find the spectrum of W_5, and for each eigenvalue, present an orthogonal basis of its eigenspace. Show your work. (You may use a solver.)

2. Find the spectrum of the Petersen graph. Show your work. (You may use a solver.)

3. Determine the eigenvalues of the complete bipartite graph $K_{m,n}$ along with their multiplicities.

4. Let G be a k-regular graph on n vertices with no loops or parallel edges, and let $k, \theta_2, \ldots, \theta_n$ be its spectrum. Prove that G and its complement $\overline{G}$ have a common set of eigenvectors which form a basis for $\mathbb{R}^n$, and that the eigenvalues of $\overline{G}$ are $n - 1 - k, -1 - \theta_2, \ldots, -1 - \theta_n$.

5. Prove Proposition 1.7.

6. Prove Proposition 1.8.

7. Denote by P_n the path on n vertices $\{1, 2, \ldots, n\}$ and edges $\{1, 2\}, \{2, 3\}, \ldots, \{n - 2, n - 1\}, \{n - 1, n\}$. Define a sequence of $n + 1$ polynomials as follows: $p_0(x) = 1, p_1(x) = x$ and

   $$p_{r+1}(x) = xp_r(x) - p_{r-1}(x) \qquad r = 1, 2, \ldots, n - 1.$$

   Prove that p_n is the characteristic polynomial of P_n.

8. The *odd girth* of a graph is the minimum length of an odd cycle. Prove that cospectral graphs have the same odd girth.

9. Prove Theorem 2.4.

10. Prove Theorem 2.5.

11. Prove Theorem 2.6.

12. The *diameter* of a connected graph is the smallest integer d such that there is a path of length at most d between every pair of vertices.

    Let G be a connected graph whose diameter is d. Prove that the spectrum of G has at least $d + 1$ distinct elements.

13. Given a graph, a *Seidel switch* is obtained by picking a vertex and swapping its neighbours with the non-neighbours. For example, the two graphs displayed in Figure 1 are related by a Seidel switch applied to the central vertex.

    Prove that the spectrum of a graph is invariant under a Seidel switch. Then conclude that the two graphs displayed in Figure 1 are cospectral.

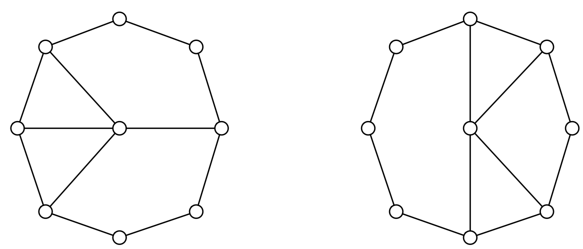

Figure 1: Two non-isomorphic cospectral graphs with the same degree sequence.

3 Cauchy's Interlacing Theorem

Consider a sequence $\theta_1 \geq \cdots \geq \theta_n$ of length n, and another $\mu_1 \geq \cdots \geq \mu_{n-1}$ of length $n - 1$. We say that the latter *interlaces* the former if

$$\theta_1 \geq \mu_1 \geq \theta_2 \geq \mu_2 \geq \cdots \geq \theta_{n-1} \geq \mu_{n-1} \geq \theta_n.$$

For instance, for every real-rooted polynomial f, it can be shown that the derivative f' is real-rooted as well, by showing that the roots of the derivative f' must interlace the roots of f, where every root is repeated according to its multiplicity. In fact, there is a useful characterisation of this phenomenon:

Theorem 3.1. Let f, g be real-rooted polynomials, where f has degree n, and g has degree $n - 1$. Then the roots of g interlace the roots of f if, and only if, $f + \alpha g$ is a real-rooted polynomial for all $\alpha \in \mathbb{R}$.

Proof. Exercise. (Hint. Use the intermediate value theorem, as well as the fact that every polynomial of degree n over the reals can be factorised into n linear factors over the complex numbers.) $\square$

Let us use this characterisation to prove the following theorem:

Proposition 3.2. Given an $n \times n$ real symmetric matrix A, and an $(n-1) \times (n-1)$ principal submatrix B, the spectrum of B interlaces the spectrum of A.

Proof. Let $f(x) := \det(xI - A)$ and $g(x) := \det(xI - B)$. We know that both f, g are real-rooted polynomials. By Theorem 3.1, we need to show that $f + \alpha g$ is real-rooted for all $\alpha \in \mathbb{R}$. Given that $A = \left(\begin{smallmatrix} B & c \\ c^\top & d \end{smallmatrix}\right)$, we have

$$
\begin{aligned}
f + \alpha g &= \det(xI - A) + \alpha \det(xI - B) \\
&= \det \begin{pmatrix} xI - B & -c \\ -c^\top & x - d \end{pmatrix} + \det \begin{pmatrix} xI - B & -c \\ \mathbf{0}^\top & \alpha \end{pmatrix} \\
&= \det \begin{pmatrix} xI - B & -c \\ -c^\top & x - d + \alpha \end{pmatrix} \quad \text{by linearity of determinant} \\
&= \det(xI - C)
\end{aligned}
$$

where $C = \left(\begin{smallmatrix} B & c \\ c^\top & d-\alpha \end{smallmatrix}\right)$. Consequently, $f + \alpha g$ is the characteristic polynomial of the real symmetric matrix C, so it is real-rooted, as required. ☐

By a repeated application of the result above to smaller principal submatrices of A, we get the following important result:

Theorem 3.3 (Cauchy's Interlacing Theorem)**.** Given an $n \times n$ real symmetric matrix A with spectrum $\theta_1 \geq \cdots \geq \theta_n$ and an $m \times m$ principal submatrix B with spectrum $\mu_1 \geq \cdots \geq \mu_m$, we have that

$$\theta_i \geq \mu_i \geq \theta_{n-m+i} \qquad i = 1, \ldots, m.$$

Proof. This follows immediately from Proposition 3.2. ☐

This motivates the following extension of the interlacing relation between sequences. Given a sequence $\theta_1 \geq \cdots \geq \theta_n$ and a shorter sequence $\mu_1 \geq \cdots \geq \mu_m$, we say that the shorter sequence *interlaces* the longer sequence if

$$\theta_i \geq \mu_i \geq \theta_{n-m+i} \qquad i = 1, \ldots, m.$$

4 Applications of Cauchy's Interlacing Theorem

Let us start with an important proposition.

Proposition 4.1. Given a graph, its spectrum is interlaced by the spectrum of any induced subgraph.

Proof. Given a graph G and an induced subgraph H, $A(H)$ is a principal submatrix of $A(G)$, so the result follows from Cauchy's Interlacing Theorem. ☐

As a consequence,

Theorem 4.2. Let G be a graph, let $A := A(G)$, and let $\rho := \rho(A)$. Then every induced subgraph has average degree, and therefore minimum degree, at most ρ.

Proof. Let H be an induced subgraph, and let $\rho' := \rho(A(H))$. By Theorem 2.5, H has average degree at most ρ'. However, by Proposition 4.1, $\rho' \leq \rho$, so the result follows. $\square$

Recall that the *chromatic number* of a graph is the least number of colours needed to colour the vertices such that adjacent vertices are coloured differently.

Theorem 4.3. Let G be a graph, let $A := A(G)$, and let $\rho := \rho(A)$. Then G has chromatic number at most $1 + \lfloor \rho \rfloor$.

Proof. Exercise. $\square$

Acknowledgements

The presentation of Sections 1 and 2 followed closely [2], Chapter 8. The short proof of Proposition 3.2 is due to Fisk [1].

References

[1] S. Fisk. A very short proof of Cauchy's interlace theorem for eigenvalues of hermitian matrices. *Amer. Math. Monthly*, 112(2):118, February 2005.

[2] C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer, New York, NY, 2000.