MA431 Spectral Graph Theory: Lecture 3

Ahmad Abdi Neil Olver

Last time, we saw the following theorem.

Theorem 3.3 (Cauchy's Interlacing Theorem)**.** Given an $n \times n$ real symmetric matrix A with spectrum $\theta_1 \geq \cdots \geq \theta_n$ and an $m \times m$ principal submatrix B with spectrum $\mu_1 \geq \cdots \geq \mu_m$, we have that

$$\theta_i \geq \mu_i \geq \theta_{n-m+i} \qquad i = 1, \ldots, m.$$

We also saw some applications to graph theory, prompted by the following consequence.

Proposition 4.1. Given a graph, its spectrum is interlaced by the spectrum of any induced subgraph.

Given a graph G, a *stable set* is any subset of pairwise non-adjacent vertices. Denote by $\alpha(G)$ the maximum cardinality of a stable set.

Theorem 4.4. Let G be a graph on n vertices, and let $\theta_1 \geq \cdots \geq \theta_n$ denote its spectrum. Then

$$\alpha(G) \leq |\{i \in [n] : \theta_i \geq 0\}| \quad \text{and} \quad \alpha(G) \leq |\{i \in [n] : \theta_i \leq 0\}|.$$

Proof. Let S be a stable set of cardinality $\alpha := \alpha(G)$. By Proposition 4.1, the spectrum of the subgraph induced on S, which is $0^{(\alpha)}$, interlaces the spectrum of G. This immediately proves the result. □

We shall provide another more powerful upper bound on the stability number.

4.1 On the Sensitivity Conjecture

Let us see a final application of Cauchy's Interlacing Theorem, this time to Boolean functions. Recall that the spectral radius of a real symmetric matrix A, denoted by $\rho(A)$, is the maximum absolute value of its eigenvalues.

Lemma 4.5. Let G be an n-vertex graph, and let A be an $n \times n$ real symmetric matrix such that $|A| \leq A(G)$. Then $\Delta(G) \geq \rho(A)$.

Proof. Exercise. □

Denote by Q_n the skeleton graph of the unit hypercube $[0,1]^n$. That is, Q_n has vertex set $\{0,1\}^n$ where two vertices are adjacent if they differ in exactly one coordinate, that is, if their Hamming distance is one.

The adjacency graph of Q_n is defined recursively as follows: $A(Q_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and for each integer $n \geq 2$, $A(Q_n) = \begin{pmatrix} A(Q_{n-1}) & I \\ I & A(Q_{n-1}) \end{pmatrix}$.

We shall use Cauchy's Interlacing Theorem in a clever way to prove that every induced subgraph of Q_n on at least $2^{n-1}+1$ vertices has a vertex of degree at least $\sqrt{n}$. We will need to work with an appropriate signing of the adjacency matrix $A(Q_n)$. Let $A_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and for each integer $n \geq 2$, let $A_n := \begin{pmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{pmatrix}$.

Lemma 4.6. The following statements hold:

1. A_n is a $2^n \times 2^n$ real symmetric matrix,
2. $|A_n| = A(Q_n)$,
3. $A_n^2 = nI$,
4. A_n has spectrum $-\sqrt{n}$ and $\sqrt{n}$, each with multiplicity 2^{n-1}.

Proof. **(1)** and **(2)** are immediate. **(3)** follows from induction combined with $A_n^2 = \begin{pmatrix} A_{n-1}^2+I & \mathbf{0} \\ \mathbf{0} & A_{n-1}^2+I \end{pmatrix}$. **(4)** It follows from (3) that every eigenvalue of A_n is either $\sqrt{n}$ or $-\sqrt{n}$. Since $\operatorname{tr}(A_n) = 0$, the result follows. □

As a consequence,

Theorem 4.7. Let G be an induced subgraph of Q_n with $2^{n-1}+1$ vertices. Then $\Delta(G) \geq \sqrt{n}$.

Proof. Note that $A(G)$ is a principal submatrix of $A(Q_n)$; let A be the corresponding principal submatrix of A_n. Then by Lemma 4.6, $|A| = A(G)$, so $\Delta(G) \geq \rho(A)$ by Lemma 4.5.

By Cauchy's Interlacing Theorem, the spectrum of A, say $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{|V(G)|}$, interlaces the spectrum of A_n, say $\theta_1 \geq \cdots \geq \theta_{2^n}$. By Lemma 4.6, the first 2^{n-1} θ_i's are equal to $\sqrt{n}$, while the last 2^{n-1} are equal to $-\sqrt{n}$. Thus, since $|V(G)| = 2^{n-1}+1$, interlacing implies that $\mu_1 \geq \sqrt{n}$.

Since $\rho(A) \geq \mu_1$, the two inequalities obtained imply that $\Delta(G) \geq \sqrt{n}$, as required. □

In the 90s, Gotsman and Linial proved that the statement above has a deep implication on the "sensitivity" of Boolean functions [1]. To elaborate, every Boolean function $f : \{0,1\}^n \to \{0,1\}$ can be uniquely expressed as a multilinear polynomial of degree at most n over the reals. The *degree* of f is then simply the degree of this polynomial. The *sensitivity* of $f : \{0,1\}^n \to \{0,1\}$ is the maximum over all inputs $x \in \{0,1\}^n$ of the number of coordinates which, when flipped in x, change f. One form of the so-called *Sensitivity Conjecture* claims that the degree of a Boolean function is polynomially upper-bounded by its sensitivity [4]. What Gotsman and Linial proved is that this conjecture is implied from Theorem 4.7. The theorem above was proved quite recently by Huang [3], and the proof we gave is almost identical to his proof.

5 The Courant-Hilbert-Haemers Theorem

In this section, we state a powerful extension of Cauchy's Interlacing Theorem, and prove it using the Courant-Fischer Theorem instead of interlacing polynomials. To this end, consider a sequence $\theta_1 \geq \cdots \geq \theta_n$ and a shorter one $\mu_1 \geq \cdots \geq \mu_m$ that interlaces the longer sequence. The interlacing is *tight* if, for some j,

$$\mu_i = \begin{cases} \theta_i & \text{for } i \leq j \\ \theta_{n-m+i} & \text{for } i \geq j+1. \end{cases}$$

In particular, the first j values of the shorter sequence are as large as possible, while the remaining $m-j$ values are as small as possible.

Theorem 5.1 (Courant-Hilbert-Haemers Theorem)**.** Take an integer $n \geq 2$, and let A be an $n \times n$ real symmetric matrix with eigenvalues $\theta_1 \geq \cdots \geq \theta_n$. For some integer $1 \leq m < n$, let S be an $n \times m$ real matrix such that $S^\top S = I_m$, and let $B := S^\top AS$. Let $v_1, \ldots, v_m$ be orthogonal eigenvectors for B corresponding to eigenvalues $\mu_1 \geq \cdots \geq \mu_m$, respectively. Then the following statements hold:

1. the eigenvalues of B interlace those of A,
2. if $\mu_i = \theta_i$ (resp. $\mu_i = \theta_{n-m+i}$), then B has a μ_i-eigenvector v such that Sv is a μ_i-eigenvector for A,
3. if $\mu_i = \theta_i$ for $i = 1, \ldots, j$ (resp. $\mu_i = \theta_{n-m+i}$ for $i = j, \ldots, m$), then Sv_i is a μ_i-eigenvector for A for $i = 1, \ldots, j$ (resp. $i = j, \ldots, m$),
4. if the interlacing is tight, then $SB = AS$.

Proof. **(1)** Let $u_1, \ldots, u_n$ be an orthogonal basis of eigenvectors for A with eigenvalues $\theta_1, \ldots, \theta_n$, respectively. For each $i \in [m]$, take a nonzero vector $w_i \in \mathbb{R}^m$ in

$$\langle v_1, \ldots, v_i \rangle \cap \langle S^\top u_1, \ldots, S^\top u_{i-1} \rangle^\perp.$$

(For $i = 1$, the RHS is $\langle v_1 \rangle$.) As $w_i \in \langle v_1, \ldots, v_i \rangle$, CFT (1) and (3) applied to B implies that

$$\frac{w_i^\top B w_i}{w_i^\top w_i} \geq \mu_i.$$

Our choice of w_i implies that $Sw_i \in \langle u_1, \ldots, u_{i-1} \rangle^\perp$, so CFT (2) and (4) applied to A implies that

$$\frac{w_i^\top S^\top AS w_i}{w_i^\top S^\top S w_i} \leq \theta_i.$$

Since $B = S^\top AS$ and $S^\top S = I_m$, we have

$$\theta_i \geq \frac{w_i^\top S^\top AS w_i}{w_i^\top S^\top S w_i} = \frac{w_i^\top B w_i}{w_i^\top w_i} \geq \mu_i.$$

A similar argument applied to $-A$ and $-B$ implies that $\mu_i \geq \theta_{n-m+i}$, thereby proving (1).

(2) If $\theta_i = \mu_i$, then equality holds throughout, so it follows from CFT parts (3) and (4) that w_i is a μ_i-eigenvector of B, while Sw_i is a μ_i-eigenvector of A, thereby proving (2).

(3) We proceed by induction on j. By the induction hypothesis, we may pick $u_i = Sv_i$ for $i = 1, \ldots, j-1$.[1] We may therefore pick $w_j = v_j$. Since $\theta_j = \mu_j$, it follows from (2) that Sv_j is a μ_j-eigenvector, thereby completing the induction step.

[1]By Lecture 0, Exercise 4 (c), for any integer $1 \leq m < n$, any set of m orthogonal eigenvectors can be extended to n orthogonal eigenvectors.

(4) For some j,

$$\mu_i = \begin{cases} \theta_i & \text{for } i \leq j \\ \theta_{n-m+i} & \text{for } i \geq j+1. \end{cases}$$

Thus, by applying (3) twice, we get that Sv_i is a μ_i-eigenvector for A for all $i \in [m]$. Consequently,

$$(SB - AS)v_i = SS^\top ASv_i - ASv_i = \mu_i SS^\top Sv_i - \mu Sv_i = 0 \quad \forall i \in [m].$$

Since $v_1, \ldots, v_m$ is a basis for $\mathbb{R}^m$, it follows that $SB = AS$, as required. □

6 Applications of the Courant-Hilbert-Haemers Theorem

We leave it as an exercise for the reader to prove Cauchy's Interlacing Theorem as an application of the Courant-Hilbert-Haemers Theorem. For now, let us see another application. Let A be an $n \times n$ real symmetric matrix, whose rows and columns

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix}$$

are partitioned according to a partition $X_1, \ldots, X_m$ of $[n]$ into nonempty parts. Observe that $A_{ij} = A_{ji}^\top$. Denote by S' the $n \times m$ matrix whose entries are defined as follows: $S'_{ij} = 1$ if $i \in X_j$, and $S'_{ij} = 0$ if $i \notin X_j$. The *quotient matrix* of the partition is the $m \times m$ matrix B' whose ij-entry is equal to the average row sum of the block A_{ij}, that is,

$$B'_{ij} = \frac{1}{|X_i|} \mathbf{1}^\top A_{ij} \mathbf{1} = \frac{1}{|X_i|} (S'^\top AS')_{ij}.$$

The partition is *equitable* if each block A_{ij} has constant row sum, that is, if $AS' = S'B'$. Since $A_{ij} = A_{ji}^\top$, each block of an equitable partition has constant column sum.

Theorem 6.1. Let A be a real symmetric matrix that is partitioned symmetrically, and let B' be the quotient matrix of the partition. Then the following statements hold:

1. the eigenvalues of B' interlace those of A,

2. if the interlacing is tight, then the partition is equitable.

Proof. Let $D := \text{Diag}(|X_1|, |X_2|, \ldots, |X_m|)$. Then $DB' = S'^\top AS'$, implying in turn that $D^{\frac{1}{2}} B' D^{-\frac{1}{2}} = D^{-\frac{1}{2}} S'^\top AS' D^{-\frac{1}{2}}$. Thus, for

$$B := D^{\frac{1}{2}} B' D^{-\frac{1}{2}}$$
$$S := S' D^{-\frac{1}{2}},$$

we have $B = S^\top AS$. As $S^\top S = D^{-\frac{1}{2}} S'^\top S' D^{-\frac{1}{2}} = D^{-\frac{1}{2}} D D^{-\frac{1}{2}} = I_m$, it follows from the Courant-Hilbert-Haemers Theorem that the eigenvalues of B interlace those of A, and if the interlacing is tight, then $SB = AS$, and so $S'B' = AS'$, meaning the partition is equitable. As B, B' are similar matrices, they have the same eigenvalues, so the theorem follows. □

Let us now present a powerful upper-bound on the stability number of a graph:

Theorem 6.2. Let G be an n-vertex graph, let A be an $n \times n$ real symmetric matrix where $A_{ij} \neq 0$ only if i and j are adjacent vertices of G, and let $\theta_1 \geq \cdots \geq \theta_n$ be the spectrum of A. Assume that A has constant row sum $k > 0$. Then

$$\alpha(G) \leq n \cdot \frac{-\theta_n}{k - \theta_n}.$$

Proof. Let $S \subseteq V$ be a stable set of cardinality $\alpha := \alpha(G)$. Consider the symmetric partitioning of A according to the partition $S, V \setminus S$ of the vertex set. The quotient matrix of this partition is

$$B' = \begin{pmatrix} 0 & k \\ \frac{k\alpha}{n-\alpha} & k - \frac{k\alpha}{n-\alpha} \end{pmatrix},$$

which has spectrum $\mu_1 = k, \mu_2 = -\frac{k\alpha}{n-\alpha}$ (note that k is the constant row sum of B', while the other eigenvalue is $\mathrm{tr}(B') - k$). By Theorem 6.1, $\mu_2 \geq \theta_{n-2+2} = \theta_n$, which in turn implies the desired inequality. □

Acknowledgements

The presentation of §5 and §6 follows Haemers [2] closely.

Exercises

1. Use Cauchy's Interlacing Theorem to prove that the Petersen graph has no Hamilton cycle.

2. Determine the graphs with smallest eigenvalue at least -1.

3. Prove Theorem 4.3.

4. Prove Lemma 4.5.

5. Prove Cauchy's Interlacing Theorem by using the Courant-Hilbert-Haemers Theorem.

6. Let $G = (V, E)$ be a k-regular graph with spectrum $k \geq \theta_2 \geq \cdots \geq \theta_n$. Prove that

   $$\alpha(G) \leq n \cdot \frac{-\theta_n}{k - \theta_n}.$$

   Moreover, prove that if S is a stable set meeting this bound, then every vertex outside of S has exactly $-\theta_n$ neighbours inside S.

7. Let G be a graph on n vertices with minimum degree δ, and let $\theta_1 \geq \cdots \geq \theta_n$ denote its spectrum. Prove that

   $$\alpha(G) \leq n \cdot \frac{-\theta_1\theta_n}{\delta^2 - \theta_1\theta_n}.$$

8. Let G be a graph on n vertices with at least one edge whose spectrum is $\theta_1 \geq \cdots \geq \theta_n$, and let x be an arbitrary eigenvector of $A := A(G)$. Let $X_1, \ldots, X_k$ be a partition of the vertex set into k nonempty stable sets, and let B be the $k \times k$ matrix where

$$B_{ij} = \frac{1}{\sum_{u \in X_i} x_u^2} \cdot \sum \left(x_u x_v : u \in X_i, v \in X_j,\ u, v \text{ are adjacent} \right).$$

   (a) Prove that the spectrum of B interlaces the spectrum of A.

   (b) Prove that $k \geq 1 - \frac{\theta_1}{\theta_n}$.

   (c) Conclude that G has chromatic number at least $1 - \frac{\theta_1}{\theta_n}$.

9. Let $G = (V, E)$ be a regular graph. Suppose S is a stable set such that every vertex in $V \setminus S$ has a unique neighbour in S. Prove that -1 is an eigenvalue of $A(G)$.

References

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