MA431 Lecture 5

Ahmad Abdi

London School of Economics and Political Science

Feb 18, 2022

Outline

1. Matrix-Tree Theorem
2. Weighted Matrix-Tree Theorem
3. Laplacian of weighted graphs
4. Cycle and cut spaces
5. Fundamental cycles and fundamental cuts
6. Bases for and the dimension of the cycle and cut spaces
7. ∇, grad, and ∇ grad linear operators

The Laplacian, the Matrix-Tree Theorem, and Consequences

Recap

- Let $G = (V, E)$ be an n-vertex graph.
- The Laplacian matrix of G is $L := \Delta(G) - A(G)$, where $\Delta(G) = \mathrm{Diag}(\deg(u) : u \in V)$.
- Denote by $\lambda_1(G) \leq \cdots \leq \lambda_n(G)$ the spectrum of L.
- Denote by $T(G)$ the number of spanning trees of G.

Proposition

- $L = \sum_{e=\{u,v\}\in E}(e_u - e_v)(e_u - e_v)^\top$,
- for each $x \in \mathbb{R}^V$, $x^\top L x = \sum_{e=\{u,v\}\in E}(x_u - x_v)^2$, ≥ 0 $\Big\} \to L$ is PSD
- $L\mathbf{1} = \mathbf{0}$ and $\lambda_1(G) = 0$,
- $\mathrm{rank}(L) = n - c$, where c is the number of connected components of G.

Recap

- Let $G = (V, E)$ be an n-vertex graph.
- The Laplacian matrix of G is $L := \Delta(G) - A(G)$, where $\Delta(G) = \mathrm{Diag}(\deg(u) : u \in V)$.
- Denote by $\lambda_1(G) \leq \cdots \leq \lambda_n(G)$ the spectrum of L.
- Denote by $T(G)$ the number of spanning trees of G.

Proposition

- $L = \sum_{e=\{u,v\}\in E}(e_u - e_v)(e_u - e_v)^\top$,
- for each $x \in \mathbb{R}^V$, $x^\top L x = \sum_{e=\{u,v\}\in E}(x_u - x_v)^2$,
- $L\mathbf{1} = \mathbf{0}$ and $\lambda_1(G) = 0$,
- $\mathrm{rank}(L) = n - c$, where c is the number of connected components of G.

Deletion-contraction formula

For each $e \in E$, $T(G) = T(G \setminus e) + T(G/e)$.

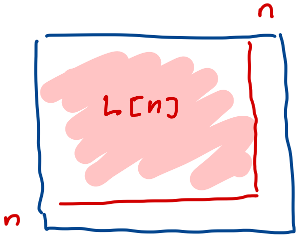
The Matrix-Tree Theorem

The Matrix-Tree Theorem

$T(G) = \det(L')$ where L' is any $(n-1)\times(n-1)$ principal submatrix of L.

- $L[i]$: the submatrix of L obtained after removing row i and column i.
- It suffices to prove that $\det(L[n]) = T(G)$.
- We proceed by induction.

$L =$

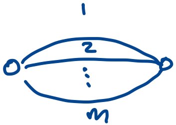


The Matrix-Tree Theorem

The Matrix-Tree Theorem

$T(G) = \det(L')$ where L' is any $(n-1)\times(n-1)$ principal submatrix of L.

Base case: $n = 2$. If G consists of m parallel edges, then $L = \begin{pmatrix} \boxed{m} & -m \\ -m & m \end{pmatrix}$, so the result follows.



$T(G) = m$

$\det(L') = m$

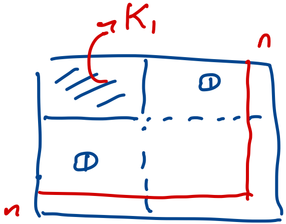
The Matrix-Tree Theorem

The Matrix-Tree Theorem

$T(G) = \det(L')$ where L' is any $(n-1) \times (n-1)$ principal submatrix of L.

Base case: G is not connected.

- $L[n]$ is singular, so $\det(L[n]) = 0$.
- Clearly, $T(G) = 0$.
- So $\det(L[n]) = T(G)$.



The Matrix-Tree Theorem

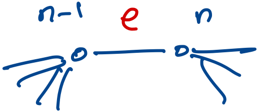
The Matrix-Tree Theorem

$T(G) = \det(L')$ where L' is any $(n-1)\times(n-1)$ principal submatrix of L.

Induction step: $n \geq 3$ and G connected.

- Pick $e \in E$ incident with $n \in V$, say $e = \{n-1, n\}$.
- We know

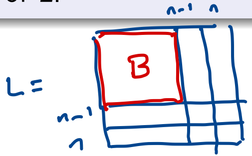
$$T(G) = T(G/e) + T(G \setminus e).$$



The Matrix-Tree Theorem

The Matrix-Tree Theorem

$T(G) = \det(L')$ where L' is any $(n-1) \times (n-1)$ principal submatrix of L.

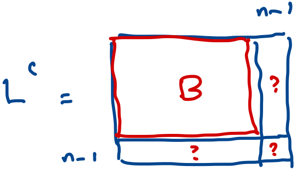


Induction step: $n \geq 3$ and G connected.

- Pick $e \in E$ incident with $n \in V$, say $e = \{n-1, n\}$.
- Denote by L^c the Laplacian matrix of G/e. By IH,

$$T(G/e) = \det(L^c[n-1])$$

$$= \det(L[n][n-1])$$



The Matrix-Tree Theorem

The Matrix-Tree Theorem

$T(G) = \det(L')$ where L' is any $(n-1) \times (n-1)$ principal submatrix of L.

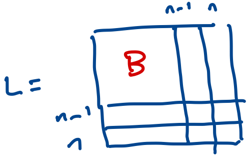
Induction step: $n \geq 3$ and G connected.

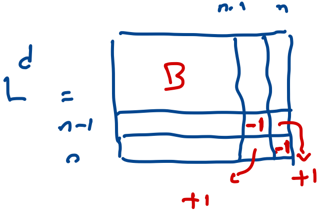
- Denote by L^d the Laplacian matrix of $G \setminus e$. By IH,

$$T(G \setminus e) = \det(L^d[n]) =$$

$$= \det(L[n]) - \det(\overbrace{L[n][n-1]}^{B})$$

by Laplace expansion along row n-1





The Matrix-Tree Theorem

The Matrix-Tree Theorem

$T(G) = \det(L')$ where L' is any $(n-1) \times (n-1)$ principal submatrix of L.

Induction step: $n \geq 3$ and G connected.

- In summary,

$$\begin{aligned} \det(L^c[n-1]) &= \det(L[n][n-1]) \\ \det(L^d[n]) &= \det(L[n]) - \det(L[n][n-1]). \end{aligned}$$

- Thus,

$$T(G) = T(G/e) + T(G \setminus e) = \det(L^c[n-1]) + \det(L^d[n]) = \det(L[n])$$

thereby completing the IS.

Consequences

complete n-vertex graph

Cayley's formula

The number of spanning trees of K_n is n^{n-2}.

Proof.

The Laplacian matrix is $L := nI_n - J_n$, so $L[n] = nI_{n-1} - J_{n-1}$ whose spectrum is

$$(n-0)^{(n-2)}, \quad (n-(n-1))^{(1)}$$

so $\det(L[n]) =$ product of its eigenvalues $= n^{n-2} \times 1$. ∎

Consequences

Theorem

$T(G) = \frac{1}{n} \prod_{i=2}^{n} \lambda_i(G)$.

Proof.

Exercise. □

Theorem

Every entry of $\mathrm{adj}(L)$ is equal to $T(G)$.

Proof.

Exercise. □

Kirchhoff polynomial

- For every $e \in E$, introduce a variable x_e.
- The Kirchhoff polynomial of G is

$$\mathrm{Kir}(G; x) = \sum_{\text{spanning tree } T} \prod_{e \in T} x_e$$

- Note $T(G) = \mathrm{Kir}(G; \mathbb{1})$

Deletion-contraction formula

For $e \in E$, $\mathrm{Kir}(G; x) = x_e \cdot \mathrm{Kir}(G/e; x^e) + \mathrm{Kir}(G \setminus e; x^e)$.

Kirchhoff polynomial

- For every $e \in E$, introduce a variable x_e.
- The Kirchhoff polynomial of G is

$$\mathrm{Kir}(G;x) = \sum_{\text{spanning tree } T} \prod_{e \in T} x_e$$

- Note $T(G) = \mathrm{Kir}(G;\mathbf{1})$.

Deletion-contraction formula

For $e \in E$, $\mathrm{Kir}(G;x) = x_e \cdot \mathrm{Kir}(G/e;x^e) + \mathrm{Kir}(G \setminus e;x^e)$.

x^e is the vector obtained from x after dropping coordinate e.

Proof.

Exercise. $\square$

Weighted Matrix-Tree Theorem

Define $L(G, x)$ to be the $V \times V$ matrix s.t.

- for $u \in V$: the (u, u)-entry is $\sum(x_e : e \in E$ is incident with $v)$,
- for distinct $u, v \in V$: the (u, v)-entry is $-\sum(x_e : e \in E$ has ends $u, v)$.

<u>Note</u>: $L(G, \mathbb{1}) = L$ the Laplacian

Weighted Matrix-Tree Theorem

Define $L(G, x)$ to be the $V \times V$ matrix s.t.

- for $u \in V$: the (u, u)-entry is $\sum(x_e : e \in E \text{ is incident with } v)$,
- for distinct $u, v \in V$: the (u, v)-entry is $-\sum(x_e : e \in E \text{ has ends } u, v)$.

The Weighted Matrix-Tree Theorem

$\mathrm{Kir}(G; x) = \det(L')$ where L' is any $(n-1) \times (n-1)$ principal submatrix of $L(G, x)$.

Proof.

Exercise. $\square$

Laplacian of weighted graph

Define $w \in \mathbb{R}^E$ such that $w \geq \mathbf{0}$.

Definition

The Laplacian matrix of the weighted graph (G, w) is $L_w := L(G, w)$.

Laplacian of weighted graph

Define $w \in \mathbb{R}^E$ such that $w \geq \mathbf{0}$.

Definition

The Laplacian matrix of the weighted graph (G, w) is $L_w := L(G, w)$.

Proposition

1. $L_w = \sum_{e=\{u,v\}\in E} w_e \cdot (e_u - e_v)(e_u - e_v)^\top$,
2. for each $x \in \mathbb{R}^V$, $x^\top L_w x = \sum_{e=\{u,v\}\in E} w_e (x_u - x_v)^2$, ≥ 0 b/c $w \geq 0$

Laplacian of weighted graph

Define $w \in \mathbb{R}^E$ such that $w \geq \mathbf{0}$.

Definition

The Laplacian matrix of the weighted graph (G, w) is $L_w := L(G, w)$.

Proposition

1. $L_w = \sum_{e=\{u,v\}\in E} w_e \cdot (e_u - e_v)(e_u - e_v)^\top$,
2. for each $x \in \mathbb{R}^V$, $x^\top L_w x = \sum_{e=\{u,v\}\in E} w_e(x_u - x_v)^2$,
3. L_w is PSD,
4. $L_w\mathbf{1} = \mathbf{0}$, and 0 is the least eigenvalue of L_w,
5. if $w > \mathbf{0}$, then $\mathrm{rank}(L_w) = n - c$, where c is the number of connected components of G.

Proof.

Exercise. $\square$

Laplacian of weighted graph

Define $w \in \mathbb{R}^E$ such that $w \geq \mathbf{0}$.

Definition

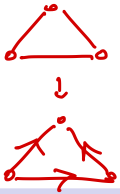
The Laplacian matrix of the weighted graph (G, w) is $L_w := L(G, w)$.

Given a weighted graph (G, w), we may assume G is simple and $w > \mathbf{0}$.

$w_e = 0$ "corresponds to" deleting edge e

The cycle and cut spaces

The incidence matrix of an orientation



- Let $G = (V, E)$ be a **connected** graph with n vertices and m edges.
- Denote by $\vec{G} = (V, \vec{E})$ an arbitrary orientation of G.

Recall

The incidence matrix of $\vec{G}$ is the $V \times \vec{E}$ matrix B where column $(v, u) \in \vec{E}$ is equal to $e_u - e_v$.

Recall

The Laplacian matrix of G is $BB^\top$.

$$B = \begin{array}{c} (v,u) \\ \begin{bmatrix} \cdots & 1 & \cdots \\ \cdots & -1 & \cdots \end{bmatrix} \end{array} \begin{array}{c} \\ u \\ v \end{array}$$

Cycle and cut spaces

Cycle space

$$W^\diamond = \left\{ f \in \mathbb{R}^{\vec{E}} : Bf = \mathbf{0} \right\}$$ Kernel of B

Cut space

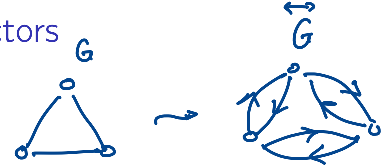
$$W^\star = \left\{ B^\top \pi : \pi \in \mathbb{R}^V \right\}$$ row space of B

Remark

1. $W^\star = (W^\diamond)^\perp$ and $W^\diamond = (W^\star)^\perp$
2. $\mathbb{R}^{\vec{E}} = W^\diamond \oplus W^\star$, that is, $\mathbb{R}^{\vec{E}}$ is decomposed into two orthogonal subspaces.

We shall count the dimensions of $W^\diamond$, $W^\star$, and provide bases for each.

The bidirection and signed characteristic vectors

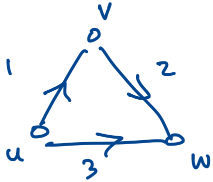


- Denote by $\overleftrightarrow{G} = (V, \overleftrightarrow{E})$ the bidirection of G.
- For each $e \in \overleftrightarrow{E}$, the signed characteristic vector of e is the vector $\chi^e \in \mathbb{R}^{\overrightarrow{E}}$ defined as

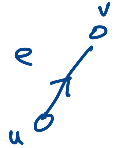
$$(\chi^e)_a = \begin{cases} 1 & \text{if } a = e, \\ -1 & \text{if } a \text{ is the reverse of } e, \\ 0 & \text{otherwise.} \end{cases}$$

- For any $F \subseteq \overleftrightarrow{E}$, define

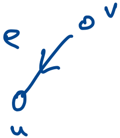
$$\chi(F) := \sum_{e \in F} \chi^e.$$



$\vec{G}$

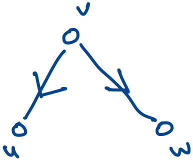


$$: \quad \chi^e = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3$$



$$: \quad \chi^e = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3$$

$F =$



$$\chi(F) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

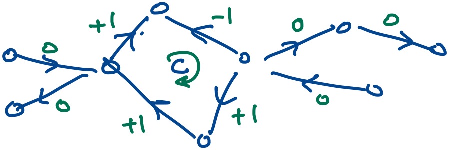
Directed cycles

Lemma

Let $C \subseteq \overleftrightarrow{E}$ be a directed cycle of $\overleftrightarrow{G}$. Then $\chi(C) \in W^{\diamond}$.

We need to show

$$B\,\chi(C) = \mathbf{0}.$$



$\vec{G}$ $\chi(C)$

Consider row $v \in V$:

$$(B\,\chi(C))_v = \begin{cases} 1-1 & \text{if } v \text{ is incident with two edges of } C \\ 0 & \text{o/w} \end{cases}$$

$$= 0$$

Directed cycles

Lemma

Let $C \subseteq \overleftrightarrow{E}$ be a directed cycle of $\overleftrightarrow{G}$. Then $\chi(C) \in W^\diamond$.

Corollary

$W^\diamond \supseteq \operatorname{span}\{\chi(C) : C \text{ is a directed cycle of } \overleftrightarrow{G}\}$.

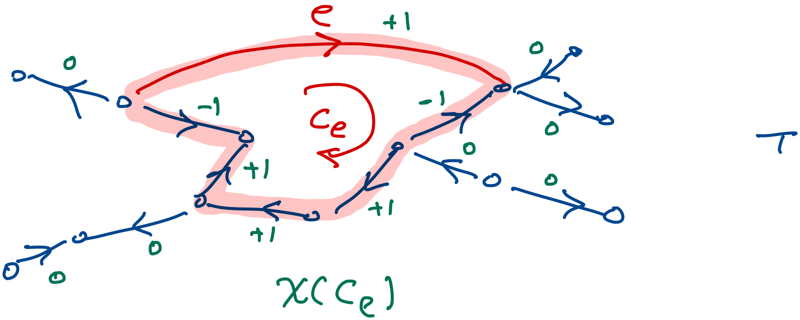
We'll see that $=$ holds here.

Fundamental cycles

Let $T \subseteq \vec{E}$ be a spanning tree. Let $e \in \vec{E} \setminus T$.

Fundamental cycle

The fundamental cycle associated with e w.r.t T is the unique cycle in $T \cup \{e\}$.
Think of this as a directed cycle $C_e \subseteq \overleftrightarrow{E}$ where e appears in the forward direction.



Fundamental cycles

Let $T \subseteq \vec{E}$ be a spanning tree. Let $e \in \vec{E} \setminus T$.

Fundamental cycle

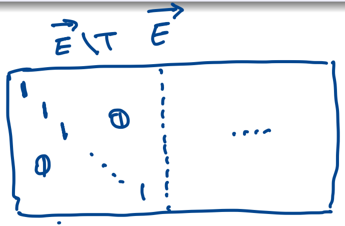
The fundamental cycle associated with e w.r.t T is the unique cycle in $T \cup \{e\}$.
Think of this as a directed cycle $C_e \subseteq \overleftrightarrow{E}$ where e appears in the forward direction.

Lemma

$\{\chi(C_e) : e \in \vec{E} \setminus T\}$ are linearly independent in $W^\diamond$.

$\chi(C_e)$

$e \in \vec{E} \setminus T$



(this is not B)

Fundamental cycles

Let $T \subseteq \vec{E}$ be a spanning tree. Let $e \in \vec{E} \setminus T$.

Fundamental cycle

The fundamental cycle associated with e w.r.t T is the unique cycle in $T \cup \{e\}$.
Think of this as a directed cycle $C_e \subseteq \overleftrightarrow{E}$ where e appears in the forward direction.

Lemma

$\{\chi(C_e) : e \in \vec{E} \setminus T\}$ are linearly independent in $W^\diamond$.

Corollary

$\dim(W^\diamond) \geq m - n + 1$. (note $|\vec{E} \setminus T| = m - n + 1$)

We'll show $\{\chi(C_e) : e \in \vec{E} \setminus T\}$ is a basis for $W^\diamond$, and $\underbrace{\dim(W^\diamond)}_{= m-n+1}$

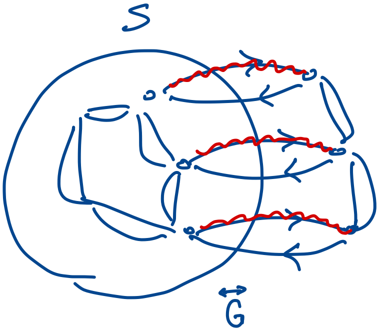
Cuts

Let S be a nonempty proper subset of V.

Definition

The cut of $\overleftrightarrow{G}$ associated with S is

$$\delta^+(S) := \{(u,v) \in \overleftrightarrow{E} : u \in S, v \notin S\}.$$



. . .

$\delta^+(S)$ ~

Cuts

Let S be a nonempty proper subset of V.

Definition

The cut of $\overleftrightarrow{G}$ associated with S is

$$\delta^+(S) := \{(u,v) \in \overleftrightarrow{E} : u \in S, v \notin S\}.$$

Lemma

$\chi(\delta^+(S)) \in W^\star$.

Cuts

Let S be a nonempty proper subset of V.

Definition

The cut of $\overleftrightarrow{G}$ associated with S is

$$\delta^+(S) := \{(u,v) \in \overleftrightarrow{E} : u \in S, v \notin S\}.$$

Remark

The row of B corresponding to node v is precisely $-\chi(\delta^+(v))$.

Lemma

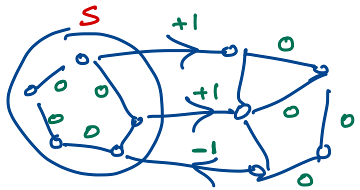
$\chi(\delta^+(S)) \in W^\star$.

$$x(\delta^+(S)) = \sum_{v \in S} x(\delta^+(v))$$

$$= \sum_{v \in S} -(\text{row } v \text{ of } B)$$

$$\in \text{row space of } B$$

$$= W^*$$



$x(\delta^+(S))$

Cuts

Let S be a nonempty proper subset of V.

Definition

The cut of $\overset{\leftrightarrow}{G}$ associated with S is

$$\delta^+(S) := \{(u,v) \in \overset{\leftrightarrow}{E} : u \in S, v \notin S\}.$$

Lemma

$\chi(\delta^+(S)) \in W^\star$.

Corollary

$W^\star \supseteq \operatorname{span}\{\chi(\delta^+(S)) : S \subseteq V, S \neq \emptyset, V\}$

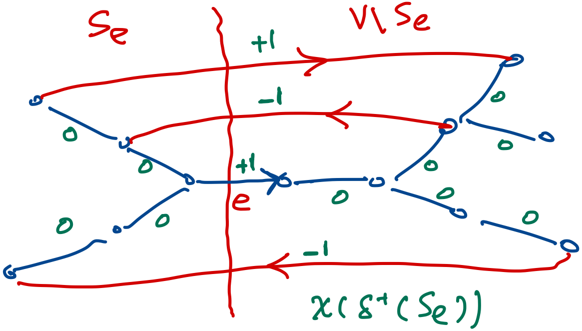
We'll see that $=$ holds here.

Fundamental cuts

Let $T \subseteq \vec{E}$ be a spanning tree. Let $e \in T$.

Fundamental cut

The fundamental cut associated with e w.r.t T is the unique cut intersecting T at exactly $\{e\}$. Think of this as a cut $\delta^+(S_e) \subseteq \overleftrightarrow{E}$ where $S_e, V \setminus S_e$ are the two connected components of $T \setminus e$, where S_e contains the tail of e.



Fundamental cuts

Let $T \subseteq \vec{E}$ be a spanning tree. Let $e \in T$.

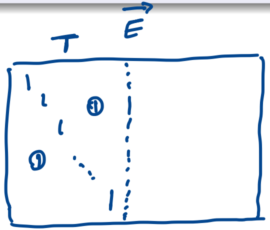
Fundamental cut

The fundamental cut associated with e w.r.t T is the unique cut intersecting T at exactly $\{e\}$. Think of this as a cut $\delta^+(S_e) \subseteq \overleftrightarrow{E}$ where $S_e, V \setminus S_e$ are the two connected components of $T \setminus e$, where S_e contains the tail of e.

Lemma

$\{\chi(\delta^+(S_e)) : e \in T\}$ are linearly independent in $W^\star$.

$\chi(\delta^+(S_e))$:
$e \in T$



(this is not B)

Fundamental cuts

Let $T \subseteq \vec{E}$ be a spanning tree. Let $e \in T$.

Fundamental cut

The fundamental cut associated with e w.r.t T is the unique cut intersecting T at exactly $\{e\}$. Think of this as a cut $\delta^+(S_e) \subseteq \overleftrightarrow{E}$ where $S_e, V \setminus S_e$ are the two connected components of $T \setminus e$, where S_e contains the tail of e.

Lemma

$\{\chi(\delta^+(S_e)) : e \in T\}$ are linearly independent in $W^\star$.

Corollary

$\dim(W^\star) \geq n - 1$.

Cycle and cut spaces

We have shown $\dim(W^\diamond) \geq m - n + 1$ and $\dim(W^\star) \geq n - 1$.

Recall

$\mathbb{R}^{\vec{E}} = W^\diamond \oplus W^\star$.

$\therefore \dim(W^\diamond) + \dim(W^*) = m$

Cycle and cut spaces

Theorem

$\dim(W^\diamond) = m - n + 1$ and $\dim(W^\star) = n - 1$.

Cycle and cut spaces

Theorem

$\dim(W^\diamond) = m - n + 1$ and $\dim(W^\star) = n - 1$.

Consequence for the cycle space

1. $\{\chi(C_e) : e \in \vec{E} \setminus T\}$ is a basis for $W^\diamond$.
2. $W^\diamond = \mathrm{span}\{\chi(C) : C$ is a directed cycle of $\overleftrightarrow{G}\}$.

Cycle and cut spaces

Theorem

$\dim(W^\diamond) = m - n + 1$ and $\dim(W^\star) = n - 1$.

Consequence for the cycle space

- $W^\diamond$ = kernel of B

1. $\{\chi(C_e) : e \in \vec{E} \setminus T\}$ is a basis for $W^\diamond$.
2. $W^\diamond = \mathrm{span}\{\chi(C) : C \text{ is a directed cycle of } \overleftrightarrow{G}\}$.

Consequence for the cut space

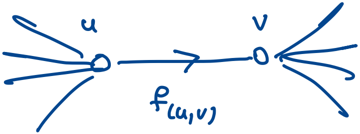
- $W^\star$ = row space of B
- row v of B = $\chi(\delta^+(v))$

1. $\{\chi(\delta^+(S_e)) : e \in T\}$ is a basis for $W^\star$.
2. $W^\star = \mathrm{span}\{\chi(\delta^+(S)) : S \subseteq V, S \neq \emptyset, V\}$.
3. $W^\star = \mathrm{span}\{\chi(\delta^+(r)) : r \in V\}$.
4. $\{\chi(\delta^+(r)) : r \in V \setminus t\}$ is a basis for $W^\star$, for any $t \in V$.

∇, grad, and ∇ grad

The linear operator ∇

Let $f \in \mathbb{R}^{\vec{E}}$. Think of f as a flow on $\vec{G}$ where $f_{(u,v)}$ is the amount of flow going from u to v.

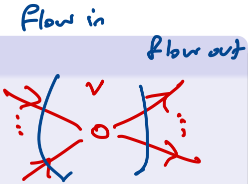


$$B = \begin{array}{c} v \\ \end{array} \begin{pmatrix} \overbrace{1 \cdots 1}^{\text{entering}} & \overbrace{-1 \cdots 1}^{\text{leaving}} \\ & \end{pmatrix}$$

Net flow

For each $v \in V$, define

$$\nabla f_v \; := \; (\nabla f)_v := \overbrace{\sum_{(u,v)\in\vec{E}} f_{(u,v)}}^{\text{flow in}} - \overbrace{\sum_{(v,w)\in\vec{E}} f_{(v,w)}}^{\text{flow out}}.$$



$\nabla : \mathbb{R}^{\vec{E}} \to \mathbb{R}^{V}$ is a linear operator represented by the matrix B. That is, for each $f \in \mathbb{R}^{\vec{E}}$, the image ∇f is the matrix product Bf.

Demand vectors

Question

Given a demand vector b $\in \mathbb{R}^V$, does there exist a flow $f \in \mathbb{R}^{\vec{E}}$ s.t. $b = \nabla f$?

Exercise

If $b = \nabla f$ for some flow f, then $\sum_{v \in V} b_v = 0$.

Demand vectors

Question

Given a demand vector b, does there exist a flow $f \in \mathbb{R}^{\vec{E}}$ s.t. $b = \nabla f$?

Exercise

If $b = \nabla f$ for some flow f, then $\sum_{v \in V} b_v = 0$.

Exercise

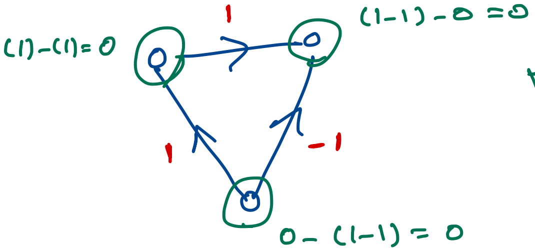
Give a characterization of all demand vectors b such that $b = \nabla f$ for some flow f.

Circulations

Definition

A circulation of $\vec{G}$ is a flow $f \in \mathbb{R}^{\vec{E}}$ s.t. $\nabla f = \mathbf{0}$.

f



∇f_v

Remark

$W^\diamond$ is precisely the set of all circulations of $\vec{G}$.

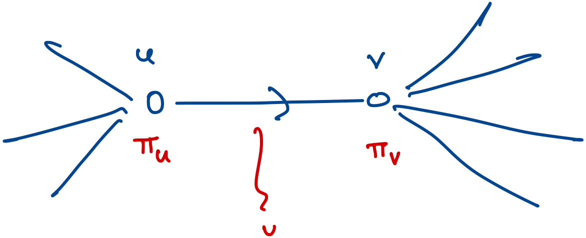
$$W^\diamond = \{f : Bf = 0\} = \{f : \nabla f = 0\}.$$

The linear operator grad

Definition (gradient)

Let grad : $\mathbb{R}^V \to \mathbb{R}^{\vec{E}}$ be the operator defined as follows: for each $\pi \in \mathbb{R}^V$,

$$(\mathrm{grad}\,\pi)_{(u,v)} = \pi_v - \pi_u$$



$$B = \begin{array}{cc} & (u,v) \\ \begin{matrix} v \\ u \end{matrix} & \begin{pmatrix} \cdots & 1 \\ \cdots & -1 \end{pmatrix} \end{array}$$

$$(\mathrm{grad}\,\pi)_{(u,v)} = \pi_v - \pi_u$$

The linear operator grad

grad is a linear operator represented by the matrix $B^\top$. That is, for each $\pi \in \mathbb{R}^V$, the image $\mathrm{grad}\,\pi$ is the matrix product $B^\top \pi$.

Range of grad

$\mathrm{range}(\mathrm{grad}) = \{B^\top \pi : \pi \in \mathbb{R}^V\} \quad = \quad W^{*}$

The Laplacian as a differential operator

Summary

1. ∇ is represented by B.
2. grad is represented by $B^\top$.

The Laplacian as a differential operator

Summary

1. ∇ is represented by B.
2. grad is represented by $B^\top$.

Lemma

∇ and grad are adjoint operators: $\langle \nabla f, \pi \rangle = \langle f, \operatorname{grad} \pi \rangle$.

$$\langle \nabla f, \pi \rangle = \langle Bf, \pi \rangle = \langle f, B^T \pi \rangle$$
$$= \langle f, \operatorname{grad} \pi \rangle$$

The Laplacian as a differential operator

Summary

1. ∇ is represented by B.
2. grad is represented by $B^\top$.

Lemma

∇ and grad are adjoint operators: $\langle \nabla f, \pi \rangle = \langle f, \operatorname{grad} \pi \rangle$.

Lemma

$\nabla \operatorname{grad} : \mathbb{R}^V \to \mathbb{R}^V$ is a linear operator represented by the Laplacian matrix L of G.

$$(\nabla \operatorname{grad})\pi = B B^T \pi = L\pi$$