# MA431 Spectral Graph Theory: Lecture 6

Ahmad Abdi    Neil Olver

## 11 Electrical flows and electrical potentials

Recall that $G = (V, E)$ is an $n$-vertex connected graph on $n$ vertices and $m$ edges, and $\vec{G} = (V, \vec{E})$ is an arbitrary orientation of $G$. Last time we defined the cut space $W^\star$ and the cycle space $W^\diamond$ of $G$, which were orthogonal complement subspaces of $\mathbb{R}^{\vec{E}}$. Let $P_\star : \mathbb{R}^E \to \mathbb{R}^E$ denote the linear operator that projects orthogonally onto $W^\star$. We will study this operator in some detail.

Let $\Pi$ be the $m \times m$ matrix that represents $P_\star$ in the basis $\{\chi^e : e \in \vec{E}\}$. That is, $\Pi_{e,e'}$ is the amount of flow on arc $e$ in the flow $P_\star \chi^{e'}$ obtained from the unit flow across $e'$. Since $P_\star$ is an orthogonal projection and hence self-adjoint, $\Pi$ is symmetric. We will mostly not need to refer to a specific basis, but we will use this on occasions (and more later).

So let $f \in \mathbb{R}^E$ be some flow, and let $i = P_\star f$. We first make the observation that $\nabla f = \nabla i$, since $f - i \in W^\diamond$. Moreover, $i$ depends only on $\nabla f$; if $\nabla f = \nabla g$, then $P_\star f = P_\star g$.

Now suppose $f$ satisfies $\nabla f = \mathrm{e}_t - \mathrm{e}_s$, for some $s, t \in V$. (In particular, if $(s,t) \in \vec{E}$, we can take $f = \chi^{(s,t)}$.) So:

- $i$ sends one unit of flow from $v$ to $w$, and satisfies flow conservation (flow in equals flow out) everywhere else.
- By Lemma 10.3, we have that $i = \operatorname{grad} \pi$ for some $\pi \in \mathbb{R}^V$. In other words, $i_e = \pi_w - \pi_v$ for all $e = (v, w) \in \vec{E}$.

What we have described are nothing more than Kirchhoff's famous laws (as well as Ohm's law) for electrical flow. We think of edges of our graph as being resistors, all with unit resistance. The values $\pi$ represent *electrical potentials*[1] corresponding to the electrical flow $i$; flow on an edge is precisely given by the potential difference. In the setup described, we have a unit current from $s$ to $t$; physically speaking, we can imagine that we have a battery set up that is connected to $s$ and to $t$. In the scaling we have chosen, we have fixed the current to be unit, with the voltage difference between $s$ and $t$ not determined. That is, $\pi_t - \pi_s$ would be the voltage of the battery. Of course, we could scale differently: if we consider $i' = i/(\pi_t - \pi_s)$, then the potentials associated with $i'$

[1] Actually, the negations of electrical potentials—the usual convention is that electrical current flows "downhill" with respect to the potential. Nevertheless we'll just call them potentials.

are $\pi' = \pi/(\pi_t - \pi_s)$, and so $\pi'_t - \pi'_s = 1$. So $i'$ would be the electrical flow associated with a battery of unit voltage.

So $P_\star$ is the operator that takes a flow $f$, and returns the electrical flow that "corresponds" to it, in the sense of having the same net flow as $f$ everywhere. We can also now read off immediately the well-known fact that electrical flows have minimum *energy*: the energy of a flow is just its squared norm.

**Lemma 11.1.** *For any flow $f \in \mathbb{R}^E$, $i = P_\star f$ is the unique minimizer of $\|g\|$ amongst all flows $g$ satisfying $\nabla g = \nabla f$ and $g = P_\star f$.*

*Proof.* Let $i = P_\star f$. For any $g$ with $\nabla g = \nabla f$, $g - i \in W^\diamond$, and so $\langle i, g - i\rangle = 0$. We deduce that $\|g\|^2 = \|i\|^2 + \|g - i\|^2$, and the claim is immediate. □

**Definition 11.2.** The *effective resistance* of an edge $e = (s,t) \in \vec{E}$ is the potential difference $\pi_t - \pi_s$ of a potential $\pi$ corresponding to the electrical flow $i = P_\star \chi^e$.

The name is motivated by imagining replacing the graph with a single resistor between the endpoints of $e$, and setting the resistance of this resistor equal to the effective resistance. Then the potential difference corresponding to a unit current through this resistor is the same as in the original circuit. There are a number of equivalent formulations of the effective resistance (the final one requires a small adjustments in the weighted case that we will discuss later).

- It is the inverse of the total amount of current that flows between the endpoints of $e$ if a unit voltage battery is applied across $e$ (just consider the rescaling $i/(\pi_t - \pi_s)$).
- It is the energy of $i$. To see this, observe that

$$\langle i, \chi^e\rangle = \langle P_\star i, \chi^e\rangle = \langle i, P_\star \chi^e\rangle = \|i\|^2.$$

- It is the current $i_e$ (since $i_e = \pi_t - \pi_s$), and hence also the entry $\Pi_{e,e}$ of the matrix associated with $P_\star$.

We can also define the effective resistance between any two nodes $u, v \in V$, even if they are not connected by an edge. Take $i$ to be the electrical flow with $\nabla i = \mathrm{e}_v - \mathrm{e}_u$, and let $\pi$ be associated potentials; the effective resistance between $u$ and $v$ is then simply $\pi_v - \pi_u$.

The effective resistance turns out to be a very important quantity. Edges with large effective resistance can be considered "more important" than ones with small effective resistance: for example, if an edge is a cut in the graph (removing the edge disconnects it), it's effective resistance will be 1, which is as large as possible (why?). We will see some applications later.

It will be very useful to have a concrete description of $P_\star$. How can we actually *compute* $i = P_\star f$? We proceed as follows. First, compute $b = \nabla f$; as already noted, this is the only information we need about $f$. Let $\pi$ be a potential associated with the current $i$ we are trying to compute; it suffices to find $\pi$, since then $i = \operatorname{grad} \pi$ follows immediately. So given $b$, we would like to compute $\pi$. To see how to do this, let's consider the much easier question of going from $\pi$ to $b$: we have that $b = \nabla i = \nabla \operatorname{grad} \pi = L\pi$, by Lemma 10.4.

So to get $\pi$ from $b$, we simply invert: $\pi = L^+ b$. (Notice, though, that any $\pi$ that solves $L\pi = b$ would also work; this would differ from $L^+ b$ by a multiple of the all-ones vector, and shifting all potentials by a constant makes no difference.) Putting this all together,

$$P_\star f = \mathrm{grad}\,\pi = \mathrm{grad}\,L^+ b = \mathrm{grad}\,L^+ \nabla f.$$

In other words, we can write the operator $P_\star$ as $\mathrm{grad}\,L^+\nabla$. Or written in terms of the standard basis, we have $\Pi = B^\top L^+ B$. (For maximal possible confusion, this can be further expanded as $\Pi = B^\top (BB^\top)^+ B \ldots$)

Note that this also gives us another way to write the effective resistance of an edge $e = (s,t)$. It is

$$\Pi_{e,e} = (B^\top L^+ B)_{e,e} = (\mathrm{e}_t - \mathrm{e}_s)^\top L^+ (\mathrm{e}_t - \mathrm{e}_s). \tag{1}$$

It is also worth interpreting the equation $L\pi = b$ in the case $b = \mathrm{e}_t - \mathrm{e}_s$. This tells us in particular that $\pi$ is *harmonic* on $V \setminus \{s,t\}$: for any $v \neq s,t$, we have that

$$\pi_v = \frac{1}{\deg(v)} \sum_{w:\{v,w\}\in E} \pi_w.$$

**Remark 11.3.** There is a very well-developed theory about solving linear Laplacian systems. That is, given a matrix $L$ that is the Laplacian of some graph, and some vector $b$, solve the system $Lx = b$. There are algorithms that solve (with some small error that can be specified) these systems in "near-linear time", meaning time $O(m \operatorname{polylog} m)$, which is basically the best you could hope for; $O(m)$ time is needed just to read the graph.

The above shows that solving such a system fast is equivalent to implementing $P_\star$ fast—and this is indeed crucially exploited in these fast algorithms. We won't develop the theory in this course; more information can be found in a monograph by Vishnoi [3].

The following is called the *Rayleigh monotonicity principle*. It says that adding edges to the graph can only decrease the effective resistance.

**Theorem 11.4.** *Let $G' = (V, E')$ with $E' \supseteq E$. Then for any two nodes $u, v \in V$, the effective resistance between $u$ and $v$ in $G'$ is smaller (or equal to) its value in $G$.*

*Proof.* Exercise. □

# 12 Kirchhoff's effective resistance theorem

So we now know how to compute the effective resistance of an edge, by (1); we simply need to solve a linear system. But let's do something that's usually not very useful, and obtain a "formula" for the effective resistance via Cramer's rule.

Assume vertices are labelled so that the edge of interest is $e = (n, n-1)$; we wish to find some $\pi$ for which $L\pi = \mathrm{e}_{n-1} - \mathrm{e}_n$, and then the desired effective resistance is $\pi_{n-1} - \pi_n$. Since $\mathbf{1}$ is in the kernel of $L$, we can restrict our attention to solutions with $\pi_n = 0$ (note that this means we are not choosing $\pi = L^+(\mathrm{e}_{n-1} - \mathrm{e}_n)$; our choice will differ by a shift). We can also drop the last row of the system, since it is a linear combination of the other constraints. Thus, we can solve the system $(L[n])y = \mathrm{e}_{n-1}$; a solution $y$ represents the restriction of $\pi$ to $\mathbb{R}^{n-1}$, and $y_{n-1}$ is the effective resistance.

We now apply Cramer's rule to the now invertible system. Let $L'$ be obtained from $L[n]$ by replacing the $(n-1)$'th column by $\mathrm{e}_{n-1}$. Then

$$y_{n-1} = \frac{\det(L')}{\det(L[n])} = \frac{\det(L[n][n-1])}{\det(L[n])}.$$

The last equality comes from expanding the determinant of $L'$ on the final column.

But now we can apply the matrix tree theorem. Recall that $\det(L[n])$ is the number of spanning trees of $G$; we also saw in the proof of the matrix tree theorem that $L[n][n-1]$ is a $(n-2) \times (n-2)$ principal submatrix of the Laplacian of $G/e$, and hence that $\det(L[n][n-1])$ is the number of spanning trees of $G/e$. Thus, the effective resistance of $e$ is simply the fraction of spanning trees that use edge $e$.

Let $\mathcal{T}$ denote the set of all spanning trees of $G$. A *uniformly random spanning tree* is simply a spanning tree chosen uniformly at random from $\mathcal{T}$. Properties of uniformly random spanning trees and related objects are objects of intense interest to probabilists; we'll only scratch the surface. We have shown

**Theorem 12.1.** *If $T$ is a uniformly random spanning tree of $G$, then $\Pr(e \in T) = \Pi_{e,e}$ for any edge $e \in E$.*

There is something a bit opaque about this proof. Both spanning trees and electrical flows are somewhat "combinatorial" objects, satisfying combinatorial or linear constraints. The above proof shows the correspondence via a detour through the algebraic world of determinants, which is difficult to interpret. So here we give a second, much more combinatorial proof, one which shows a bit more: random spanning trees give us a way of constructing $i = P_\star \chi^e$ in its entirety, not just its value on edge $e$.

*Second proof of Theorem 12.1.* Let $N$ denote the number of spanning trees of $G$. Let $(s,t) \in \vec{E}$ be the orientation of $e$. For any spanning tree $T$, let $f^T$ denote the flow that sends 1 unit of flow from $s$ to $t$ along the unique path in $T$ from $s$ to $t$. That is, $f^T_{e'}$ is either 1 (if $e'$ is on this unique path and is traversed in the forward direction), $-1$ (if $e'$ is on the path but traversed in the backward direction), or 0 otherwise.

Let $i = P_\star \chi^e$. We will prove that $i = g$, where

$$g = \frac{1}{N} \sum_{T \in \mathcal{T}} f^T = \mathbb{E}_{T \in_u \mathcal{T}}[f^T].$$

Considering then $i_e$ proves the theorem, since for any spanning tree $T$ with $e \in T$, the unique path from $s$ to $t$ in $T$ is simply the edge $e$, which is used in the forward direction.

Observe that $\nabla g = \mathrm{e}_t - \mathrm{e}_s$. Thus to show that $g = i$, it suffices to show the following claim:

**Claim.** $\sum_{a \in C} g_a = 0$ for any directed cycle $C$ in $\overleftrightarrow{G}$.

Fix $C$, and observe that the claim is clearly invariant under the choice of orientation of $G$. So we may assume for convenience that $C \subseteq \vec{E}$. Let

$$\mathcal{Q} = \{(T, a) : T \in \mathcal{T}, a \text{ is on the path from } s \text{ to } t \text{ in } T\},$$

and for any $(T, a) \in \mathcal{Q}$, let $\sigma_{T,a}$ be 1 if $a$ is used in the forward direction in the path from $s$ to $t$ in $T$, and $-1$ if it is used in the backwards direction. Note that $Ng_a = \sum_{T \in \mathcal{T}:(T,a) \in \mathcal{Q}} \sigma_{T,a}$. Our goal is thus to show that

$$\sum_{(T,a) \in \mathcal{Q}: a \in C} \sigma_{T,a} = 0. \tag{2}$$

Let $\phi(T, a) = T \setminus \{a\}$ for every $(T, a) \in \mathcal{Q}$. This defines a partition of $\mathcal{Q}$. Fix any $F \subseteq E$ in the range of $\phi$, which will be a forest consisting of two components; we will show that

$$\sum_{(T,a) \in \phi^{-1}(F): a \in C} \sigma_{T,a} = 0, \tag{3}$$

which proves (2) and hence the claim.

Let $S$ be the set of nodes in the connected component of $(V, F)$ containing $s$; then $V \setminus S$ is the connected component containing $t$. Observe then that we can add *any* edge crossing $S$ to $F$, and the result is a spanning tree. That is, $\phi^{-1}(F) = \{(F \cup \{e\}, e) : \ e \text{ has one endpoint in } S\}$. Further, $C$ obviously has an equal number of arcs entering $S$ as leaving $S$ (recall our choice of orientation). It follows that we get a contribution to (3) of $+1$ for each $(v, w) \in \vec{E}$ with $v \in S, w \notin S$ and a contribution of $-1$ for each $(v, w) \in \vec{E}$ with $v \notin S, w \in S$. This completes the proof of the claim, and hence the lemma. $\square$

## Acknowledgements

The book by Lyons and Peres [2] is an excellent source for a lot of this, from the more probabilistic viewpoint. The second proof of Kirchhoff's effective resistance formula is based on Grimmett [1].

## References

[1] G. Grimmett. *Probability on Graphs*. Cambridge University Press, 2010.

[2] R. Lyons and Y. Peres. *Probability on Trees and Networks*, volume 42 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, New York, 2016. Available at `http://rdlyons.pages.iu.edu/`.

[3] N. Vishnoi. $Lx = b$. In *Foundations and Trends in Theoretical Computer Science*, volume 8, pages 1–141. now, 2013.