MA431 Lecture 6

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Outline

1. Recap
2. Flows of minimum energy
3. Projection onto cut space
4. Electrical flows and electrical potentials
5. Computing i and Π
6. Effective resistance
7. Kirchhoff's effective resistance theorem
8. A strengthening of Kirchhoff's effective resistance theorem
9. Uniform sampling of spanning trees

The incidence matrix of an orientation

- Let $G = (V, E)$ be a **connected** graph with n vertices and m edges.
- Denote by $\vec{G} = (V, \vec{E})$ an arbitrary orientation of G.

Recall

The incidence matrix of $\vec{G}$ is the $V \times \vec{E}$ matrix B where column $(v, u) \in \vec{E}$ is equal to $e_u - e_v$.

Recall

The Laplacian matrix of G is $BB^\top$.

Cycle and cut spaces

Cycle space

$$W^\diamond = \left\{ f \in \mathbb{R}^{\vec{E}} : Bf = \mathbf{0} \right\}$$

Cut space

$$W^\star = \left\{ B^\top \pi : \pi \in \mathbb{R}^V \right\}$$

Remark

1. $W^\star = (W^\diamond)^\perp$ and $W^\diamond = (W^\star)^\perp$
2. $\mathbb{R}^{\vec{E}} = W^\diamond \oplus W^\star$.

Cycle and cut spaces

T spanning tree

Theorem

G connected

$\dim(W^\diamond) = m - n + 1$ and $\dim(W^\star) = n - 1$.

Consequence for the cycle space

1. $\{\chi(C_e) : e \in \vec{E} \setminus T\}$ is a basis for $W^\diamond$.
2. $W^\diamond = \mathrm{span}\{\chi(C) : C$ is a directed cycle of $\overleftrightarrow{G}\}$.

Consequence for the cut space

1. $\{\chi(\delta^+(S_e)) : e \in T\}$ is a basis for $W^\star$.
2. $W^\star = \mathrm{span}\{\chi(\delta^+(S)) : S \subseteq V, S \neq \emptyset, V\}$.
3. $W^\star = \mathrm{span}\{\chi(\delta^+(r)) : r \in V\}$.
4. $\{\chi(\delta^+(r)) : r \in V \setminus t\}$ is a basis for $W^\star$, for any $t \in V$.

∇, grad, and ∇ grad

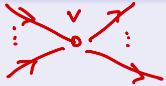
The linear operator ∇

Let $f \in \mathbb{R}^{\vec{E}}$. Think of f as a flow on $\vec{G}$ where $f_{(u,v)}$ is the amount of flow going from u to v.

Net flow

For each $v \in V$, define

$$\nabla f_v = (\nabla f)_v := \sum_{(u,v)\in\vec{E}} f_{(u,v)} - \sum_{(v,w)\in\vec{E}} f_{(v,w)}.$$



$\nabla : \mathbb{R}^{\vec{E}} \to \mathbb{R}^V$ is a linear operator represented by the matrix B. That is, for each $f \in \mathbb{R}^{\vec{E}}$, the image ∇f is the matrix product Bf.

Definition

A circulation of $\vec{G}$ is a flow $f \in \mathbb{R}^{\vec{E}}$ s.t. $\nabla f = \mathbf{0}$.

Remark

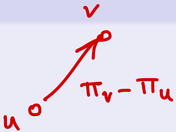
$W^\diamond$ is precisely the set of all circulations of $\vec{G}$.

The linear operator grad

Definition (gradient)

Let $\text{grad} : \mathbb{R}^V \to \mathbb{R}^{\vec{E}}$ be the operator defined as follows: for each $\pi \in \mathbb{R}^V$,

$$(\text{grad}\, \pi)_{(u,v)} = \pi_v - \pi_u$$



grad is a linear operator represented by the matrix $B^\top$. That is, for each $\pi \in \mathbb{R}^V$, the image $\text{grad}\, \pi$ is the matrix product $B^\top \pi$.

Range of grad

$\mathrm{range}(\text{grad}) = \{B^\top \pi : \pi \in \mathbb{R}^V\}$ $= \omega^*$

The Laplacian as a differential operator

Summary

1. ∇ is represented by B.
2. grad is represented by $B^\top$.

Lemma

∇ and grad are adjoint operators: $\langle \nabla f, \pi \rangle = \langle f, \operatorname{grad} \pi \rangle$.

Lemma

$\nabla \operatorname{grad} : \mathbb{R}^V \to \mathbb{R}^V$ is a linear operator represented by the Laplacian matrix L of G.

$L = B B^T$

Electrical flows

Energy

Let $b \in \mathbb{R}^V$ be a demand vector such that $b = \nabla f$ for some flow $f \in \mathbb{R}^{\vec{E}}$.

Energy

The energy of f is $\|f\|^2 = \sum_{e \in \vec{E}} f_e^2$.

Goal

Find a flow of minimum energy satisfying the demands b. That is, solve the quadratic program:

$$\min \left\{ \|f'\|^2 : \nabla f' = b, f' \in \mathbb{R}^{\vec{E}} \right\}$$

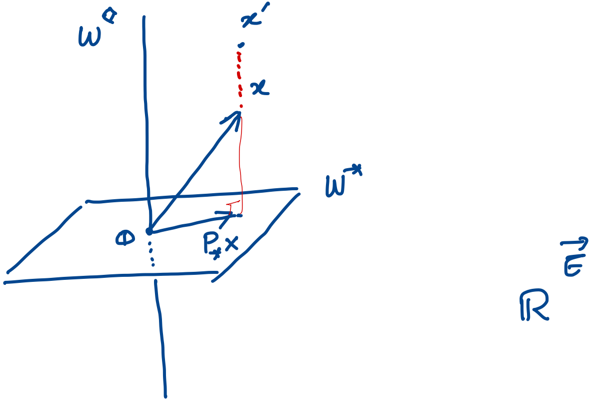
Answer: The flow turns out to be unique!

let's find it.

Orthogonal projection onto cut space

Definition

Denote by $P_\star : \mathbb{R}^{\vec{E}} \to \mathbb{R}^{\vec{E}}$ the linear operator that projects orthogonally onto $W^\star$.



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Denote by $P_\star : \mathbb{R}^{\vec{E}} \to \mathbb{R}^{\vec{E}}$ the linear operator that projects orthogonally onto $W^\star$.

Lemma

If $\nabla f' = \nabla f$, then $P_\star f' = P_\star f$.

Orthogonal projection onto cut space

Definition

Denote by $P_\star : \mathbb{R}^{\vec{E}} \to \mathbb{R}^{\vec{E}}$ the linear operator that projects orthogonally onto $W^\star$.

Lemma

If $\nabla f' = \nabla f$, then $P_\star f' = P_\star f$.

Proof.

- $\nabla(f' - f) = \nabla f' - \nabla f = \mathbf{0}$, so $f' - f \in W^\diamond$.
- Thus, f', f have the same orthogonal projection onto $(W^\diamond)^\perp = W^\star$.

$\square$

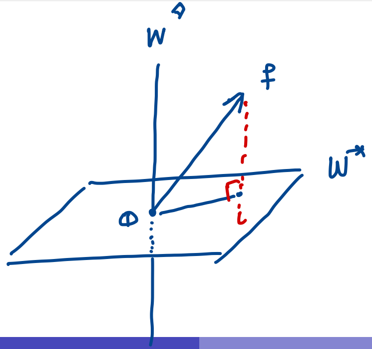
Flow of minimum energy

Let $b \in \mathbb{R}^V$ be a demand vector such that $b = \nabla f$ for some flow $f \in \mathbb{R}^{\vec{E}}$.

Theorem

Let $\iota := P_\star f$. Then ι is the unique optimal solution to

$$\min \left\{ \|f'\|^2 : \nabla f' = b, f' \in \mathbb{R}^{\vec{E}} \right\}$$



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Theorem

Let $\iota := P_\star f$. Then ι is the unique optimal solution to

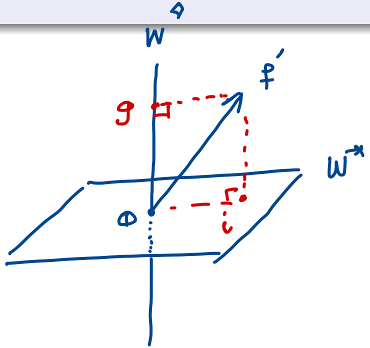
$$\min \left\{ \|f'\|^2 : \nabla f' = b, f' \in \mathbb{R}^{\vec{E}} \right\}$$

Proof.

- Let f' be an arbitrary feasible solution.
- We know $P_\star f' = \iota$.
- Let $g \in \mathbb{R}^{\vec{E}}$ be the orthogonal projection of f' onto $W^\diamond$.
- Then

$$f' = g + \iota$$

$$\|f'\|^2 = \|g\|^2 + \|\iota\|^2$$



Flow of minimum energy

Claim 1

$\nabla\iota = b$. That is, ι is also a feasible solution to the quadratic program.

$$\nabla i = \nabla (f' - g') = \nabla f' - \underbrace{\nabla g}_{0} = \nabla f = b$$

Flow of minimum energy

Claim 1

$\nabla\iota = b$. That is, ι is also a feasible solution to the quadratic program.

g is a circulation so $\nabla g = \mathbf{0}$, so $\nabla\iota = \nabla f - \nabla g = b$.

Claim 2

$\|f'\|^2 \geq \|\iota\|^2$ with equality iff $f' = \iota$.

Follows from $\|f'\|^2 = \|g\|^2 + \|i\|^2$.

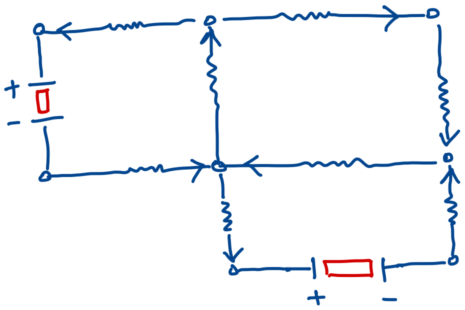
$\square$

Electrical flow

Definition

ι is called the electrical flow satisfying the demands b.

Think of $\vec{G}$ as an electrical network, the vertices as nodes, and the arcs as resistors of unit resistance, and therefore unit conductance, connecting the end nodes.



Electrical flow

Definition

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Think of $\vec{G}$ as an electrical network, the vertices as nodes, and the arcs as resistors of unit resistance, and therefore unit conductance, connecting the end nodes.

Lemma

$\langle \iota, f \rangle = \|\iota\|^2$.

Proof.

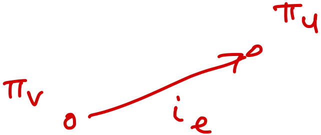
Recall $f = i + g$ for some $g \in W^\diamond$. Then

$$\langle \iota, f \rangle = \langle i, i+g \rangle = \langle i, i \rangle + \overset{0}{\cancel{\langle \underset{W^*}{\underset{\in}{i}}, \underset{W^\diamond}{\underset{\in}{g}} \rangle}} = \langle i, i \rangle$$

□

Electrical potentials

We know $\iota \in W^\star = \mathrm{range}(\mathrm{grad})$. Thus, $\iota = \mathrm{grad}\,\pi$ for some $\pi \in \mathbb{R}^V$.



$$i_e = \pi_u - \pi_v \qquad e = (u, v)$$

Electrical potentials

We know $\iota \in W^\star = \mathrm{range}(\text{grad})$. Thus, $\iota = \text{grad}\,\pi$ for some $\pi \in \mathbb{R}^V$.

Definition

$\pi_v, v \in V$ are called the electrical potentials.

Kirchhoff's second law (potential difference)

$\iota_{(u,v)} = \pi_v - \pi_u$ for all $(u, v) \in \vec{E}$.

Electrical potentials

We know $\iota \in W^\star = \mathrm{range}(\mathrm{grad})$. Thus, $\iota = \mathrm{grad}\,\pi$ for some $\pi \in \mathbb{R}^V$.

Definition

$\pi_v, v \in V$ are called the electrical potentials.

Kirchhoff's second law (potential difference)

$\iota_{(u,v)} = \pi_v - \pi_u$ for all $(u,v) \in \vec{E}$.

Remark

The electrical potentials are unique up to shifting by a constant.

$\mathrm{grad} : \mathbb{R}^V \to \mathbb{R}^{\vec{E}}$

Matrix of projection onto $W^\star$

Recall

$P_\star : \mathbb{R}^{\vec{E}} \to \mathbb{R}^{\vec{E}}$ is the linear operator that projects orthogonally onto $W^\star$.

Let Π be the $\vec{E} \times \vec{E}$ matrix where column e is precisely $P_\star \chi^e$.

Remark

Π represents the linear operator $P_\star$ in the basis $\{\chi^e : e \in \vec{E}\}$.

Matrix of projection onto $W^\star$

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Let Π be the $\vec{E} \times \vec{E}$ matrix where column e is precisely $P_\star \chi^e$.

Remark

Π represents the linear operator $P_\star$ in the basis $\{\chi^e : e \in \vec{E}\}$.

Remark

Π is a orthogonal projection matrix, so $\underbrace{\Pi^2 = \Pi}_{\text{idempotent}}$ and $\underbrace{\Pi^\top = \Pi}_{\text{self-adjoint}}$.

$$\boxed{\langle x, \Pi y \rangle = \langle \Pi x, y \rangle}$$

Computing ι and Π

Question

Given $b = \nabla f$, how do we compute ι?

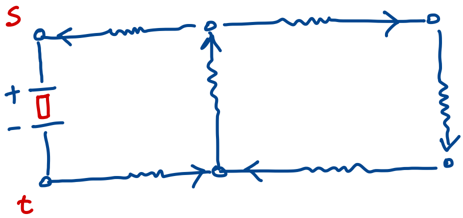
1. We know $\iota_e = \pi_v - \pi_u$ for each $e = (u, v) \in \vec{E}$. That is, $\iota = \text{grad}\,\pi$.
2. So how do we compute π?
3. We know
$$b = \nabla \iota = \nabla\, \text{grad}\,\pi = L\pi$$
4. We can get one solution π by solving
$$\pi = L^+ b$$
L^+: pseudoinverse of L
5. Putting it altogether we get
$$\Pi f = \iota = \text{grad}\,\pi = \text{grad}\,L^+ b = \text{grad}\,L^+ \nabla f$$
$$= B^\top L^+ B f$$
6. Then
$$\Pi = B^\top L^+ B$$

Effective resistance between two nodes

Take distinct vertices $s, t \in V$. Send an electrical unit flow ι' from s to t, with electrical potentials $\pi'_v, v \in V$.

Definition

The potential difference $\pi'_t - \pi'_s$ is the effective resistance between s and t.

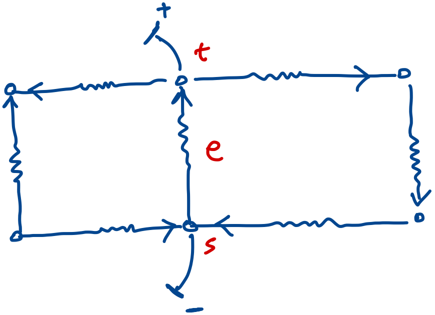


Effective resistance of an edge

Take an edge $e = (s, t) \in \vec{E}$.

Definition

The effective resistance of edge e is the effective resistance between s and t.



Effective resistance of an edge

$s \xrightarrow{e} t$

Take an edge $e = (s, t) \in \vec{E}$.

Definition

The effective resistance of edge e is the effective resistance between s and t.

Theorem (Characterization of Effective Resistance)

Let ι' be the electrical unit flow from s to t. Let $\pi'_v, v \in V$ be the electrical potentials. Then

1. $\pi'_t - \pi'_s = \iota'_e$ by Kircchoff's 2nd law
2. the effective resistance of e is ι'_e

Effective resistance of an edge

Take an edge $e = (s, t) \in \vec{E}$.

Definition

The effective resistance of edge e is the effective resistance between s and t.

Theorem (Characterization of Effective Resistance)

Let ι' be the electrical unit flow from s to t. Let $\pi'_v, v \in V$ be the electrical potentials. Then

1. $\pi'_t - \pi'_s = \iota'_e$
2. the effective resistance of e is ι'_e
3. the effective resistance of e is $\Pi_{e,e}$
4. $\iota'_e = \|\iota'\|^2$.

Proof of Theorem

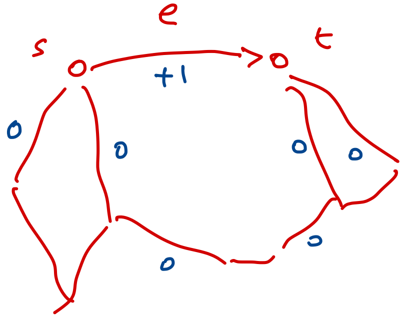
x^e

③ $\Pi_{e,e} = (P_* x^e)_e$

Note that x^e is a flow of unit value from s to t!

$= (i')_e$

$= i'_e$



another flow satisfying the same demand,

④ $\|i'\|^2 = \langle i', i' \rangle = \langle i', x^e \rangle$

$= i'_e$.

Effective resistance and connectivity

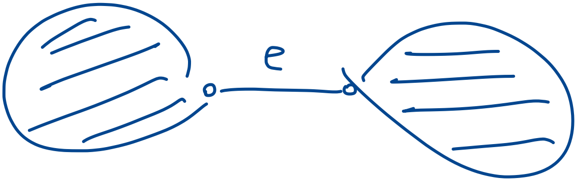
Lemma

1. The effective resistance of $e \in \vec{E}$ is at most 1.
2. The effective resistance of $e \in \vec{E}$ is 1 if and only if $G \setminus e$ is not connected.

Proof.

Exercise. $\square$

(2)



Intuition: The higher the effective resistance, the "more important" the edge.

Effective resistance and connectivity

Lemma

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Proof.

Exercise. $\square$

Rayleigh monotonicity principle

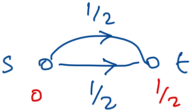
Let $G' = (V, E')$ with $E' \supseteq E$. Then for any two vertices $u, v \in V$, the effective resistance between u and v in G' is smaller than or equal to the effective resistance between u and v in G.

Proof.

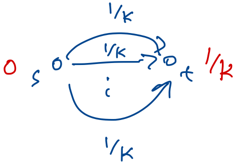
Exercise. $\square$

π

i



$\vdots$



k edges

Kirchhoff's effective resistance theorem

Take an edge $e = (s,t) \in \vec{E}$.

Recall

$T(G)$ is the number of spanning trees of G.

Theorem

The effective resistance of e is $\frac{T(G/e)}{T(G)}$.

$T(G/e) = $ # spanning trees of G that contain e

$\therefore \dfrac{T(G/e)}{T(G)} = $ the proportion of spanning trees that contain e

Kirchhoff's effective resistance theorem

Take an edge $e = (s, t) \in \vec{E}$.

Recall

$T(G)$ is the number of spanning trees of G.

Theorem

The effective resistance of e is $\frac{T(G/e)}{T(G)}$.

Relabel $V = \{1, \ldots, n\}$ and $e = (n, n-1)$.

$n \longrightarrow n-1$

$\pi'_n \qquad \pi'_{n-1}$

Recall

Let ι' be the electrical unit flow from n to $n-1$. Let $\pi'_v, v \in V$ be the electrical potentials. Then the effective resistance of e is $\pi'_{n-1} - \pi'_n$.

Proof of Kirchhoff's effective resistance theorem

- Let us apply Cramer's rule to solve

$$L\pi' = \underbrace{e_{n-1} - e_n}_{\text{demand vector}}$$

Proof of Kirchhoff's effective resistance theorem

- Let us apply Cramer's rule to solve

$$L\pi' = e_{n-1} - e_n$$

- The rows of L add up to $\mathbf{0}$, so we may drop the last row of $L\pi' = e_{n-1} - e_n$.

. Force $\pi'_n = 0$.

Proof of Kirchhoff's effective resistance theorem

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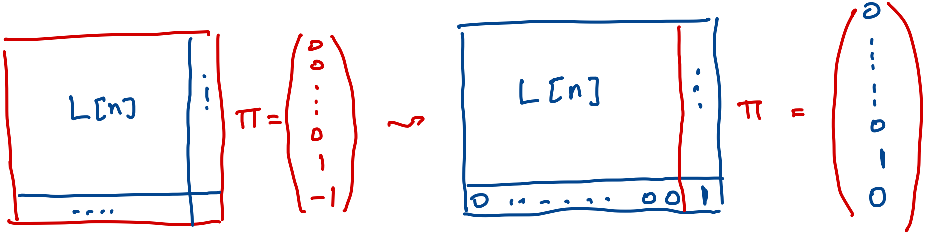
- The rows of L add up to $\mathbf{0}$, so we may drop the last row of $L\pi' = e_{n-1} - e_n$.
- To ensure that π' is unique, we enforce in addition

$$\pi'_n = 0$$



Proof of Kirchhoff's effective resistance theorem

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- To ensure that π' is unique, we enforce in addition

$$\pi'_n = 0$$

- As a result, π' is obtained by solving

See the Matrix-Tree Thm & its proof

$$L[n]\, y = e_{n-1}$$

- By Cramer's rule,

$$\pi'_{n-1} - \pi'_n = \pi'_{n-1} = y_{n-1} = \frac{\det(L[n][n-1])}{\det(L[n])} = \frac{T(G/e)}{T(G)}$$

The diagonal entries of Π

Kirchhoff's effective resistance theorem

The effective resistance of e is $\frac{T(G/e)}{T(G)}$.

Recall

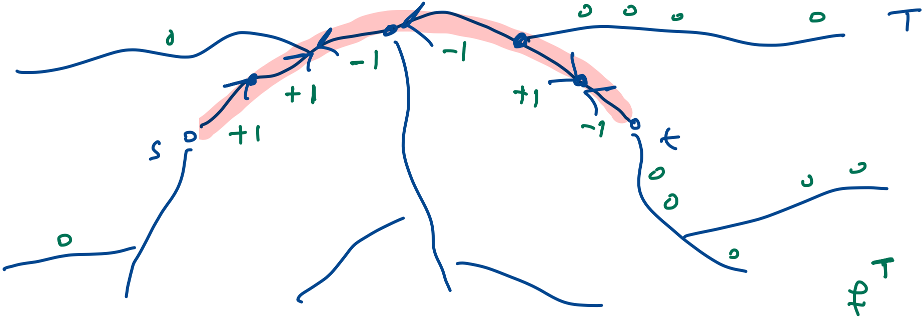
The effective resistance of e is $\Pi_{e,e}$.

Corollary

$$\Pi_{e,e} = \frac{T(G/e)}{T(G)}.$$

A strengthening of Kirchhoff's effective resistance theorem

- Take two vertices $s, t \in V$.
- For each spanning tree $T \subseteq \vec{E}$, let $f^T \in \mathbb{R}^{\vec{E}}$ be the flow that sends 1 unit of flow from s to t along the unique st-path in T.



A strengthening of Kirchhoff's effective resistance theorem

- Take two vertices $s, t \in V$.
- For each spanning tree $T \subseteq \vec{E}$, let $f^T \in \mathbb{R}^{\vec{E}}$ be the flow that sends 1 unit of flow from s to t along the unique st-path in T.
- Let
$$g := \frac{1}{T(G)} \sum_{T \in \mathcal{T}} f^T$$
where $\mathcal{T}$ is the set of all spanning trees.

Theorem

g is the electrical unit flow from s to t.

This implies Kirscchoff's effective resistance theorem
(set $e = (s,t)$ & look at g_e).

Proof

Claim 1

$\nabla g = e_t - e_s$.

Proof

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$\nabla g = e_t - e_s$.

$$\nabla g = \frac{1}{T(G)} \sum_{T \in \mathcal{T}} \nabla f^T = \frac{1}{T(G)} \cdot T(G) \cdot (e_t - e_s) = e_t - e_s$$

Claim 2

$g \in W^\star = (W^\diamond)^\top$.

- Let $C \subseteq \vec{E}$ be a directed cycle in $\overleftrightarrow{G}$.
- It suffices to show $g^\top \chi(C) = 0$
- Thm is invariant under the orientation of G, so WMA that C is a directed cycle in $\vec{G}$.

Proof

Thus,

$$g^\top \chi(C) = \sum_{a \in C} g_a$$

Proof

Thus,

$$
\begin{aligned}
g^\top \chi(C) &= \sum_{a\in C} g_a \\
&= \frac{1}{T(G)} \sum_{a\in C} \sum_{T\in\mathcal{T}} f_a^T
\end{aligned}
$$

Proof

Thus,

$$\begin{aligned} g^\top \chi(C) &= \sum_{a \in C} g_a \\ &= \frac{1}{T(G)} \sum_{a \in C} \sum_{T \in \mathcal{T}} f_a^T \\ &= \frac{1}{T(G)} \sum \left(f_a^T : a \in C, T \in \mathcal{T}, a \in T \right) \end{aligned}$$

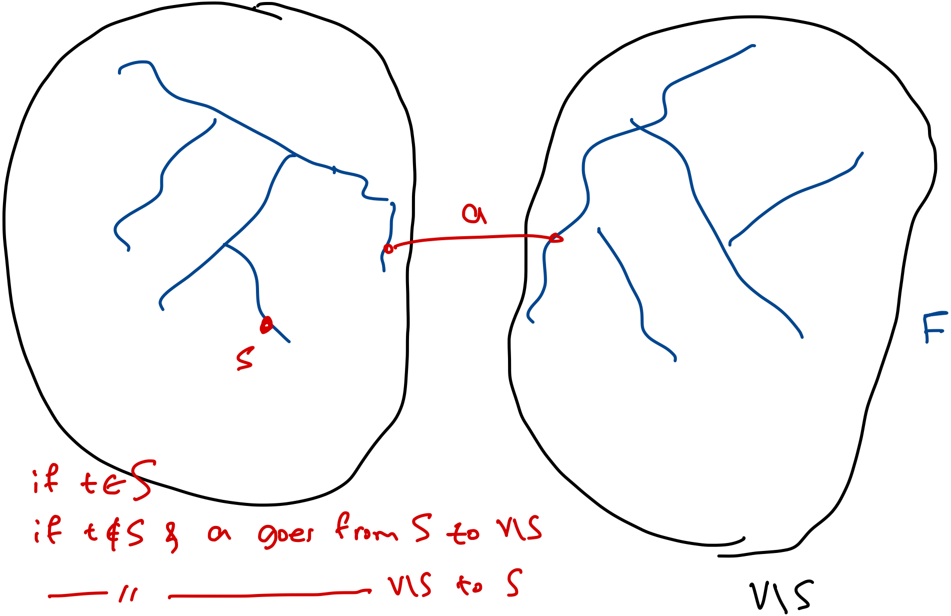
Proof

Thus,

$$
\begin{aligned}
g^\top \chi(C) &= \sum_{a \in C} g_a \\
&= \frac{1}{T(G)} \sum_{a \in C} \sum_{T \in \mathcal{T}} f_a^T \\
&= \frac{1}{T(G)} \sum \left(f_a^T : a \in C, T \in \mathcal{T}, a \in T \right) \\
&= \frac{1}{T(G)} \sum_F \sum \left(f_a^T : a \in C, T \in \mathcal{T}, T \setminus a = F \right)
\end{aligned}
$$

where $F \subseteq \vec{E}$ is any spanning forest with two connected components.

Proof



$$f_a^T = \begin{cases} 0 & \text{if } t \in S \\ +1 & \text{if } t \notin S \text{ \& } a \text{ goes from } S \text{ to } V\setminus S \\ -1 & \text{---}\ ''\ \text{---} \ V\setminus S \text{ to } S \end{cases}$$

Proof

Fix F. Let $S, V \setminus S$ be the two connected components where $s \in S$.

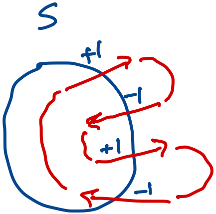
- Case 1: $t \in S$. Then, if $T \in \mathcal{T}$ and a satisfies $T \setminus a = F$, then $f_a^T = 0$, so

$$\sum \left(f_a^T : a \in C, T \in \mathcal{T}, T \setminus a = F \right) = 0$$

Proof

Fix F. Let $S, V \setminus S$ be the two connected components where $s \in S$.

- Case 2: $t \in V \setminus S$. Then



$$\sum \left(f_a^T : a \in C, T \in \mathcal{T}, T \setminus a = F \right)$$
$$= \sum (1 : a \in C, \mathrm{tail}(a) \in S, \mathrm{head}(a) \in V \setminus S)$$
$$+ \sum (-1 : a \in C, \mathrm{tail}(a) \in V \setminus S, \mathrm{head}(a) \in S)$$

$$= |C \cap \delta^+(S)| - |C \cap \delta^-(S)|$$

$= 0$ b/c C is a directed cycle!

Proof

Thus,

$$\begin{aligned}
g^\top \chi(C) &= \sum_{a\in C} g_a \\
&= \frac{1}{T(G)} \sum_{a\in C} \sum_{T\in\mathcal{T}} f_a^T \\
&= \frac{1}{T(G)} \sum \left(f_a^T : a \in C, T \in \mathcal{T}, a \in T \right) \\
&= \frac{1}{T(G)} \sum_{F} \underbrace{\sum \left(f_a^T : a \in C, T \in \mathcal{T}, T \setminus a = F \right)}_{0} \\
&= 0
\end{aligned}$$

Proof

Theorem

g is the electrical unit flow from s to t.

Proof.

Claim 1

$\nabla g = e_t - e_s$.

Claim 2

$g \in W^\star = (W^\diamond)^\top$.

Thus, g must be the electrical unit flow from s to t.

The columns of Π

Take $e = (s,t) \in \vec{E}$.

Recall

$P_\star \chi^e$ is the electrical unit flow from s to t.

Recall

$P_\star \chi^e$ is column e of Π.

The columns of Π

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Recall

$P_\star \chi^e$ is the electrical unit flow from s to t.

Recall

$P_\star \chi^e$ is column e of Π.

We just showed

$\frac{1}{T(G)} \sum_{T \in \mathcal{T}} f^T$ is the electrical unit flow from s to t.

Corollary

Column e of Π is equal to $\frac{1}{T(G)} \sum_{T \in \mathcal{T}} f^T$.

Uniform sampling of spanning trees

Recall

$\Pi_{e,e} = \frac{T(G/e)}{T(G)}$.

Remark

$\frac{T(G/e)}{T(G)}$ is the fraction of spanning trees of G that use the edge e.

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Kirchhoff's effective resistance theorem (rephrased)

If T is a uniformly random spanning tree, then $\Pr[e \in T] = \Pi_{e,e}$.

The Transfer-Current Theorem

The Transfer-Current Theorem

If T is a uniformly random spanning tree, then for any $F \subseteq E$,

$$\Pr[F \subseteq T] = \det(\Pi_F)$$

where Π_F denotes the principal submatrix of Π indexed by F.