MA431 Spectral Graph Theory: Lecture 7

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Last time we proved Kirchhoff's effective resistance theorem:

Theorem 12.1. *If T is a uniformly random spanning tree of G, then $\Pr(e \in T) = \Pi_{e,e}$ for any edge $e \in E$.*

Today we will prove a significant generalization of Theorem 12.1.

13 The Transfer-Current Theorem

Theorem 13.1. *Let T be a uniformly random spanning tree of G. Then for any $F \subseteq E$,*

$$\Pr(F \subseteq T) = \det \Pi_F,$$

where Π_F denotes the principal submatrix of Π induced by F.

One reason that this theorem is important is that it relates to certain strong negative dependence properties that the uniform spanning tree measure has. Let's see how it implies a somewhat weak (though useful) notion of negative dependence. Consider the case $|F| = 2$; let $F = \{e_1, e_2\}$. The theorem then tells us that (with T denoting a uniformly random spanning tree)

$$\Pr(e_1, e_2 \in T) = \Pi_{e_1 e_1}\Pi_{e_2 e_2} - \Pi_{e_1 e_2}\Pi_{e_2 e_1}.$$

But now by Theorem 12.1 (i.e., Theorem 13.1 with $|F| = 1$) and the symmetry of Π, we have

$$\Pr(e_1, e_2 \in T) = \Pr(e_1 \in T)\Pr(e_2 \in T) - \Pi_{e_1 e_2}^2 \leq \Pr(e_1 \in T)\Pr(e_2 \in T).$$

This says that the uniform distribution on random spanning trees is *negatively correlated*: if we condition on e_1 being in the spanning tree, the probability that any other edge e_2 is in the spanning tree can only decrease.

The transfer-current theorem is much stronger than this, and in fact implies a much stronger form of negative dependence. It shows that the uniform distribution on spanning trees is a *determinental measure*, which is in turn a subclass of *strongly Rayleigh measures*. We may come across some related notions later in the course.

In order to prove this theorem, we first investigate the impact of contraction on electrical flows.

Lemma 13.2. *Let $F \subseteq E$ and let $\hat{G} = G/F$. For any edge e of $\hat{G}$, let $\hat{\imath}^e$ denote the vector in $\mathbb{R}^E$ which is equal to the unit electrical flow on $\hat{G}$ across edge e, extended by setting $\hat{\imath}^e_a = 0$ for all $a \in E$ not present in $\hat{G}$.*

Let $i^e = P_\star \chi^e$ for every $e \in E$. Let $Z = \mathrm{span}(\{i^a : a \in F\})$, and let $P_Z^\perp$ denote the orthogonal projection onto $Z^\perp$. Then

$$\hat{i}^e = P_Z^\perp i^e \qquad \textit{for any edge } e \textit{ of } \hat{G}.$$

Here is some intuition for this lemma. The reason i^e is not the correct current for $\hat{G}$ is that it has flow on the edges that become loops (and are then deleted) upon contraction; an electrical flow necessarily has zero flow on loops, because the endpoints have the same potential. So we need to cancel out these flows, which we do by adding some linear combination of electrical flows across the edges which are contracted, in order to cancel out this flow. The result of this is an electrical flow j in G, with $j_a = 0$ for all $a \in E \setminus E'$, but where $\nabla j \neq \nabla i^e$. But each electrical flow that we added to i^e has its source and destination contracted together in $\hat{G}$, which is a wash; this ensures that the restriction of j^e to $\hat{G}$ has the correct net flow everywhere (0, except ± 1 at the endpoints of e).

Proof. Let $\mathcal{P}$ be the partition of V corresponding to $\hat{G}$ (so each part of $\mathcal{P}$ corresponds to a vertex of $\hat{G}$). Fix any edge e in $\hat{G}$, and let $j = P_Z^\perp i^e$. Note that $j \in W^\star$, since both i^e and $P_Z i^e$ (the orthogonal projection of i^e onto Z) are in $W^\star$.

First, we observe that $j_a = 0$ for every $a \in F$. This is because

$$\langle j, \chi^a \rangle = \langle P_\star j, \chi^a \rangle = \langle j, P_\star \chi^a \rangle = \langle j, i^a \rangle = 0.$$

Now let π be a potential corresponding to j. Then we must have $\pi_u = \pi_v$ for any two vertices u, v in the same part of $\mathcal{P}$ (since along any path in F between the vertices, the current and hence potential difference is zero). This implies that j restricted to the edges of $\hat{G}$ does define an electrical flow, with corresponding potential $\hat{\pi}$ on a vertex of $\hat{G}$ given by the common value of π_v on the vertices that contract to it.

All that remains is to check that the net flow into any node of $\hat{G}$ is correct (0, except for the endpoints of e in $\hat{G}$). Since $j - i^e \in Z$, we can write $j = i^e + \sum_{a \in F} \alpha_a i^a$. But for any part S of $\mathcal{P}$ and $a \in F$, $\sum_{u \in S} \nabla i^a_u = 0$. Further, $\sum_{u \in S} \nabla i^e_u$ is zero except for the part containing the head and tail of e, in which case it is $+1$ and -1 respectively. The claim follows. □

Proof of Theorem 13.1. First, note that if F is not a forest, then $\Pr(F \subseteq T)$ and $\det \Pi_F$ are both 0. The former is trivial, and the latter follows from the fact that if C is a directed cycle contained in the bidirection of F, then $\Pi\chi(C) = 0$, implying a linear dependence between the rows of Π indexed by edges in C.

So assume F is a forest. We proceed by induction on $|F|$. If $|F| = 1$, then this is precisely Theorem 12.1, so assume $|F| > 1$. Let $e \in F$ and $\hat{F} = F \setminus \{e\}$. Let $(s, t) \in \vec{E}$ be the orientation of e in $\vec{G}$. Observe that

$$\Pr(F \subseteq T) = \Pr(e \in T | \hat{F} \subseteq T) \cdot \Pr(\hat{F} \subseteq T) = \hat{i}^e \det \Pi_{\hat{F}},$$

where $\hat{i}^e$ is the unit electrical flow from s to t in $G/\hat{F}$. Here we have exploited the inductive claim for $\hat{F}$, and the fact that a uniformly random spanning tree conditioned on containing $\hat{F}$ is precisely a uniformly random spanning tree in $G/\hat{F}$.

So all we need to show is that $\hat{i}^e \det \Pi_{\hat{F}} = \det \Pi_F$. We do this by applying some row operations to Π_F which do not affect the value of the determinant. By Lemma 13.2, we have

$$\hat{i}^e = i^e - \sum_{a \in \hat{F}} \alpha_a i^a$$

for some multipliers α_a. Recall that the a'th row of Π is precisely i^a. So if we apply the corresponding row operations to Π_F, subtracting α_a times row a from row e for each $a \in \hat{F}$, we obtain a new matrix $\hat{\Pi}$ differing from Π_F only on row e, which is then given by $\hat{\Pi}_{e,a} = \hat{i}^e_a$ for all $a \in F$; that is,

$$\hat{\Pi}_{e,a} = \begin{cases} 0 & \text{if } a \in \hat{F}, \\ \hat{i}^e_e & \text{if } a = e. \end{cases}$$

Expanding the determinant of $\hat{\Pi}$ along row e gives the result. □

14 The weighted case

We have restricted our attention so far to unweighted graphs, but the situation with weights is not much more difficult. Let $w : E \to \mathbb{R}_{++}$ denote the weights, which we will also call *conductances* (note that we require them to be strictly positive; a zero-weight edge we can generally think of as removing the edge entirely from the graph). The *resistance* of an edge e is defined to be the inverse weight w_e^{-1}.

The main adjustment to be made is in the choice of inner product. Instead of the standard inner product $\langle f, g \rangle = \sum_{e \in E} f_e g_e$ on $\mathbb{R}^E$, we use the inner product

$$\langle f, g \rangle_w := \sum_{e \in E} \tfrac{1}{w_e} f_e g_e. \tag{1}$$

This changes our notion of orthogonality. The definition of $W^\diamond$ is unchanged from the unweighted case; it remains the space of circulations. The definition of $W^\star$ remains the orthogonal complement of $W^\diamond$—but with respect to $\langle \cdot, \cdot \rangle_w$. Similarly, $P_\star$ is orthogonal with respect to $\langle \cdot, \cdot \rangle_w$. The matrix Π representing $P_\star$ in the standard basis $\{\chi^e : e \in E\}$ will thus no longer be symmetric, since this basis is no longer orthonormal. Instead, we have $w_f \Pi_{e,f} = w_e \Pi_{f,e}$.

Kirchhoff's current laws change slightly: the current on an edge $e = (i, j)$ is no longer equal to the potential difference $\pi_j - \pi_i$, but to the product $w_e(\pi_j - \pi_i)$.

We will still use B to denote the *unweighted* vertex-edge incidence matrix of $\vec{G}$. This means that $BB^\top$ is not the weighted Laplacian L_w (it remains the unweighted Laplacian). Instead, we have $L_w = BWB^\top$, where (here and throughout) W denotes the diagonal matrix with entries $W_{ee} = w_e$. (We will abuse notation and use W also to denote the corresponding linear operator.)

We give a brief list of the most important changes, and leave it as an exercise to the reader to confirm these, and to check that all the proofs can be straightforwardly modified to the weighted case.

- The stated definition of the effective resistance of an edge e remains correct, but this is no longer equal to $\Pi_{e,e}$, the current of the electrical flow $i = P_\star \chi^e$ on edge e. Since $i_e = w_e(\pi_t - \pi_s)$ for potentials π corresponding to i, we do have that the effective resistance is equal to $\frac{1}{w_e}\Pi_{e,e}$.
- The definition of energy is with respect to the new norm: the energy of a flow f is $\|f\|_w^2$. It remains true that the electrical flow minimizes energy, and that the energy of $P_\star \chi^e$ is equal to the effective resistance of e.
- The formula for Π becomes $\Pi = WB^\top L_w^+ B$.[1]
- Rayleigh monotonicity becomes more powerful: increasing the weights of any edge can only decrease effective resistances.
- The spanning tree in Kirchhoff's effective resistance theorem is now the weighted uniform spanning tree measure, where a tree is chosen with probability proportional to the product of the weights in the spanning tree. With T chosen randomly according to this measure, the statement remains

  $$\Pr(e \in T) = \Pi_{e,e} = i_e, \qquad \text{where } i = P_\star \chi^e.$$

  (Note that this probability is no longer equal to the effective resistance.)

Exercises

For all these exercises, graphs are connected and undirected unless otherwise specified.

1. Show that effective resistances form a metric (i.e., are symmetric and satisfy the triangle inequality). You can assume that all resistances are unit.
2. A *strong orientation* of an undirected graph G is an orientation of G that is strongly connected as a digraph. Show that a connected graph with no cut edge (i.e., no edge whose removal disconnects the graph) has a strong orientation.
3. Let G be any connected graph with no cut vertex, i.e., there is no vertex whose removal disconnects the graph. Let e be any edge of G. Show that the cycle space of G has a basis consisting of the characteristic vectors of a set of cycles that all contain e.
4. Let G be any 2-connected planar graph. Let $\mathcal{F}$ consist of the directed cycles in $\overleftrightarrow{G}$ corresponding to the faces of G in some planar embedding (excluding the outer face); there are two possible choices of orientation for each such cycle, choose arbitrarily. Show that $\{\chi(C) : C \in \mathcal{F}\}$ is a basis for the cycle space of G.

[1] There is a subtlety here. If you look at the definition of the pseudoinverse, you will note that in the form we defined it, it implicitly makes reference to an inner product (via self-adjointness). So given an inner differing from the standard one, such as $\langle \cdot, \cdot \rangle$, one could define the pseudoinverse with respect to this inner product. That is *not* what is being used here; L_w^+ is with respect to the standard inner product. In particular, L_w^+ is a symmetric matrix.

5. Suppose (G, w) is a weighted graph where all weights are rational. Describe a way of encoding the weighted instance by an unweighted instance, such that electrical properties (currents, effective resistances etc.) correspond in a natural way.

6. Consider an unweighted connected graph G, and fix an edge e of G. Define $R : \mathbb{R}^E_{++} \to \mathbb{R}_+$ by defining $R(w)$ to be the effective resistance of edge e in the weighted graph (G, w). Show that R is a concave function.

7. Let $G = (V, E)$ be an unweighted graph and let e be any edge of G. Let i be the unit current across the endpoints of e, i.e., $i = P_\star \chi^e$. Show that for any edge $e' \in E$, $i_{e'} \leq i_e$.

8. Prove the Rayleigh monotonicity principle for unweighted graphs (**??**). Then state and prove a version of the property for weighted instances.

9. Prove Lemma 10.2.

10. Let M_n be the $n \times n$ 2-dimensional grid. Show that the effective resistance between opposite corners of the grid is $\Theta(\log n)$.

11. Consider an unweighted connected graph G, and fix an edge e of G. Define $R : \mathbb{R}^E_{++} \to \mathbb{R}_+$ by defining $R(r)$ to be the effective resistance of edge e in the weighted graph (G, w), where $w_e = 1/r_e$ for each $e \in E$. Define $C : \mathbb{R}^E_{++} \to \mathbb{R}_+$ by defining $C(w)$ to be the inverse of the effective resistance of edge e in the weighted graph (G, w).

    Show that R and C are both concave functions.

12. **Note:** I do not know how to do this, or if it is possible (though I suspect it is).

    Is there a direct proof of the transfer-current theorem directly from the matrix-tree theorem, in a similar way to the proof of Kirchhoff's effective resistance formula from the matrix-tree theorem?

15 Simple random walks and electrical networks

Consider a connected graph $G = (V, E)$. Let A denote its adjacency matrix, and L its Laplacian. An infinite random sequence $(X_0, X_1, X_2, \ldots)$ is called a *simple random walk* on G if, for each $i \geq 0$, the condition distribution of X_{i+1} given X_i is the uniform distribution over the neighbours of X_i, and the choice of neighbour is independent of the history of the process before step i (that is, the process is Markovian).

Hitting probabilities. Suppose we run a simple random walk (or any of the random walks discussed) starting from some node v in the graph, and stop it when it hits either some vertex s, or some other vertex t. What can we say about the probability that we hit t before s?

Lemma 15.1. *Fix $s \neq t \in V$. Let q_v denote the probability that a random walk starting from v hits t before hitting s. Then q is precisely given by the voltages obtained upon putting a unit-voltage battery across s and t, chosen so that the voltage at s is 0 and the voltage at t is 1.*

Proof. We observe that q is harmonic on $V \setminus \{s,t\}$. For suppose $(X_0, X_1, \ldots)$ is a simple random walk with $X_0 = v$ for some $v \neq s, t$. Then

$$\begin{aligned} q_v &= \Pr((X_j)_{j\geq 0} \text{ hits } t \text{ before } s) \\ &= \sum_{w:\{v,w\}\in E} \Pr(X_1 = w \text{ and } (X_j)_{j\geq 1} \text{ hits } t \text{ before } s) \\ &= \sum_{w:\{v,w\}\in E} \frac{1}{\deg(v)} q_w. \end{aligned}$$

It follows that $i = \operatorname{grad} q$ satisfies $\nabla i_v = 0$ for all $v \neq s,t$. Since q describes potentials corresponding to i, and these are unique up to shifts, the claim follows. ☐

Related to this is a very nice "physics" (sort of) interpretation of electrical flow.

Lemma 15.2. *Let $(X_0, X_1, \ldots)$ be a simple random walk on G, with $X_0 = s$. Let $\tau = \min\{j : X_j = t\}$. Define $i \in \mathbb{R}^{\vec{E}}$ by setting $i_e = \mathbb{E}[F_e]$ for each $e \in \vec{E}$, where*

$$F_{(v,w)} = |\{0 \leq j < \tau : (X_j, X_{j+1}) = (v,w)\}| - |\{0 \leq j < \tau : (X_j, X_{j+1}) = (w,v)\}| \qquad \forall (v,w) \in \vec{E}.$$

Then i is precisely the unit electrical flow from s to t (i.e, $i \in W^\star$ and $\nabla i = \mathrm{e}_t - \mathrm{e}_s$).

Proof. We have that i is a flow from s to t of unit value, essentially by the definition of i. For any realization $(X_j)_{j\geq 0}$ of the walk, and any $v \neq s,t$, $\nabla F_v = 0$; each time the walk enters and then leaves v, its contribution upon entering and upon leaving cancel. Similarly, $\nabla F_s = -1$ and $\nabla F_t = 1$.

It remains to show that $i \in W^\star$. Fix any cycle $C \subseteq \overleftrightarrow{E}$; for notational convenience, assume that the orientation is such that all arcs of C are oriented in a forward direction. Let $\overleftarrow{C}$ denote the reverse cycle to C (so all arcs of $\overleftarrow{C}$ are oriented in a backwards direction). We can write

$$\sum_{e\in C} i_e = \sum_{j\geq 0} \mathbb{E}\big[\mathbf{1}_{j<\tau} \cdot \big(\mathbf{1}_{(X_j,X_{j+1})\in C} - \mathbf{1}_{(X_j,X_{j+1})\in \overleftarrow{C}}\big)\big].$$

Now observe that for any fixed j and conditioned on X_j, the events $(X_j, X_{j+1}) \in C$ and $(X_j, X_{j+1}) \in \overleftarrow{C}$ are equally likely. So the expectation is 0 for each j, and the claim holds. ☐

Hitting times and commute times. Fix $s,t \in V$, and consider a simple random walk $(X_i)_{i\geq 0}$ starting from $X_0 = s$. What is the expected number of steps for the random walk to reach t for the first time? This is called the *expected hitting time* (of t from s).

A related quantity is the *commute time* between s and t: the expected time required for a random walk starting from s to reach t, and then return to s. Unlike the expected hitting time, this is a symmetric quantity.

Both of these can be computed quite directly using electrical networks; the result for the commute time is particularly pleasing.

Lemma 15.3. *Fix $t \in V$. Let h_v be the expected number of steps a simple random walk started at v takes until reaching t. Then h are potentials corresponding to the electrical flow i satisfying $\nabla i = b$, where $b_v = \deg(v)$ for $v \neq t$, and $b_t = -\sum_{v \neq t} \deg(v)$.*

Proof. For $v \neq t$, we have the identity

$$h_v = 1 + \frac{1}{\deg(v)} \sum_{w:\{v,w\} \in E} h_w.$$

Rearranging and reinterpreting in terms of the Laplacian, we have

$$(Lh)_v = \deg(v) \qquad \forall v \neq t.$$

We thus have $Lh = b$ (since $\sum_v b_v = 0$, which ensures that the constraint for t is also satisfied), as required. $\square$

Lemma 15.4. *The commute time between s and t is equal $2|E|$ times the effective resistance between s and t.*

Proof. Let $b^{(1)}$ be defined by $b_v^{(1)} = \deg(v)$ for $v \neq t$, and $b_t^{(1)} = -\sum_{v \neq t} \deg(v)$. Let $b^{(2)}$ be defined by $b_v^{(2)} = \deg(v)$ for $v \neq s$, and $b_s^{(2)} = -\sum_{v \neq s} \deg(v)$. Let $\pi^{(1)}$ and $\pi^{(2)}$ be potentials corresponding to the electrical flows with net flows $b^{(1)}$ and $b^{(2)}$ respectively. Then by the previous lemma, the expected time for simple random walk to reach t starting from s is $\pi_s^{(1)} - \pi_t^{(1)}$, and the same to reach s starting from t is $\pi_t^{(2)} - \pi_s^{(2)}$. Let $\hat{\pi} = \pi^{(2)} - \pi^{(1)}$; the commute time is then $\hat{\pi}_t - \hat{\pi}_s$.

But by linearity, $\hat{\pi}$ are potentials corresponding to the electrical flow $\hat{i}$ satisfying $\nabla \hat{i} = b^{(1)} - b^{(2)} =: \hat{b}$. Note that $\hat{b}_v = 0$ for $v \neq s, t$, whereas

$$\hat{b}_t = \deg(t) + \sum_{v \neq t} \deg(v) = 2|E|,$$

and similarly $\hat{b}_s = -2|E|$. It follows that $\hat{\pi}_t - \hat{\pi}_s$ is precisely $2|E|$ times the potential difference between t and s for a unit electrical flow from s to t, that is, $2|E|$ times the effective resistance. $\square$

Acknowledgements

The book by Lyons and Peres [1] is an excellent source for more on electrical networks from the more probabilistic viewpoint.

References

[1] R. Lyons and Y. Peres. *Probability on Trees and Networks*, volume 42 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, New York, 2016. Available at `http://rdlyons.pages.iu.edu/`.