Idealness and 2-resistant sets

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Abstract

A subset of the unit hypercube $\{0,1\}^n$ is *cube-ideal* if its convex hull is described by hypercube and generalized set covering inequalities. In this note, we study sets $S \subseteq \{0,1\}^n$ such that, for any subset $X \subseteq \{0,1\}^n$ of cardinality at most 2, $S \cup X$ is cube-ideal.

Keywords. Cube-ideal sets, cropped cubes, ideal clutters, generalized set covering inequalities, resistant sets, structure theorem.

1 Introduction

Take an integer $n \geq 1$. Denote by $\{0,1\}^n$ the extreme points of the n-dimensional unit hypercube $[0,1]^n$. A *sub-hypercube of* $\{0,1\}^n$ is a subset of the form

$$\left\{x \in \{0,1\}^n : x_i = 0 \;\; i \in I, x_j = 1 \;\; j \in J\right\} \quad I, J \subseteq \{1, \ldots, n\}, I \cap J = \emptyset;$$

its *rank* is $n - |I| - |J|$. For a coordinate $i \in [n] := \{1, \ldots, n\}$, we refer to $x_i \geq 0$ and $x_i \leq 1$ as *hypercube* inequalities. *Generalized set covering* inequalities are inequalities of the form

$$\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1 \qquad I, J \subseteq [n], I \cap J = \emptyset,$$

which are precisely the inequalities that cut off sub-hypercubes of $\{0,1\}^n$. Interpreted as clause satisfaction inequalities for the Boolean satisfiability problem, generalized set covering inequalities are prevalent in the literature. Also referred to as *cropping* inequalities [7, 13], these inequalities have surfaced as *cocycle* inequalities valid for cycle polytopes of binary matroids [5], as *set covering* inequalities ($J = \emptyset$) for various set covering problems [6, 11, 8], and as *cover* inequalities ($I = \emptyset$) for the knapsack problem [4, 12, 16].

Take a set $S \subseteq \{0,1\}^n$. S is *cube-ideal* if its convex hull, denoted $\mathrm{conv}(S)$, can be described by hypercube and generalized set covering inequalities. This notion was introduced and studied in [1]. Cube-ideal sets form a rich class of objects: Basic classes include the cycle space of a graph [15] and the up-monotone set associated

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with an ideal clutter (see [1]). Ideal clutters arise from T-joins and T-cuts in grafts (Edmonds-Johnson [10]), from dijoins and dicuts in digraphs (Lucchesi-Younger [14]) and other combinatorial structures. In this note, we introduce a new class of cube-ideal sets that is geometric in nature. We need a few definitions first.

Given points $a, b \in \{0,1\}^n$, the *distance* between a and b, denoted $\mathrm{dist}(a,b)$, is the number of coordinates a and b differ on. Denote by G_n the *skeleton graph* of $[0,1]^n$, whose vertices are the points in $\{0,1\}^n$, where two vertices $a, b \in \{0,1\}^n$ are adjacent if $\mathrm{dist}(a,b) = 1$. For a subset $X \subseteq \{0,1\}^n$, denote by $G_n[X]$ the subgraph of G_n induced on vertices X.

Given $S \subseteq \{0,1\}^n$, we refer to the points in S as *feasible* and to the points in $\overline{S} := \{0,1\}^n - S$ as *infeasible*. The connected components of $G_n[S]$ are *feasible components*, while the components of $G_n[\overline{S}]$ are *infeasible components*.

Theorem 1 ([2]). *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. If each infeasible component is a sub-hypercube or has maximum degree at most two, then S is cube-ideal.*

The various basic classes of cube-ideal sets suggest that finding a structure theorem for cube-ideal sets is a daunting task. In this note, however, we provide a structure theorem for cube-ideal sets $S \subseteq \{0,1\}^n$ that remain cube-ideal even after adding one or two points to S.

Theorem 2. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$ where, for every subset $X \subseteq \{0,1\}^n$ of cardinality at most two, $S \cup X$ is cube-ideal. Then every infeasible component is a sub-hypercube or has maximum degree at most two.*

To prove this theorem, it will be more convenient to work with the more concrete concept of 2-*resistance*. We define and study 2-resistance in §2, and then prove Theorem 2 as well as other applications in §3. In the latter section we will also introduce and discuss the concept of *k-resistance* for integers $k \geq 1$.

2 A characterization of 2-resistant sets

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Take a coordinate $i \in [n]$. The set obtained from $S \cap \{x : x_i = 0\}$ after dropping coordinate i is called the 0-*restriction of S over coordinate i*, and the set obtained from $S \cap \{x : x_i = 1\}$ after dropping coordinate i is called the 1-*restriction of S over coordinate i*. A *restriction* of S is a set obtained after a series of 0- and 1-restrictions. The *projection of S over coordinate i* is the set obtained from S after dropping coordinate i. A *minor of S* is what is obtained after a series of restrictions and projections. A minor is *proper* if at least one operation is applied. Denote by e_i the i^{th} unit vector. To *twist coordinate* $i \in [n]$ is to replace S by

$$S \triangle e_i := \{x \triangle e_i : x \in S\},$$

where the second $\triangle$ denotes coordinate-wise addition modulo 2. $S' \subseteq \{0,1\}^n$ is *isomorphic* to S, written as $S' \cong S$, if S' is obtained from S after relabeling and twisting some coordinates.

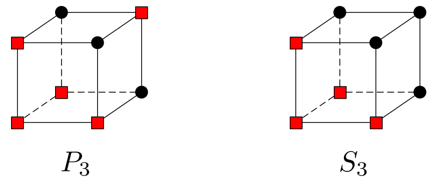

Figure 1: An illustration of P_3 and S_3. Round points are feasible while square points are infeasible.

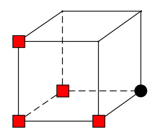

Figure 2: The excluded minor, and restriction, defining 2-resistance.

Let $P_3 := \{110, 101, 011\} \subseteq \{0,1\}^3$ and $S_3 := \{110, 101, 011, 111\} \subseteq \{0,1\}^3$, as displayed in Figure 1. We say that S is 2-*resistant* if, for every subset $X \subseteq \{0,1\}^n$ of cardinality at most two, $S \cup X$ has no P_3, S_3 isomorphic minor. (Going forward, the prefix "isomorphic" will be omitted from "isomorphic minor" and "isomorphic restriction".) The following is straightforward:

Remark 3. *If a set is* 2*-resistant, then so is every minor of it.*

How is 2-resistance relevant? Notice that

$$\mathrm{conv}(P_3) = \left\{x \in [0,1]^3 : x_1 + x_2 + x_3 = 2\right\} \quad \text{and} \quad \mathrm{conv}(S_3) = \left\{x \in [0,1]^3 : x_1 + x_2 + x_3 \geq 2\right\},$$

implying in turn that P_3, S_3 are not cube-ideal. In fact, up to isomorphism, P_3, S_3 are the only non-cube-ideal sets of dimension at most 3.

Remark 4 ([1])**.** *If a set is cube-ideal, then so is every minor of it.*

As a consequence, a cube-ideal set has no P_3, S_3 minor. In particular, if $S \cup X$ is cube-ideal for every set X of cardinality at most two, then S must be 2-resistant.

We are now ready to prove the following characterization of 2-resistant sets:

Theorem 5. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then the following statements are equivalent:*

(i) S is 2*-resistant,*

(ii) S has no restriction $F \subseteq \{0,1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$,

(iii) S has no minor $F \subseteq \{0,1\}^3$ such that $F \cap \{000, 100, 010, 001, 110\} = \{110\}$,

(iv) every infeasible component of S is a sub-hypercube or has maximum degree at most two.

Proof. **(i)** $\Rightarrow$ **(ii)**: Observe that F is not 2-resistant, because $F \cup \{101, 011\}$ is either P_3 or S_3. Thus, a 2-resistant set has no F restriction by Remark 3. **(ii)** $\Rightarrow$ **(iv)**: Assume that S has no F restriction.

Claim 1. *Let x be an infeasible point with at least three infeasible neighbors. If $x\triangle e_i, x\triangle e_j$ are infeasible for some distinct $i, j \in [n]$, then $x\triangle e_i\triangle e_j$ is also infeasible.*

Proof of Claim. Suppose for a contradiction that $x\triangle e_i\triangle e_j$ is feasible. Since x has at least three infeasible neighbors, there is a coordinate $k \in [n] - \{i, j\}$ such that $x\triangle e_k$ is infeasible. After a possible twisting and relabeling of the coordinates, we may assume that $x = \mathbf{0}$ and $i = 1, j = 2, k = 3$. Let $F \subseteq \{0,1\}^3$ be the restriction of S obtained after 0-restricting coordinates $4, \ldots, n$. Then $F \cap \{000, 100, 010, 001, 110\} = \{110\}$, a contradiction. $\Diamond$

Claim 2. *Let x be an infeasible point with at least three infeasible neighbors. Let $k \geq 3$ be the number of infeasible neighbors of x. Then the sub-hypercube of rank k containing x and its infeasible neighbors is infeasible.*

Proof of Claim. After a possible twisting and relabeling, if necessary, we may assume that $x = \mathbf{0}$ and its infeasible neighbors are $e_1, \ldots, e_k$. We need to show that for all subsets $I \subseteq [k]$, $\sum_{i\in I} e_i \in \overline{S}$. We will proceed by induction on $|I| \geq 0$. The base cases $|I| \in \{0, 1\}$ hold by assumption, and the case $|I| = 2$ follows from Claim 1. For the induction step, assume that $|I| \geq 3$. After a possible relabeling, if necessary, we may assume that $I = [\ell]$. Let $y := \sum_{i=1}^{\ell-2} e_i$. By the induction hypothesis, y and its three neighbors $y\triangle e_{\ell-2}, y\triangle e_{\ell-1}, y\triangle e_\ell$ are infeasible. It therefore follows from Claim 1 that $y\triangle e_{\ell-1}\triangle e_\ell = \sum_{i=1}^{\ell} e_i$ is infeasible, thereby completing the induction step. $\Diamond$

Let K be an infeasible component, and let k be the maximum number of infeasible neighbors of a point in K. If $k \leq 2$, then K has maximum degree at most two. Otherwise, $k \geq 3$. It then follows from Claim 2 that K contains a sub-hypercube of rank k. Our maximal choice of k in turn implies that K is in fact the sub-hypercube of rank k. Thus, every infeasible component is a sub-hypercube or has maximum degree at most two. **(iv)** $\Rightarrow$ **(iii)**: Assume that every infeasible component is a sub-hypercube or has maximum degree at most two.

Claim 3. *If S' is a minor of S, then every infeasible component of S' is a sub-hypercube or has maximum degree at most two.*

Proof of Claim. It suffices to prove this for single restrictions and single projections. The claim clearly holds for single restrictions. As for projections, assume that S' is obtained from S after projecting away coordinate n. Let $K' \subseteq \{0,1\}^{n-1}$ be an infeasible component of S'. Clearly, $\{(x, 0), (x, 1) : x \in K'\} \subseteq \{0,1\}^n$ is connected in G_n and infeasible for S, so it is contained in an infeasible component K of S. If K has maximum degree at most two, then so does $\{(x, 0), (x, 1) : x \in K'\}$, implying in turn that K' has maximum degree at most two. Otherwise, K is a sub-hypercube. In this case, as K' is an infeasible component of S', it must be that $K = \{(x, 0), (x, 1) : x \in K'\}$, implying in turn that K' is a sub-hypercube. Thus, K' is a sub-hypercube or has maximum degree at most two, as claimed. $\Diamond$

Thus, since the infeasible component of F containing 000 is neither a sub-hypercube or of maximum degree at most two, S does not have an F minor. **(iii)** $\Rightarrow$ **(i)**: Assume that S is not 2-resistant. Then there is a subset

$X \subseteq \{0,1\}^n$ of cardinality at most two such that $S \cup X$ has a P_3, S_3 minor. Thus there is a subset $Y \subseteq \{0,1\}^3$ of cardinality at most two such that S has a $P_3 - Y, S_3 - Y$ minor. After relabeling the coordinates, if necessary, we see that both $P_3 - Y, S_3 - Y$ are the desired minor. □

3 Consequences of Theorem 5

The first application of Theorem 5 is Theorem 2:

Proof of Theorem 2. Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$ where, for every subset $X \subseteq \{0,1\}^n$ of cardinality at most two, $S \cup X$ is cube-ideal. In particular, S is 2-resistant, so by Theorem 5, every infeasible component of S is a sub-hypercube or has maximum degree at most two, as required. □

Using Theorem 1 we get the following immediate consequence:

Corollary 6. *A* 2*-resistant set is cube-ideal.*

For the third application, we need another concept. Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. S is *polar* if either it contains antipodal points, or all of its points agree on a coordinate:

$$\{x, \mathbf{1} - x\} \subseteq S \quad \text{for some } x \in \{0,1\}^n \qquad \text{or} \qquad S \subseteq \{x : x_i = a\} \quad \text{for some } i \in [n] \text{ and } a \in \{0,1\}.$$

S is *strictly polar* if every restriction of it, including S itself, is polar. Introduced and studied in [1], strict polarity is a notion closely tied with cube-idealness as the authors used the two notions to reformulate the $\tau = 2$ Conjecture of Cornuéjols, Guenin and Margot [9]. We will characterize when 2-resistant sets are strictly polar. To this end, consider the sets

$$\begin{aligned} R_{1,1} &:= \{000, 110, 101, 011\} \subseteq \{0,1\}^3 \\ R_{2,1} &:= \{0000, 1110, 1001, 0101, 0011, 1101, 1011, 0111\} \subseteq \{0,1\}^4 \\ R_5 &:= \{00000, 10000, 11000, 11100, 11110, 01110, 00110, 00010\} \\ &\quad \cup \{01001, 01101, 00101, 10101, 10111, 10011, 11011, 01011\} \subseteq \{0,1\}^5, \end{aligned}$$

as displayed in Figure 3. Notice that these sets are 2-resistant, as every infeasible component has maximum degree at most two, and non-polar. (These three sets are part of an infinite class $\{R_{k,1} : k \geq 1\}$ of non-polar sets, introduced and studied in [2], and "correspond" to an infinite class $\{Q_{k,1} : k \geq 1\}$ of ideal minimally non-packing clutters [9].) We will prove the following characterization:

Theorem 7. *A* 2*-resistant set is strictly polar if, and only if, it has no* $R_{1,1}, R_{2,1}, R_5$ *restriction.*

S is *strictly non-polar* if it is not polar, but every proper restriction is polar. Notice that a set is strictly polar if, and only if, it has no strictly non-polar set as a restriction. Examples of strictly non-polar sets include the sets $R_{1,1}, R_{2,1}, R_5$ [3]. As an application of Theorem 5, we will prove that up to isomorphism, these three sets are the only 2-resistant strictly non-polar sets, thereby proving Theorem 7. We will need the following result:

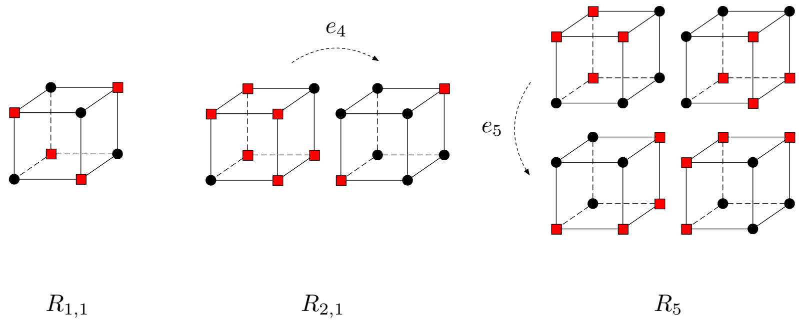

Figure 3: The 2-resistant strictly non-polar sets.

Theorem 8 ([3]). *Up to isomorphism, $R_{1,1}, R_{2,1}, R_5$ are the only strictly non-polar sets where every infeasible point has at most two infeasible neighbors.*

We will also need the following two lemmas:

Lemma 9. *Take an integer $n \geq 5$ and a set $S \subseteq \{0,1\}^n$, where every infeasible point has at most two infeasible neighbors. Then $|S| \geq 2^{n-1}$.*

Proof. Let us proceed by induction on $n \geq 5$. The base case is the crux of the proof, as the induction step is straightforward. Assume that $n = 5$. Suppose for a contradiction that $|S| \leq 15$. For $i, j \in \{0,1\}$, let $S_{ij} \subseteq \{0,1\}^3$ be the restriction of S obtained after i-restricting coordinate 4 and j-restricting coordinate 5. As $|S| \leq 15$, we may assume after a possible relabeling and twisting of coordinates $4, 5$ that $|S_{00}| + |S_{10}| \leq 7$ and $|S_{00}| \leq 3$. Since every infeasible point of S_{00} has at most two infeasible neighbors, it follows that S_{00} has antipodal points. Thus, as $|S_{00}| \leq 3$, we may assume after a possible twisting of coordinates $1, 2, 3$ that $\{000, 111\} \subseteq S_{00} \subseteq \{000, 111, 110\}$. Since

($\star$) every infeasible point of S has at most two infeasible neighbors,

it follows that $\{001, 101, 011\} \subseteq S_{10}$. If $S_{00} = \{000, 111\}$, then $\{001, 101, 011, 100, 010, 110\} \subseteq S_{10}$, which is not possible because $|S_{00}| + |S_{10}| \leq 7$. As a result, $S_{00} = \{000, 111, 110\}$. This implies that $3 \leq |S_{10}| \leq 4$, and together with ($\star$), we see that in fact $S_{10} = \{001, 101, 011, 110\}$. It now follows from ($\star$) that $\{100, 010, 001, 101, 011\} \subseteq S_{01}$ and $\{000, 100, 010\} \subseteq S_{11}$, implying in turn that $|S_{01}| + |S_{11}| \geq 8$. Our contrary assumption implies that $|S| = 15$ and $S_{01} = \{100, 010, 001, 101, 011\}$ and $S_{11} = \{000, 100, 010\}$. But then the infeasible point 11111 of S has 5 infeasible neighbors, contradicting ($\star$). This proves the base case $n = 5$. For the induction step, assume that $n \geq 6$. For $i \in \{0,1\}$, let $S_i \subseteq \{0,1\}^{n-1}$ be the i-restriction of S over coordinate n; note that every infeasible component of S_i has maximum degree at most two. By the induction hypothesis, $|S| = |S_0| + |S_1| \geq 2^{n-2} + 2^{n-2} = 2^{n-1}$, thereby completing the induction step. □

Lemma 10. *Take an integer $n \geq 5$ and a nonempty set $S \subseteq \{0,1\}^n$, where every infeasible component is a sub-hypercube or has maximum degree at most two. If S has no $R_{1,1}$ restriction and one of its infeasible components is a sub-hypercube of rank at least 3, then*

- *$|S| \geq 2^{n-1}$, and*

- *if $|S| = 2^{n-1}$, then S is either a sub-hypercube of rank $n-1$ or the union of antipodal sub-hypercubes of rank $n-2$.*

Proof. We will prove this by induction on $n \geq 5$.

Consider first the base case $n = 5$ is clear. For $i, j \in \{0,1\}$, let $S_{ij} \subseteq \{0,1\}^3$ be the restriction of S obtained after i-restricting coordinate 4 and j-restricting coordinate 5. After a possible twisting and relabeling of the coordinates, we may assume that $S_{00} = \emptyset$. If $S_{10} = \emptyset$, then $S = \{x \in \{0,1\}^5 : x_5 = 1\}$, so S is a sub-hypercube of rank 4. Similarly, if $S_{01} = \emptyset$, then S is a sub-hypercube of rank 4. Otherwise, $S_{10} = S_{01} = \{0,1\}^3$, so $|S| \geq 16$. Moreover, if equality holds, then $S = \{x \in \{0,1\}^5 : x_4 = x_5\}$, so S is the union of antipodal hypercubes of rank 3. This proves the base case.

For the induction step, assume that $n \geq 6$. For $i \in \{0,1\}$, let $S_i \subseteq \{0,1\}^{n-1}$ be the i-restriction of S over coordinate n. If one of S_0, S_1 is empty, then the other one must be $\{0,1\}^{n-1}$, so S is a sub-hypercube of rank $n-1$ and the induction step is complete. We may therefore assume that S_0, S_1 are nonempty.

Assume in the first case that S has an infeasible sub-hypercube of rank ≥ 4 active in, say, direction e_n; that is, the infeasible sub-hypercube intersects both $\{x : x_n = 0\}$ and $\{x : x_n = 1\}$. Then both S_0, S_1 have infeasible sub-hypercubes of rank ≥ 3. Thus by the induction hypothesis, $|S_0| \geq 2^{n-2}$ and $|S_1| \geq 2^{n-2}$, implying in turn that $|S| = |S_0| + |S_1| \geq 2^{n-1}$. Assume next that $|S| = 2^{n-1}$. Then $|S_0| = |S_1| = 2^{n-2}$. By the induction hypothesis, one of the following cases holds:

- S_0 is a sub-hypercube of rank $n-2 \geq 4$: In this case, we may assume that $S \cap \{x : x_n = 0\} = \{x : x_{n-1} = x_n = 0\}$. Since every infeasible component of S is a sub-hypercube or has maximum degree at most two, the sub-hypercube $\{x : x_{n-1} = 0, x_n = 1\}$ is either totally feasible or totally infeasible. Since $|S_1| = 2^{n-2}$, it follows that $S \cap \{x : x_n = 1\}$ is either

  $$\{x : x_{n-1} = 0, x_n = 1\} \quad \text{or} \quad \{x : x_{n-1} = x_n = 1\}.$$

  Thus, S is either a sub-hypercube of rank $n-1$ or the union of antipodal sub-hypercubes of rank $n-2$.

- S_1 is the union of two antipodal sub-hypercubes of rank $n-3 \geq 3$: In this case, we may assume that $S \cap \{x : x_n = 0\} = \{x : x_{n-2} = x_{n-1}, x_n = 0\}$. Since every infeasible component of S is a sub-hypercube or has maximum degree at most two, and $|S_1| = 2^{n-2}$, it follows that $S \cap \{x : x_n = 1\}$ is either

  $$\{x : x_{n-2} = x_{n-1}, x_n = 1\} \quad \text{or} \quad \{x : x_{n-2} + x_{n-1} = 1, x_n = 1\}.$$

  However, since S has no $R_{1,1}$ restriction, the latter is not possible. Thus, $S = \{x : x_{n-2} = x_{n-1}\}$, so S is the union of antipodal sub-hypercubes of rank $n-2$.

Thus, S is either a sub-hypercube of rank $n-1$ or the union of antipodal sub-hypercubes of rank $n-2$, thereby completing the induction step in this case.

Assume in the remaining case that every infeasible component of S has maximum degree at most two or is a cube (i.e. a sub-hypercube of rank 3). By assumption, one of the infeasible components is a cube, which we may assume is contained in S_0. By the induction hypothesis, $|S_0| \geq 2^{n-2}$ and if equality holds, then S_0 is either a sub-hypercube of rank $n-2$ or the union of antipodal sub-hypercubes of rank $n-3$. If S_1 has an infeasible component that is a cube, then the induction hypothesis implies that $|S_1| \geq 2^{n-2}$, and if not, S_1 has maximum degree at most two, so by Lemma 9, $|S_1| \geq 2^{n-2}$. Either way, $|S_1| \geq 2^{n-2}$, so $|S| = |S_0| + |S_1| \geq 2^{n-1}$. We claim that equality does not hold. Suppose for a contradiction that $|S| = 2^{n-1}$. Then $|S_0| = |S_1| = 2^{n-2}$. So S_0 is either a sub-hypercube of rank $n-2 \geq 4$ or the union of antipodal sub-hypercubes of rank $n-3 \geq 3$. As S has no infeasible sub-hypercube of rank ≥ 4, it follows that $n = 6$ and S_0 is the union of antipodal cubes, say

$$S \cap \{x : x_6 = 0\} = \{x : x_4 = x_5, x_6 = 0\},$$

and so

$$S \cap \{x : x_6 = 1\} = \{x : x_4 + x_5 = 1, x_6 = 1\}$$

as $|S_1| = 2^{n-2}$. But then S has an $R_{1,1}$ restriction, a contradiction to our assumption. This completes the induction step. □

We are now ready the following characterization:

Theorem 11. *Up to isomorphism, $R_{1,1}, R_{2,1}, R_5$ are the only 2-resistant strictly non-polar sets.*

Proof. We know that $R_{1,1}, R_{2,1}, R_5$ are strictly non-polar sets, and since their infeasible components have maximum degree at most two, they are 2-resistant by Theorem 5. To prove that they are up to isomorphism the only 2-resistant strictly non-polar sets, pick an integer $n \geq 1$ and a 2-resistant set $S \subseteq \{0,1\}^n$ without an $R_{1,1}, R_{2,1}, R_5$ restriction. It suffices to show that S is polar. By Theorem 5, every infeasible component is a sub-hypercube or has maximum degree at most two. If S has maximum degree at most two, then by Theorem 8, S is polar. Otherwise, S has an infeasible sub-hypercube of rank at least 3. If $n = 4$ or $S = \emptyset$, then S is clearly polar. Otherwise, $n \geq 5$ and $S \neq \emptyset$. By Lemma 10, $|S| \geq 2^{n-1}$; if equality holds, then S is either a sub-hypercube or the union of antipodal sub-hypercubes, so S is clearly polar. Otherwise, $|S| > 2^{n-1}$, implying in particular that there are antipodal feasible points, so S is polar, as required. □

Theorem 7 follows as an immediate consequence.

For the fourth and final application of Theorem 5, take an integer $k \geq 1$. We say that S is *k-resistant* if $S \cup X$ has no P_3, S_3 minor, for every subset $X \subseteq \{0,1\}^n$ of cardinality at most k.

Theorem 12. *Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. Then for any integer $k \geq 3$, S is k-resistant if, and only if, every infeasible component of S has maximum degree at most two.*

Proof. ($\Leftarrow$) Assume that every infeasible component of S has maximum degree at most two. Take an integer $k \geq 3$. To prove that S is k-resistant, let X be a subset of $\{0,1\}^n$ of cardinality at most k. Then every infeasible component of $S \cup X$ also has maximum degree at most two, so by Theorem 5, S has no P_3, S_3 minor. Thus, S

is k-resistant. ($\Rightarrow$) Assume that S is k-resistant. In particular, S is 2-resistant by Theorem 5, so every infeasible component of S is a sub-hypercube or has maximum degree at most two. Notice however that S cannot have an infeasible component that is a sub-hypercube of rank at least 3, for if not, then $S \cup X$ would have a P_3 restriction for some $X \subseteq \{0,1\}^n - S$ of cardinality 3, which is not possible as S is 3-resistant. Thus, every infeasible component of S has maximum degree at most two. $\square$

In particular, $R_{1,1}, R_{2,1}, R_5$ are k-resistant for any integer $k \geq 3$, so

Corollary 13. *For an integer $k \geq 3$, a k-resistant set is strictly polar if, and only if, it has no $R_{1,1}, R_{2,1}, R_5$ restriction.*

What about 1-resistant sets? It turns out that these sets, simply referred to as *resistant* sets, form a very rich class of cube-ideal sets and are much more complex than k-resistant sets for any integer $k \geq 2$. These sets are studied in detail in [2].

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