Strongly connected orientations and integer lattices

Ahmad Abdi[1], Gérard Cornuéjols[2], Siyue Liu[2], and Olha Silina[2]

[1] Department of Mathematics, LSE, London, UK
a.abdi1@lse.ac.uk
[2] Tepper School of Business, Carnegie Mellon University, Pittsburgh, USA
gc0v@andrew.cmu.edu,siyueliu@andrew.cmu.edu,osilina@andrew.cmu.edu

Abstract. Let $D = (V, A)$ be a digraph whose underlying graph is 2-edge-connected, and let P be the polytope whose vertices are the incidence vectors of arc sets whose reversal makes D strongly connected. We study the lattice theoretic properties of the integer points contained in a proper face F of P not contained in $\{x : x_a = i\}$ for any $a \in A, i \in \{0, 1\}$. We prove under a mild necessary condition that $F \cap \{0, 1\}^A$ contains an *integral basis* B, i.e., B is linearly independent, and any integral vector in the linear hull of F is an integral linear combination of B. This result is surprising as the integer points in F do not necessarily form a Hilbert basis. In proving the result, we develop a theory similar to Matching Theory for degree-constrained dijoins in bipartite digraphs. Our result has consequences for head-disjoint strong orientations in hypergraphs, and also to a famous conjecture by Woodall that the minimum size of a dicut of D, say τ, is equal to the maximum number of disjoint dijoins. We prove a relaxation of this conjecture, by finding for any prime number $p \geq 2$, a p-adic packing of dijoins of value τ and of support size at most $2|A|$. We also prove that the all-ones vector belongs to the lattice generated by $F \cap \{0, 1\}^A$, where F is the face of P satisfying $x(\delta^+(U)) = 1$ for every minimum dicut $\delta^+(U)$.

1 Introduction

Let $D = (V, A)$ be a digraph whose underlying undirected graph is 2-edge-connected. A *strengthening set* is an arc subset J such that the digraph obtained from D after reversing the arcs in J is strongly connected. Observe that $J \subseteq A$ is a strengthening set if, and only if, its indicator vector x satisfies the following *generalized set covering inequalities*:

$$\sum_{a \in \delta^+(U)} x_a + \sum_{b \in \delta^-(U)} (1 - x_b) \geq 1 \quad \forall U \subset V, U \neq \emptyset. \tag{CUT}$$

In words, (CUT) asks that after reversing the arcs of J in D, every nonempty proper vertex subset U has at least one incoming arc. Observe that (CUT) can be rewritten as $x(\delta^+(U)) - x(\delta^-(U)) \geq 1 - |\delta^-(U)|$; as the right-hand sides correspond to a crossing supermodular function, the system above may be viewed as a *supermodular flow system*. Let

$$\mathrm{SCR}(D) := [0,1]^A \cap \{x : x \text{ satisfies (CUT)}\}.$$

It is well-known that $\mathrm{SCR}(D)$ is a nonempty integral polytope, and so its vertices are precisely the indicator vectors of the strengthening sets of D ([9], see [17], §60.1). This polytope and its variants have played an important role in graph orientations, combinatorial and matroid optimization; see ([17], Chapters 60-61) and ([10], Chapter 16).

In this paper, we study the lattice theoretic properties of the integer points in $\mathrm{SCR}(D)$. Given a rational linear subspace $S \subseteq \mathbb{R}^A$, an *integral basis for* S is a subset $B \subseteq S \cap \mathbb{Z}^A$ of linearly independent vectors such that every vector in $S \cap \mathbb{Z}^A$ is an integral linear combination of B.

Theorem 1. *Let $D = (V, A)$ be a digraph whose underlying undirected graph is 2-edge-connected. Let $\mathcal{F}$ be a nonempty family over ground set V such that $\emptyset, V \notin \mathcal{F}$, and the following face of* $\mathrm{SCR}(D)$ *is nonempty:*

$$F := \mathrm{SCR}(D) \cap \left\{x \in \mathbb{R}^A : x(\delta^+(U)) - x(\delta^-(U)) = 1 - |\delta^-(U)|,\ \forall U \in \mathcal{F}\right\}.$$

Suppose $\gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\} = 1$. *Then $F \cap \{0,1\}^A$ contains an integral basis for* $\mathrm{lin}(F)$.

Above, $\mathrm{lin}(F)$ refers to the linear hull of F. It can be readily checked that the GCD condition is necessary for $F \cap \{0,1\}^A$ to contain an integral basis for $\mathrm{lin}(F)$. Theorem 1 is a consequence of a more general theorem about the lattice generated by the integer points in any face of $\mathrm{SCR}(D)$ where the GCD condition is replaced by '$1 - |\delta^-(U)| \neq 0$ for some $U \in \mathcal{F}$'. This theorem is stated in §3.

Theorem 1 is best possible in two different ways. First, the result does not extend to faces F involving both $\mathbf{0} \le x \le \mathbf{1}$ and (CUT) inequalities. Secondly, for the face F from Theorem 1, the integer points in F do not necessarily form a *Hilbert basis*, so the result cannot be strengthened in this direction either. These points are explained further in the full version of this extended abstract [2].

1.1 Three applications of the main theorem

Woodall's conjecture. Let $D = (V, A)$ be a digraph whose underlying undirected graph is connected. A *dicut* is the set of arcs leaving a nonempty proper vertex subset with no incoming arc, i.e., it is of the form $\delta^+(U)$ where $U \subset V, U \neq \emptyset$ and $\delta^-(U) = \emptyset$. A *dijoin* is an arc subset whose contraction makes the digraph strongly connected. Subsequently, every strengthening set is a dijoin. It can be readily checked that J is a dijoin if, and only if, J intersects every dicut at least once.

A famous conjecture by Douglas Woodall states that the maximum number of disjoint dijoins is equal to the minimum size of a dicut [19]. This conjecture has a convenient reformulation that appears in an unpublished note by Lex Schrijver.

Conjecture 2 ([16]). Let $\tau \geq 2$ be an integer, and let $D = (V, A)$ be a digraph, where every dicut has size at least τ. Then A can be partitioned into τ strengthening sets.

Note the difference between the original formulation of Woodall's conjecture and Conjecture 2. While the former is concerned with *packing* dijoins, the latter seeks a *partition* into strengthening sets. This subtle difference comes from the key distinction that while every superset of a dijoin is also a dijoin, a superset of a strengthening set may not necessarily remain a strengthening set.

As a consequence of Theorem 1, we obtain the following relaxation of this conjecture. For a subset $J \subseteq A$, $\mathbf{1}_J \in \{0,1\}^A$ denotes the indicator vector of J.

Theorem 3. *Let $\tau \geq 2$ be an integer, and let $D = (V, A)$ be a digraph where the minimum size of a dicut is τ. Then there exists an assignment $\lambda_J \in \mathbb{Z}$ to every strengthening set J that intersects every minimum dicut exactly once, such that $\sum_J \lambda_J \mathbf{1}_J = \mathbf{1}$, $\mathbf{1}^\top \lambda = \tau$, and $\{\mathbf{1}_J : \lambda_J \neq 0\}$ is an integral basis for its linear hull.*

Observe that Conjecture 2 states that one can replace $\lambda_J \in \mathbb{Z}$ by $\lambda_J \in \mathbb{Z}_{\geq 0}$ in this theorem. This result does not extend to the capacitated setting [2].

p-adic programming. Given a prime number $p \geq 2$, a rational number is *(finitely) p-adic* if it is of the form r/p^k for some integer r and nonnegative integer k, and a vector is *p-adic* if each entry is a p-adic rational number. The 2-adic, or *dyadic*, rationals are important for numerical computations because they have a finite binary representation, and therefore can be represented exactly on a computer in floating-point arithmetic. Recent research has characterized when a linear program admits an optimal solution that is p-adic, and furthermore, it has provided a polynomial algorithm for solving a linear program whose domain is restricted to the set of p-adic vectors [1].

Theorem 1 implies, for any prime number $p \geq 2$, the existence of a sparse p-adic optimal solution to a linear program related to packing dijoins. To elaborate, let $D = (V, A)$ be a digraph whose underlying undirected graph is connected. Denote by M the matrix whose columns are labeled by A, and whose rows are the indicator vectors of the dijoins of D. Consider the following pair of dual linear programs,

$$(P) \quad \min\left\{\mathbf{1}^\top x : Mx \geq \mathbf{1}, x \geq \mathbf{0}\right\} \qquad\qquad (D) \quad \max\left\{\mathbf{1}^\top y : M^\top y \leq \mathbf{1}, y \geq \mathbf{0}\right\}$$

where $\mathbf{1}, \mathbf{0}$ denote the all-ones and all-zeros vectors of appropriate dimensions, respectively. A seminal theorem is that the primal linear program (P) models exactly the *minimum dicut problem*, i.e., (P) admits an integral optimal solution ([14], see [5], §1.3.4). Woodall's conjecture equivalently states that the dual linear program (D), in turn, computes the maximum number of pairwise disjoint dijoins, that is, (D) admits an integral optimal solution [19]. The main result of this paper implies some number-theoretic evidence for this conjecture, as it has the following consequence. A † means the proof appears in [2].

Theorem 4 (†). *For any prime number $p \geq 2$, (D) admits a p-adic optimal solution with at most $2|A|$ nonzero entries.*

Observe that Carathéodory's theorem guarantees an optimal solution to (D) with at most $|A|$ nonzero entries. Theorem 4 guarantees a p-adic optimal solution to (D), all the while losing only a factor 2 in the guarantee for the number of nonzero entries.

This theorem does not extend to the capacitated setting. More specifically, if the objective function of (P) is replaced by $c^\top x$ for a nonnegative integral vector c, then (D) may not necessarily have a p-adic optimal solution, for any prime number $p \neq 2$ [2]. Interestingly, it has very recently been shown that (D) always admits a dyadic optimal solution in the capacitated setting [12]; the techniques do not seem to yield a guarantee on the number of nonzero entries of a solution.

Hypergraph orientations. Let $H = (V, \mathcal{E})$ be a hypergraph. An *orientation of H* consists in designating to each hyperedge $E \in \mathcal{E}$ a node inside as the *head* of E, i.e., it is a mapping $O : \mathcal{E} \to V$ such that $O(E) \in E$ for each $E \in \mathcal{E}$. The orientation is *strongly connected* if for each $X \subset V, X \neq \emptyset$, there exists a hyperedge whose designated head is inside X, and has at least one node outside X.

Two orientations of H are *head-disjoint* if no hyperedge has the same head in both orientations. It is well-known that a graph, which is simply a 2-uniform hypergraph, has 2 head-disjoint strongly connected orientations if, and only if, the graph is 2-edge-connected. The following unpublished conjecture by Bérczi and Chandrasekaran attempts to extend one direction of this to general τ-uniform hypergraphs.

Given two subsets $X, E \subseteq V$, we say that X *separates* E if $E \cap X \neq \emptyset$ and $E \notin X$. For $X \subseteq V$, denote by $d_H(X)$ the sum of $|X \cap E|$ ranging over all hyperedges $E \in \mathcal{E}$ separated by X.

Conjecture 5. Let $H = (V, \mathcal{E})$ be a τ-uniform hypergraph such that $d_H(X) \geq \tau$ for all $X \subset V, X \neq \emptyset$. Then H has τ pairwise head-disjoint strongly connected orientations.

For $\tau = 3$, a weaker form of this conjecture appears explicitly in ([10], Conjecture 9.4.15). We prove the following relaxation of Conjecture 5; the conjecture states that one can replace $\lambda_O \in \mathbb{Z}$ by $\lambda_O \in \mathbb{Z}_{\geq 0}$ below.

Theorem 6 (†). *Let $\tau \geq 2$ be an integer, and let $H = (V, \mathcal{E})$ be a τ-uniform hypergraph such that $d_H(X) \geq \tau$ for all $X \subset V, X \neq \emptyset$. Then there exists an assignment $\lambda_O \in \mathbb{Z}$ to every strongly connected orientation $O : \mathcal{E} \to V$ such that*

$$\sum (\lambda_O : \textit{strongly connected orientation } O, O(E) = v) = 1 \qquad \forall E \in \mathcal{E},\ \forall v \in E,$$

and $|\{O : \lambda_O \neq 0\}| \leq (\tau - 1)|\mathcal{E}| + 1$.

1.2 The dijoin polyhedron and digrafts

Theorem 1 is a consequence of a lattice theoretic result about the dijoin polyhedron of bipartite digraphs. To this end, for a digraph $D = (V, A)$, let

$$\mathrm{DIJ}(D) \coloneqq \left\{x \in \mathbb{R}^A : x(\delta^+(U)) \geq 1,\ \forall \text{ dicut } \delta^+(U); x \geq \mathbf{0}\right\}.$$

It is known that $\mathrm{DIJ}(D)$ is an integral polyhedron, and its vertices are precisely the indicator vectors of the (inclusionwise) minimal dijoins of D ([14], see [5], §1.3.4).

A digraph is *bipartite* if every node is a source or a sink. Recent research has demonstrated the importance of bipartite digraphs in studying Woodall's conjecture, by reducing the conjecture to nearly regular bipartite digraphs [3]. We follow this approach by studying faces of $\mathrm{DIJ}(D)$ of a bipartite digraph D.

Definition 7 (digraft). *A* digraft *is a pair $(D = (V, A), \mathcal{F})$ where D is a bipartite digraph, the underlying undirected graph of D is 2-edge-connected, and $\mathcal{F}$ is a family over ground set V such that (a) $\emptyset, V \notin \mathcal{F}$, (b) if $U \in \mathcal{F}$ then $\delta^-(U) = \emptyset$, (c) $V \setminus v \in \mathcal{F}$ for every sink v of D, and (d) the following face of* $\mathrm{DIJ}(D)$ *is nonempty:*

$$F(D, \mathcal{F}) \coloneqq \mathrm{DIJ}(D) \cap \left\{x \in \mathbb{R}^A : x(\delta^+(U)) = 1,\ \forall U \in \mathcal{F}\right\} \subseteq [0, 1]^A.$$

The choice of the 'digraft' terminology mirrors that of a 'graft', an object that shows up in the context of the *minimum T-join problem*, and is loosely related to the *minimum dijoin problem* (see [5], §1.3.5). We prove the following theorem about digrafts.

Theorem 8. *Let $(D = (V, A), \mathcal{F})$ be a digraft. Then $F(D, \mathcal{F}) \cap \{0, 1\}^A$ contains an integral basis for* $\mathrm{lin}(F(D, \mathcal{F}))$.

Theorem 1 and Theorem 8 are in fact equivalent. To de-mystify the connection between these results, let $(D = (V, A), \mathcal{F})$ be a digraft. As the underlying undirected graph of D is 2-edge-connected, every minimal dijoin is a strengthening set (see [17], Theorem 55.1). Furthermore, every strengthening set that has exactly one arc incident with every sink, is also a minimal dijoin. Subsequently, $F(D, \mathcal{F}) = \mathrm{SCR}(D) \cap \left\{x \in \mathbb{R}^A : x(\delta^+(U)) - x(\delta^-(U)) = 1 - |\delta^-(U)|,\ \forall U \in \mathcal{F}\right\}$. Furthermore, $\gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\} = 1$. Thus, Theorem 8 follows from Theorem 1. The converse implication is shown in §3, after we prove Theorem 8 directly.

We find Theorem 8 more convenient to work with than Theorem 1. At the highest level, one explanation for this is that every point in $F(D, \mathcal{F}) \cap \{0, 1\}^A$ is the indicator vector of an arc subset J that has degree one at every sink of D, and degree at least one at every source of D; so that J may be viewed as a perfect b_J-matching in a bipartite graph, for some degree vector b_J. Fixing the degree of J at each sink to one has advantages: first, the cardinality of J becomes invariant and equal to the number of sinks of D; secondly, in this case, J is a minimal dijoin if and only if it is a strengthening set, an equivalence which we utilized above.

2 Proof overview of Theorem 8

Let us provide an overview of the proof of Theorem 8. To this end, let $(D = (V,A),\mathcal{F})$ be a digraft. Our goal is to find an integral basis in $F(D,\mathcal{F})\cap\{0,1\}^A$ for $\text{lin}(F(D,\mathcal{F}))$.

Dicut contractions. A dicut $\delta^+(U)$ is *tight* for $(D,\mathcal{F})$ if $F(D,\mathcal{F}) \subseteq \{x : x(\delta^+(U)) = 1\}$. A tight dicut $\delta^+(U)$ is *non-trivial* if $1 < |U| < |V| - 1$, otherwise it is *trivial.* The proof proceeds by first decomposing the digraft into 'basic' pieces along non-trivial tight dicuts. Once we find integral bases for the basic pieces, then by composing the bases together in a natural manner, we obtain an integral basis in $F(D,\mathcal{F})\cap\{0,1\}^A$. Though this part of the proof is fairly routine for the most part, there are some subtle points about when and how a digraft can be decomposed; let us give a following flash summary of these subtleties. A dicut $\delta^+(U)$ is *contractible* for $(D,\mathcal{F})$ if $1 < |U| < |V| - 1$ and $F(D,\mathcal{F})\cap\{x : x(\delta^+(U)) = 1\} \neq \emptyset$. The *closure of* $\mathcal{F}$ for $(D,\mathcal{F})$ is the family of subsets $U \subset V, U \neq \emptyset$ such that $\delta^-(U) = \emptyset$ and $F(D,\mathcal{F}) \subseteq \{x : x(\delta^+(U)) = 1\}$.

Definition 9 **($(U, V\setminus U)$-contractions).** *Suppose $\delta^+(U)$ is a contractible dicut of $(D,\mathcal{F})$. Let $\overline{\mathcal{F}}$ be the closure of $\mathcal{F}\cup\{U\}$ for the digraft $(D,\mathcal{F}\cup\{U\})$. Let $U_1 := U$ and $U_2 := V\setminus U$. Let $D_i = (V_i, A_i)$ be the bipartite digraph obtained from D after shrinking U_i to a single node u_i; so $V_i = \{u_i\}\cup U_{3-i}$. Let*

$$\mathcal{F}_i := \{W : W\cap U_i = \emptyset, W \in \overline{\mathcal{F}}\} \cup \{(W\setminus U_i)\cup\{u_i\} : U_i \subseteq W, W \in \overline{\mathcal{F}}\}.$$

We refer to $(D_i,\mathcal{F}_i), i = 1,2$ as the $(U, V\setminus U)$-contractions of $(D,\mathcal{F})$.

Thus, a digraft can be decomposed along not only non-trivial tight dicuts but any contractible dicut; this will be particularly useful in Phase 3 of the proof. More specifically, a $(U, V\setminus U)$-contraction decomposes the digraft $(D,\mathcal{F})$, as well as the $0,1$ points in $F(D,\mathcal{F})\cap\{x : x(\delta^+(U)) = 1\}$, into digrafts $(D_i,\mathcal{F}_i)$, and $0,1$ points in $F(D_i,\mathcal{F}_i)$, respectively. Furthermore, thanks to our careful choice of $\mathcal{F}_i, i = 1,2$, we can ensure that under some mild conditions, two $0,1$ points in $F(D_i,\mathcal{F}_i), i = 1,2$ can be composed to give a $0,1$ point in $F(D,\mathcal{F})\cap\{x : x(\delta^+(U)) = 1\}$. This composition is crucial in the following summary of a more technical ingredient that appears in the full version of this article [2].

Lemma 10 (†). *Let $(D = (V,A),\mathcal{F})$ be a digraft, and suppose $\delta^+(U)$ is a contractible dicut. Let $(D_i = (V_i,A_i),\mathcal{F}_i), i = 1,2$ be the $(U,V\setminus U)$-contractions of $(D,\mathcal{F})$. Suppose $B_i \subseteq F(D_i,\mathcal{F}_i)\cap\{0,1\}^{A_i}$ is an integral basis for* $\text{lin}(F(D_i,\mathcal{F}_i))$, *$i = 1,2$. Then there exists an integral basis $B \subseteq F(D,\mathcal{F})\cap\{0,1\}^A\cap\{x : x(\delta^+(U)) = 1\}$ for* $\text{lin}(B)$ *where $|B| = |B_1|+|B_2|-|\delta^+(U)|$. Moreover, if $\delta^+(U)$ is a tight dicut, then* $\text{lin}(B) = \text{lin}(F(D,\mathcal{F}))$.

The proof of Theorem 8 is inductive and divided into three phases. Before explaining this, note that without loss of generality we may assume that whenever $\delta^+(U)$ is tight for $(D,\mathcal{F})$, then $U \in \mathcal{F}$; otherwise we add U to $\mathcal{F}$ without changing $F(D,\mathcal{F})$. (Maximality of $\mathcal{F}$ ensures we obtain a basic digraft in Phase 1.)

Phase 1. In this phase, we look for a non-trivial tight dicut. If there is one, say $\delta^+(U)$, then let $D_i = (V_i, A_i), i = 1, 2$ be the $(U, V \setminus U)$-contractions of $(D, \mathcal{F})$. By induction, $F(D_i, \mathcal{F}_i) \cap \{0,1\}^{A_i}$ contains an integral basis B_i for $\mathrm{lin}(F(D_i, \mathcal{F}_i))$, for $i \in \{1, 2\}$. Then by Lemma 10, there exists $B \subseteq F(D, \mathcal{F}) \cap \{0,1\}^A$ that is an integral basis for $\mathrm{lin}(B) = \mathrm{lin}(F(D, \mathcal{F}))$, thus completing the induction step, and proving Theorem 8 in this phase.

Otherwise, $(D, \mathcal{F})$ has no non-trivial tight dicut, thus yielding the following.

Definition 11 (basic digraft). *A digraft $(D, \mathcal{F})$ is* basic *if every tight dicut $\delta^+(U)$ is trivial and $U \in \mathcal{F}$.*

The challenge for the next two phases is to find an integral basis for a basic digraft. Here comes a key idea of the proof: to study the facet-defining inequalities of $F(D, \mathcal{F})$.

Definition 12 (basic robust digraft). *A basic digraft $(D = (V, A), \mathcal{F})$ is* robust *if every facet-defining inequality for $F(D, \mathcal{F})$ is equivalent to $x_a \geq 0, a \in A$, or $x(\delta^+(u)) \geq 1$ for some source u of D.*

Phase 2 (base case). In this phase, which is the base case of our induction, we assume that the basic digraft is robust. In this case, we prove that $F(D, \mathcal{F})$ is a very special polyhedron. To elaborate, for a polyhedron $P \subseteq \mathbb{R}^n$ and $k \geq 0$, define kP as the set of all points of the form $\sum_{p \in P} \lambda_p p$ where $\lambda \in \mathbb{R}^P_{\geq 0}$ and $\mathbf{1}^\top \lambda = k$. P has the *integer decomposition property* if for every integer $k \geq 1$, every integral point in kP can be written as the sum of k integral points in P. The inequality description of $F(D, \mathcal{F})$, along with a classic result of de Werra [18] on balanced edge-colourings of bipartite graphs, allows for the following theorem.

Theorem 13 (†). *Let $(D = (V, A), \mathcal{F})$ be a basic robust digraft. Then $F(D, \mathcal{F})$ has the integer decomposition property, and $\mathrm{aff}(F(D, \mathcal{F})) = \{x : Mx = \mathbf{1}\}$ for some $M \in \mathbb{Z}^{m \times n}$ with $m \geq 1$.*

Theorem 8 for basic robust digrafts now follows from the following general-purpose result about polyhedra with the integer decomposition property. The theorem below is obtained by first proving that $P \cap \mathbb{Z}^n$ forms an *integral generating set for a cone*, better known as a *Hilbert basis*, and then using a result of Gerards and Sebő [11] about such sets to finish the proof.

Theorem 14 (†). *Let $P \subseteq \mathbb{R}^n$ be a pointed polyhedron with the integer decomposition property, where $\mathrm{aff}(P) = \{x : Ax = b\}$ for $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$ such that $m \geq 1$, $b \neq \mathbf{0}$, and $\gcd\{b_i : i \in [m]\} = 1$. Then $P \cap \mathbb{Z}^n$ contains an integral basis for $\mathrm{lin}(P)$.*

Phase 3. Thus comes the most challenging yet insightful part of our proof, where the basic digraft $(D = (V, A), \mathcal{F})$ is not robust, so there is a facet-defining inequality $x(\delta^+(U)) \geq 1$ that is not equivalent to $x(\delta^+(u)) \geq 1$ for any source u. We decompose $(D, \mathcal{F})$ along the contractible dicut $\delta^+(U)$ into two smaller digrafts. By the induction hypothesis and Lemma 10, we then compose the two integral bases of the $(U, V \setminus U)$-contractions to obtain a linearly independent set $B' \subseteq F(D,\mathcal{F}) \cap \{0,1\}^A$. However, there are two key challenges to turn B' into an integral basis B for $\mathrm{lin}(F(D,\mathcal{F}))$. First, B' is at least one vector away from forming a linear basis for $\mathrm{lin}(F(D,\mathcal{F}))$, and a priori, we do not know the number of extra vectors we would need to add. Secondly, a linear basis is a long way from an integral one, so we need to extend B' very carefully. We have two lemmas that address these issues.

Phase 3, Issue 1. The first issue stems from the fact that the two $(U, V \setminus U)$-contractions are not necessarily basic digrafts. This can be fatal as we could lose our guarantee on the size of $|B \setminus B'|$. However, we will be able to prove that both of these pieces share a key property with basic digrafts, thus allowing us to guarantee that $|B \setminus B'| = 1$. To describe this property, we need a few notions. A node $u \in V$ is *tight* for the digraft if $F(D,\mathcal{F}) \subseteq \{x : x(\delta(u)) = 1\}$; the node is *active* for the digraft if it is not tight. Note that all sinks of a digraft are tight.

Definition 15 (affine critical digraft). *Let $(D = (V,A),\mathcal{F})$ be a digraft, and let V^t be the set of tight nodes. The digraft is* affine critical *if* $\mathrm{aff}(F(D,\mathcal{F})) = \left\{x : x(\delta(v)) = 1,\ \forall v \in V^t\right\}$.

In particular, a basic digraft is affine critical (the converse does not necessarily hold). The following important lemma addresses the first issue.

Lemma 16 (Affine Critical Lemma †). *Let $(D = (V,A),\mathcal{F})$ be a basic digraft that is not robust. Let $x(\delta^+(U)) \geq 1$ be a facet-defining dicut inequality for $F(D,\mathcal{F})$ not equivalent to $x(\delta^+(u)) \geq 1$ for any active source u. Then $(D,\mathcal{F})$ and its $(U, V \setminus U)$-contractions $(D_i,\mathcal{F}_i), i = 1,2$ are affine critical digrafts each of which contains at least one active source. Furthermore, for $i \in \{1,2\}$, every active source for $(D,\mathcal{F})$ in U_{3-i} is an active source for $(D_i,\mathcal{F}_i)$, and vice versa.*

This lemma is a byproduct of a careful analysis of the dimension of $F := F(D,\mathcal{F})$. Let V^t be the set of tight nodes. Define $\kappa^t(D,\mathcal{F})$ to be 1 if $V = V^t$, and 0 otherwise. Denote by $\dim(F)$ the dimension of the affine hull $\mathrm{aff}(F)$, which satisfies the inequality $|A| - |V^t| + \kappa^t(D,\mathcal{F}) \geq \dim(F)$.

Definition 17 (slack). *The* slack *of $(D = (V,A),\mathcal{F})$ is*

$$s(D,\mathcal{F}) := |A| - |V^t| + \kappa^t(D,\mathcal{F}) - \dim(F(D,\mathcal{F})) \geq 0.$$

The slack is a characteristic quantity associated with a digraft, which has an intuitive interpretation. The affine hull of $F(D,\mathcal{F})$ is described by two types of constraints: $x(\delta(v)) = 1,\ \forall v \in V^t$, and $x(\delta^+(U)) = 1$ for a non-trivial tight dicut $\delta^+(U)$. The slack computes the additional contribution of non-trivial tight dicuts —in terms of rank increase— in defining the affine hull. In particular, a digraft is affine critical if and only if it has slack zero. Furthermore, if $s := s(D,\mathcal{F}) \geq 1$, then there exists a cross-free family of s non-trivial tight dicuts, which can be used to decompose the digraft into $s+1$ pieces partitioning the active sources of $(D,\mathcal{F})$ into $s+1$ nonempty parts, such that each piece of the form $(D',\mathcal{F}')$ satisfies $s(D',\mathcal{F}') = \kappa^t(D',\mathcal{F}') = 0$.

Phase 3, Issue 2. The Affine Critical Lemma guarantees that to turn B' into a linear basis for $\mathrm{lin}(F(D,\mathcal{F}))$, we just need to add one more vector b from $F(D,\mathcal{F}) \cap \{0,1\}^A$, which must inevitably satisfy $b(\delta^+(U)) > 1$. However, we need not just a linear basis but an integral one. As it turns out, integrality of the basis can be guaranteed if $b(\delta^+(U)) = 2$, whose existence is implied by the following lemma.

Lemma 18 (Jump-Free Lemma †). *Let $(D = (V, A), \mathcal{F})$ be a digraft, let $\delta^+(U)$ be a dicut, and let $x, y \in F(D,\mathcal{F}) \cap \{0,1\}^A$ where $\lambda_1 := x(\delta^+(U)) < y(\delta^+(U)) =: \lambda_2$. Then for any integer $\lambda \in (\lambda_1, \lambda_2)$, there exists $z \in F(D,\mathcal{F}) \cap \{0,1\}^A$ such that $z(\delta^+(U)) = \lambda$.*

To prove this lemma, we map $F(D,\mathcal{F}) \subseteq \mathbb{R}^A$ to a base polyhedron in $\mathbb{R}^V$ via a suitable affine function f, which also maps $F(D,\mathcal{F}) \cap \{0,1\}^A$ to an M-convex set in $\mathbb{Z}^V$. The exchange axiom for M-convex set then allows us to construct a 'jump-free' sequence from $f(x)$ to $f(y)$ in $\mathbb{Z}^V$. From there, the theory of perfect b-matchings in bipartite graphs allows us to construct a (not necessarily unique) inverse to f, thus obtaining a jump-free sequence from x to y in $F(D,\mathcal{F}) \cap \{0,1\}^A$, and proving the Jump-Free Lemma.

Now that we have dealt with the two issues, we are ready to prove Theorem 8 for basic non-robust digrafts. Recall that the basic digraft $(D = (V, A), \mathcal{F})$ is not robust, and $x(\delta^+(U)) \geq 1$ is a facet-defining inequality that is not equivalent to $x(\delta^+(u)) \geq 1$ for any active source u. Let $(D_i = (V_i, A_i), \mathcal{F}_i), i = 1, 2$ be the $(U, V \setminus U)$-contractions of $(D,\mathcal{F})$. By the induction hypothesis, $F(D_i,\mathcal{F}_i) \cap \{0,1\}^{A_i}$ contains an integral basis B_i for $\mathrm{lin}(F(D_i,\mathcal{F}_i)), i = 1, 2$. Then by Lemma 10, there exists $B' \subseteq F(D,\mathcal{F}) \cap \{0,1\}^A$ forming an integral basis for $\mathrm{lin}(B')$.

By the Jump-Free Lemma, there exists $b \in F(D,\mathcal{F}) \cap \{0,1\}^A$ such that $b(\delta^+(U)) = 2$. We claim that $B := B' \cup \{b\} \subseteq F(D,\mathcal{F}) \cap \{0,1\}^A$ is an integral basis for $\mathrm{lin}(F(D,\mathcal{F}))$, thus finishing the proof.

First, we prove that B is a linear basis for $\mathrm{lin}(F(D,\mathcal{F}))$. Linear independence can be checked through a routine argument. To show that B is a linear basis, we count the linear dimension of $F := F(D,\mathcal{F})$, which is $d := 1 + \dim(F)$. We claim that $d = |B|$. To this end, let $F_i := F(D_i,\mathcal{F}_i), i = 1, 2$ and $d_i := 1 + \dim(F_i), i = 1, 2$.

By the Affine Critical Lemma, $(D,\mathcal{F}), (D_i,\mathcal{F}_i), i = 1, 2$ are affine critical digrafts each of which contains an active source. Thus, $\kappa^t(D,\mathcal{F}) = \kappa^t(D_1,\mathcal{F}_1) = \kappa^t(D_2,\mathcal{F}_2) = 0$, and $s(D,\mathcal{F}) = s(D_1,\mathcal{F}_1) = s(D_2,\mathcal{F}_2) = 0$. Subsequently, by the slack formula in Definition 17, $d = 1 + |A| - |V^t|$, $d_1 = 1 + |A_1| - |V_1^t|$, and $d_2 = 1 + |A_2| - |V_2^t|$, where $V^t, V_i^t, i = 1, 2$ denote the sets of tight nodes of $(D,\mathcal{F}), (D_i,\mathcal{F}_i), i = 1, 2$, respectively. By the Affine Critical Lemma, $V_1^t \cup V_2^t = V^t \cup \{u_1, u_2\}$, implying in turn the first equality below:

$$d = d_1 + d_2 - |\delta^+(U)| + 1 = |B_1| + |B_2| - |\delta^+(U)| + 1 = |B'| + 1 = |B|.$$

The third equality follows from Lemma 10.

It remains to prove that B is an *integral* basis. To this end, pick an integral vector f in $\mathrm{lin}(F(D,\mathcal{F}))$. We now know that f can be expressed as a linear combination of the vectors in B; let $\lambda_z \in \mathbb{R}$ be the coefficient of $z \in B$. Given that f is integral, $f(\delta^+(U))$ is an integer, which can be calculated alternatively as follows, where we have used $z(\delta^+(U)) = 1$ for all $z \in B'$, guaranteed by Lemma 10.

$$f(\delta^+(U)) = \sum_{z \in B} \lambda_z z(\delta^+(U)) = 2\lambda_b + \sum_{z \in B'} \lambda_z = \lambda_b + \mathbf{1}^\top \lambda.$$

As $B \subseteq F(D, \mathcal{F})$, we have $z(\delta(v)) = 1, \forall z \in B$ for any fixed tight node v. Subsequently, $\mathbf{1}^\top \lambda = \sum_{z \in B} \lambda_z = f(\delta(v))$ is an integer, implying in turn that $\lambda_b = f(\delta^+(U)) - \mathbf{1}^\top \lambda$ is an integer. Now let $f' := f - \lambda_b b \in \mathbb{Z}^A$. Evidently, $f' \in \mathrm{lin}(B') \cap \mathbb{Z}^A$, so given that B' is an integral basis for its linear hull, f' is an integer linear combination of the vectors in B', implying in turn that λ is an integral vector. Thus, $B \subseteq F(D, \mathcal{F}) \cap \{0,1\}^A$ is an integral basis for $\mathrm{lin}(F(D, \mathcal{F}))$, thereby completing the induction step, hence proving Theorem 8. □

3 Proofs of Theorem 1 and Theorem 3

In this section, we prove a general theorem about proper faces of $\mathrm{SCR}(D)$ not contained in proper faces of the unit hypercube. We then prove Theorem 1 and one of the three applications, namely Theorem 3. We start off with a useful mapping of the strengthening sets of a digraph to dijoins of another digraph. The mapping is routine and has appeared before in the literature, e.g., [16,6].

Theorem 19 (†). *Let $D = (V, A)$ be a digraph whose underlying undirected graph is 2-edge-connected. Let $\mathcal{F}$ be a family over ground set V such that $\emptyset, V \notin \mathcal{F}$, and the following face of $\mathrm{SCR}(D)$ is nonempty:*

$$F := \mathrm{SCR}(D) \cap \left\{ x \in \mathbb{R}^A : x(\delta^+(U)) - x(\delta^-(U)) = 1 - |\delta^-(U)|, \ \forall U \in \mathcal{F} \right\}.$$

Then there exists a digraft $(D', \mathcal{F}')$ such that the mapping $x \mapsto \binom{x}{\mathbf{1}-x}$ defines a bijection between the face F of $\mathrm{SCR}(D)$ and the face $F(D', \mathcal{F}')$ of $\mathrm{DIJ}(D')$.

Primer on integer lattices. Let us recall some basic concepts of the theory of integer lattices; for a reference textbook we recommend ([15], Chapter 1). A subset $L \subseteq \mathbb{R}^A$ is a *lattice* if it is the set of integer linear combinations of finitely many vectors. Alternatively, L is a lattice if it forms a subgroup of $\mathbb{R}^A$ under addition that is *discrete*, that is, there exists an $\varepsilon > 0$ such that every pair of distinct vectors in L are at distance $\geq \varepsilon$. Given a finite subset $G \subset \mathbb{R}^A$, the *lattice generated by* G, denoted $\mathrm{lat}(G)$, is the set of all integer linear combinations of the vectors in G. A *lattice basis for* L is a set B of linearly independent vectors that generates the lattice, i.e., $L = \mathrm{lat}(B)$. A nontrivial fact is that a lattice basis always exists.

Suppose now L is an *integer lattice*, that is, L is a lattice and $L \subseteq \mathbb{Z}^A$. Let $\overline{L} := \mathrm{lin}(L) \cap \mathbb{Z}^A$ which is another integer lattice that contains L. Note that $\overline{L}$ is the 'densest' integer lattice in $\mathrm{lin}(L)$. It is known that $\overline{L}$ can be partitioned into a finite number of lattices, each of which is an integral shift of L, i.e., of the form $L + w := \{v + w : v \in L\}$ for some $w \in \mathrm{lin}(L) \cap \mathbb{Z}^A$. We refer to the number of parts in this partition as the *index of* L and denote it by $\operatorname{ind}(L) \in \mathbb{Z}_{\geq 1}$. Thus, the smaller the index of L, the denser the lattice is. Of particular interest is the case when L is densest possible. Observe that L has index 1 if, and only if, L contains an integral basis for $\operatorname{lin}(L)$.

Theorem 20. *Let $D = (V, A)$ be a digraph whose underlying undirected graph is 2-edge-connected. Let $\mathcal{F}$ be a family over ground set V such that $\varnothing, V \notin \mathcal{F}$, $1 - |\delta^-(U)| \neq 0$ for some $U \in \mathcal{F}$, and the following face of $\mathrm{SCR}(D)$ is nonempty:*

$$F := \mathrm{SCR}(D) \cap \left\{x \in \mathbb{R}^A : x(\delta^+(U)) - x(\delta^-(U)) = 1 - |\delta^-(U)|,\ \forall U \in \mathcal{F}\right\}.$$

Then the following statements hold:

1. *The lattice generated by $F \cap \{0,1\}^A$ has a lattice basis contained in $F \cap \{0,1\}^A$.*
2. *Let $g := \gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\}$. Then $gx \in \operatorname{lat}\left(F \cap \{0,1\}^A\right)$ for all $x \in \operatorname{lin}(F) \cap \mathbb{Z}^A$.*

Proof. Let L be the lattice generated by $F \cap \{0,1\}^A$, $\overline{L} := \operatorname{lin}(F) \cap \mathbb{Z}^A$, and $g := \gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\}$. By Theorem 19, there exists a digraft $(D' = (V', A'), \mathcal{F}')$ such that for $F' := F(D', \mathcal{F}')$, the mapping $f : F \to F'$ defined as $f(x) = \binom{x}{\mathbf{1}-x}$ is a bijection. Let L' be the lattice generated by $F(D', \mathcal{F}') \cap \{0,1\}^{A'}$. By Theorem 8, there is an integral basis $B' \subseteq F' \cap \{0,1\}^{A'}$ for $\operatorname{lin}(F')$.

Claim. *Let w be an integral vector in $\operatorname{lin}(F)$, expressed as $w = \sum_{x \in F} \lambda_x x$. Then $\mathbf{1}^\top \lambda$ is $\frac{1}{g}$-integral. Furthermore, if $w = \mathbf{0}$, then $\mathbf{1}^\top \lambda = 0$.*

Proof of Claim. Let $\tau := \sum_{x \in F} \lambda_x$. Note that

$$w(\delta^+(U)) - w(\delta^-(U)) = \sum_{x \in F} \lambda_x (1 - |\delta^-(U)|) = \tau(1 - |\delta^-(U)|) \quad \forall U \in \mathcal{F}.$$

As w is integral, we have $\tau(1 - |\delta^-(U)|) \in \mathbb{Z}$ for all $U \in \mathcal{F}$, and so since $\gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\} = g$, it follows that τ is $\frac{1}{g}$-integral. Furthermore, if $w = \mathbf{0}$, then as $1 - |\delta^-(U)| \neq 0$ for some $U \in \mathcal{F}$, we have $0 = w(\delta^+(U)) - w(\delta^-(U)) = \tau(1 - |\delta^-(U)|)$, implying in turn that $\tau = 0$. ◇

Let B be the pre-image of B' under f. Observe that $B \subseteq F \cap \{0,1\}^A$. We shall prove that (a) B is linearly independent, (b) B is a lattice basis for L, and (c) $gw \in L$ for all $w \in \operatorname{lin}(F) \cap \mathbb{Z}^A$.

(a) Suppose $\sum_{b \in B} \lambda_b b = \mathbf{0}$. It follows from the claim above that $\mathbf{1}^\top \lambda = 0$, so $\sum_{b \in B} \lambda_b f(b) = \mathbf{0}$. The linear independence of $B' = \{f(b) : b \in B\}$, along with the bijectivity of f, implies that $\lambda = \mathbf{0}$.

(b) By (a), it suffices to show that $\operatorname{lat}(B) = L$. Clearly, $\operatorname{lat}(B) \subseteq L$. For the reverse inclusion, let $w \in L$. Then $w = \sum_{x \in F \cap \{0,1\}^A} \lambda_x x$ for some integers $\lambda_x, x \in F \cap \{0,1\}^A$. Let $w' := \sum_{x \in F \cap \{0,1\}^A} \lambda_x f(x)$. As λ is integral, $w' \in L'$, so $w' = \sum_{b \in B} \alpha_b f(b)$ for some integers $\alpha_b, b \in B$. Restricting to the coordinates in A, we obtain that $w = \sum_{b \in B} \alpha_b b \in \operatorname{lat}(B)$.

(c) Let $w \in \text{lin}(F) \cap \mathbb{Z}^A$. Write $w = \sum_{x \in F \cap \{0,1\}^A} \lambda_x x$, and let $\tau := \mathbf{1}^\top \lambda$ which is $\frac{1}{g}$-integral by the claim above. Let $w' := \sum_{x \in F \cap \{0,1\}^A} \lambda_x f(x)$, which is $\frac{1}{g}$-integral as $\tau \in \frac{1}{g}\mathbb{Z}$. Subsequently, gw' is an integral vector in $\text{lin}(F')$, so $gw' = \sum_{b \in B} \alpha_b f(b)$ for some integers $\alpha_b, b \in B$, as B' is an integral basis for $\text{lin}(F')$. Restricting to the coordinates in A, we obtain that $gw = \sum_{b \in B} \alpha_b b \in L$, as promised.

Observe that (b) proves part (1), and (c) proves part (2) of the theorem. □

Let us point out a subtle detail about part (1) of Theorem 20. A set G of generators may not necessarily contain a lattice basis for $\text{lat}(G)$. For instance, $\text{lat}(\{2,3\}) = \mathbb{Z}$, yet $\{2,3\}$ does not contain a lattice basis for $\mathbb{Z}$. Thus, the claim that $F \cap \{0,1\}^A$ contains a lattice basis is non-trivial.

Let's look at part (2) of Theorem 20. This part equivalently states that, for $L := \text{lat}\left(F \cap \{0,1\}^A\right)$ and $\overline{L} := \text{lin}(F) \cap \mathbb{Z}^A$, the quotient group $\overline{L}/L$ is an abelian group where the order of every element divides g. Subsequently, every elementary divisor of $\overline{L}/L$ divides g. This implies in turn that $\text{ind}(L)$ is the product of some divisors of g. Furthermore, if g is a prime number, then $\overline{L}/L$ is an elementary p-primary group. For more on concepts relating to group theory, we refer the interested reader to [7], more specifically, Chapter 5, Theorem 5.

We are ready to prove the main theorem.

Proof of Theorem 1. Let $D = (V, A)$ be a digraph whose underlying undirected graph is 2-edge-connected. Let $\mathcal{F}$ be a nonempty family over ground set V such that $\varnothing, V \notin \mathcal{F}$, and the following face of $\text{SCR}(D)$ is nonempty:

$$F := \text{SCR}(D) \cap \left\{x \in \mathbb{R}^A : x(\delta^+(U)) - x(\delta^-(U)) = 1 - |\delta^-(U)|,\ \forall U \in \mathcal{F}\right\}.$$

Suppose $\gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\} = 1$. It then follows from Theorem 20 part (1) that the lattice L generated by $F \cap \{0,1\}^A$ has a lattice basis $B \subseteq F \cap \{0,1\}^A$. Furthermore, by part (2), $L = \text{lin}(F) \cap \mathbb{Z}^A$, so $\text{ind}(L) = 1$, implying in turn that B is an integral basis for $\text{lin}(F)$ contained in $F \cap \{0,1\}^A$. □

We are also ready to prove one of the applications of the main theorem.

Proof of Theorem 3. Let $\tau \geq 2$ be an integer, and let $D = (V, A)$ be a digraph where the minimum size of a dicut is τ. We will prove that there exists an assignment $\lambda_J \in \mathbb{Z}$ to every strengthening set J intersecting every minimum dicut exactly once, such that $\sum_J \lambda_J \mathbf{1}_J = \mathbf{1}$, $\mathbf{1}^\top \lambda = \tau$, and $\left\{\mathbf{1}_J : \lambda_J \neq 0\right\}$ will be an integral basis for its linear hull.

Let $\mathcal{F}$ be the family of sets $U \subset V, U \neq \varnothing$ such that $\delta^-(U) = \varnothing$ and $|\delta^+(U)| = \tau$. Let $F := \text{SCR}(D) \cap \{x : x(\delta^+(U)) - x(\delta^-(U)) = 1 - |\delta^-(U)|,\ \forall U \in \mathcal{F}\}$. Observe that $F \cap \{0,1\}^A$ corresponds to the strengthening sets of D that intersect every minimum dicut exactly once.

Since every dicut of D (if any) has size at least τ, it follows that $|\delta^+(U)| + (\tau - 1)|\delta^-(U)| \geq \tau$ for all $U \subset V, U \neq \varnothing$, implying in turn that $x^\star := \frac{1}{\tau}\mathbf{1} \in F$.

Since $\mathcal{F} \neq \emptyset$, then $\gcd\{1 - |\delta^-(U)| : U \in \mathcal{F}\} = 1$, so we may apply Theorem 1 to conclude that $F \cap \{0,1\}^A$ contains an integral basis B for $\text{lin}(F)$. This implies that $\mathbf{1} = \tau x^\star \in \tau F$ is an integral linear combination of the vectors in B, say $\sum_{b \in B} \lambda_b \cdot b$. Furthermore, $\mathbf{1}^\top \lambda = \tau$, because $\mathbf{1}^\top \lambda = \sum_{b \in B} \lambda_b \cdot b(\delta^+(U)) = \mathbf{1}(\delta^+(U)) = \tau$ for any given $U \in \mathcal{F}$. Given that B is an integral basis for $\text{lin}(F)$, it follows that $\{b : \lambda_b \neq 0\}$ is also an integral basis for its linear hull, so we are done. □

Acknowledgement. Our work is inspired by the works of Edmonds, Lovász, and Pulleyblank [8], Lovász [13], and de Carvalho, Lucchesi, and Murty [4] on the matching lattice of a matching-covered graph. We would like to thank Kristóf Bérczi, Karthekeyan Chandrasekaran, Bertrand Guenin, and Levent Tunçel for fruitful discussions about various aspects of this work. This work was supported in part by EPSRC grant EP/X030989/1 and ONR grant N00014-22-1-2528.

Data Availability Statement. No data are associated with this article. Data sharing is not applicable to this article.

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