Lower bounds for cube-ideal set-systems

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Abstract

A set-system $S \subseteq \{0,1\}^n$ is cube-ideal if its convex hull can be described by capacity and generalized set covering inequalities. In this paper, we use combinatorics, convex geometry, and polyhedral theory to give exponential lower bounds on the size of cube-ideal set-systems, and linear lower bounds on their VC dimension. We then provide applications to graph theory and combinatorial optimization, specifically to strong orientations, perfect matchings, dijoins, and ideal clutters.

Keywords. Cube-ideal set-system, VC dimension, centrally symmetric hypercube, mixed graph, strong orientation, perfect matching.

1 Introduction

A set-system $S \subseteq \{0,1\}^n$ is *cube-ideal* if its convex hull can be described by capacity inequalities $\mathbf{0} \leq x \leq \mathbf{1}$ and *generalized set covering (GSC)* inequalities, which are of the form $\sum_{i\in I} x_i + \sum_{j\in J}(1-x_j) \geq 1$ for disjoint subsets $I, J \subseteq [n]$.

Cube-ideal set-systems form a rich class with examples coming from different corners of mathematics. For example, if the Hamming graph of $\{0,1\}^n \setminus S$ has degree at most 2, then S is cube-ideal [CL18]. A second example is the cycle space of any graph [BG86], while strongly connected re-orientations of any digraph give a third class of examples [EG77]. More recent examples have also been found, see [ACL20] for instance.

In propositional logic, a cube-ideal set-system corresponds to the solutions of a Boolean formula in clausal normal form whose linearization intersected with the unit hypercube forms an integral polytope [Hoo88, Hoo96]. In integer and linear programming, cube-idealness plays a key role in the study of a basic class of objects known as *ideal clutters* [CN94, Cor01]. More specifically, several important conjectures about ideal clutters can be reduced to or equivalently reformulated in terms of cube-ideal set-systems [ACGL20, ACHL21].

In this paper, we give exponential lower bounds on the size of cube-ideal set-systems, and linear lower bounds on their VC dimension. We then provide applications to strong orientations, perfect matchings, dijoins, and ideal clutters. To elaborate, we need two definitions.

Let $S \subseteq \{0,1\}^n$ be a set-system. The *connectivity of* S, denoted by $\lambda(S)$, is the minimum number of variables used in a GSC inequality valid for $\mathrm{conv}(S)$; if $S = \{0,1\}^n$ then $\lambda(S) := +\infty$.[1] Denote by $H : [0, \frac{1}{2}] \to [0,1]$ the *binary entropy* function, defined as $H(\varepsilon) := -\varepsilon \log_2(\varepsilon) - (1-\varepsilon)\log_2(1-\varepsilon)$ for $\varepsilon > 0$, and $H(0) := 0$. Note that H is a strictly increasing continuous function, $H(1/3) \approx 0.9183$, and $H(1/2) = 1$. We prove the following lower bound.

Theorem 1.1 (proved in §2)**.** *Let $S \subseteq \{0,1\}^n$ be a cube-ideal set-system with connectivity λ, where $\lambda \geq 3$. Then $|S| \geq 2^{(1-H(1/\lambda))n}$.*

Observe that $1 - H(1/\lambda) \geq 1 - H(1/3) \geq 0.0817$, so the theorem gives a lower bound of $2^{0.0817n}$ on the size of every cube-ideal set-system $S \subseteq \{0,1\}^n$ with connectivity at least 3. In fact, under certain conditions on

[1] This notion is closely related to the recently defined notion of *notch* – more precisely, when connectivity is finite, it is equal to the notch minus 1 [BFHW18].

S, we will show that a particular face of $\mathrm{conv}(S)$ also contains exponentially many $0,1$ points. To this end, a *set covering (SC)* inequality is an inequality of the form $\sum_{i\in I} x_i \geq 1$ for some subset $I \subseteq [n]$. For an integer $k \geq 2$, a *k-SC inequality* is an SC inequality with exactly k variables.

Theorem 1.2 (proved in §3)**.** *For every integer $\lambda \geq 3$ and constant $\beta \in (0,1]$, there is a constant $\theta := \theta(\lambda,\beta) > 0$ such that the following statement holds:*

Let $S \subseteq \{0,1\}^n$ be a cube-ideal set-system with connectivity λ such that every variable appears in a valid λ-SC inequality. Suppose the minimal face F of $\mathrm{conv}(S)$ containing $\frac{1}{\lambda}\mathbf{1}$ has dimension at least βn. Then $|S \cap F| \geq e^{\theta n}$.

What about cube-ideal set-systems $S \subseteq \{0,1\}^n$ with connectivity 2? Note that neither Theorem 1.1 nor Theorem 1.2 include the $\lambda = 2$ case. One explanation for this disparity is that, in contrast to the $\lambda \geq 3$ case, a cube-ideal set-system with connectivity 2 can have size linear in n even if $\mathrm{conv}(S)$ is full-dimensional, as we shall see in Example 5.1 in §5. Nonetheless, for cube-ideal set-systems of connectivity 2, we can still provide a lower bound on the size of S that is exponential not in n but in the dimension of the minimal face of $\mathrm{conv}(S)$ containing $\frac{1}{2}\mathbf{1}$. To elaborate, for an integer $k \geq 2$, a *k-GSC inequality* is a GSC inequality that involves exactly k variables.

Definition 1.3. *Let $S \subseteq \{0,1\}^n$ be a set-system with connectivity at least 2. The* 2-cover graph *of S, denoted $G(S)$, is the graph on vertex set $[n]$ with an edge $\{i,j\}$ for every pair i,j of indices such that one of the 2-GSC inequalities*

$$\begin{array}{ccccc} x_i & + & (1-x_j) & \geq & 1 \\ (1-x_i) & + & x_j & \geq & 1 \\ x_i & + & x_j & \geq & 1 \\ (1-x_i) & + & (1-x_j) & \geq & 1 \end{array}$$

is valid for $\mathrm{conv}(S)$. The core *of S, denoted $\mathrm{core}(S)$, is the set of points in S that satisfy every 2-GSC inequality valid for $\mathrm{conv}(S)$ at equality. A GSC inequality $\sum_{i\in I} x_i + \sum_{j\in J}(1-x_j) \geq 1$ valid for $\mathrm{conv}(S)$ is* rainbow *if $I \cup J$ intersects every connected component of $G(S)$ at most once.*

Observe that every rainbow inequality involves at least 3 variables.

When S is a cube-ideal set-system, we shall prove in §4 that $G(S)$ is the comparability graph of a preorder, and $\mathrm{core}(S)$ is also a cube-ideal set-system. We shall also prove the following lower bound on $|S|$.

Theorem 1.4 (proved in §4)**.** *Let $S \subseteq \{0,1\}^n$ be a cube-ideal set-system with connectivity at least 2, and fix an inequality description of $\mathrm{conv}(S)$ comprised of capacity and GSC inequalities. Let d be the number of connected components of $G(S)$, and let κ be the minimum number of variables used in a rainbow inequality in the fixed description for $\mathrm{conv}(S)$. Then $|\mathrm{core}(S)| \geq 2^{(1-H(1/\kappa))d}$.*

Note that $\kappa := +\infty$ if there is no rainbow inequality in the fixed description. Note further that if S has connectivity at least 3, then every valid GSC inequality is rainbow, so κ is simply the connectivity λ of S, so Theorem 1.4 extends Theorem 1.1.

It can be readily checked that in Theorem 1.4, d is simply the dimension of the minimal face F of $\mathrm{conv}(S)$ containing $\frac{1}{2}\mathbf{1}$, and that $\mathrm{core}(S) = S \cap F$. In this light, we conjecture the following generalization of Theorem 1.2, in which the condition lower bounding the dimension of the face has been dropped, though the lower bound on the number of $0,1$ points in the face depends instead on this dimension.

Conjecture 1.5. *For every integer $\lambda \geq 3$, there is a constant $\theta > 0$ such that the following statement holds:*

Let $S \subseteq \{0,1\}^n$ be a cube-ideal set-system with connectivity λ such that every variable appears in a valid λ-SC inequality. Let d be the dimension of the minimal face F of $\mathrm{conv}(S)$ containing $\frac{1}{\lambda}\mathbf{1}$. Then $|S \cap F| \geq e^{\theta d}$.

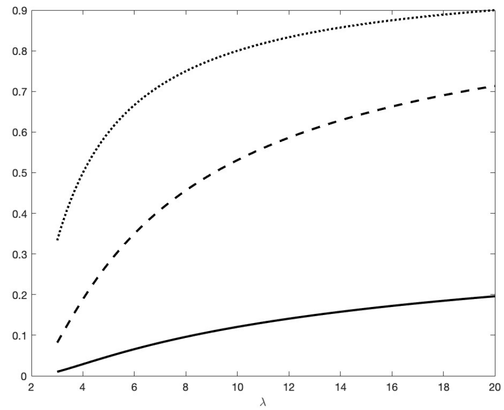

Figure 1: Graphs of f, g, h (from bottom to top) over the domain $[3, 20]$.

1.1 Lower bounding the VC dimension of cube-ideal set-systems

Theorem 1.1 may be used to lower bound the 'VC dimension' of cube-ideal set-systems with connectivity at least 3. To this end, let $S \subseteq \{0,1\}^n$ be a set-system, and let $I \subseteq [n]$. The *projection of S onto I* is the set-system $\{p_I : p \in S\}$, where $p_I \in \{0,1\}^I$ denotes the restriction of p to the index set I. The *Vapnik-Chervonenkis (VC) dimension* of S is the largest integer d such that S has $\{0,1\}^d$ as a projection; we set $d := 0$ if $S = \emptyset$ [VC71]. We shall use Theorem 1.1 to prove the following lower bound, where $H^{-1} : [0,1] \to [0, \frac{1}{2}]$ denotes the inverse of the binary entropy function.

Theorem 1.6 (proved in §2)**.** *Let $S \subseteq \{0,1\}^n$ be a cube-ideal set-system with connectivity λ, where $\lambda \geq 3$. Then S has VC dimension at least $f(\lambda) \cdot n$ where $f(\lambda) = H^{-1}(1 - H(1/\lambda))$.*

The function $f : (2, \infty) \to [0,1]$ is increasing (see Figure 1) and $f(3) \geq 0.01013$. In particular, every cube-ideal set-system $S \subseteq \{0,1\}^n$ with connectivity at least 3 has VC dimension at least $0.01013n$. We conjecture that Theorem 1.6 can be strengthened as follows.

Conjecture 1.7. *Let $S \subseteq \{0,1\}^n$ be a cube-ideal set-system with connectivity λ, where $\lambda \geq 3$. Then S has VC dimension at least $h(\lambda) \cdot n + 1$ where $h(\lambda) = 1 - 2/\lambda$.*

An intermediate result would be to prove the above for the function $g : (2, \infty) \to [0,1]$ defined as $g(\lambda) := 1 - H(1/\lambda)$. Given that every set-system of VC dimension d has size at least 2^d, this result would imply Theorem 1.1.

Conjecture 1.7 holds for an important class of examples from graphs. Let $\lambda \geq 3$ be an integer, let $G = (V, E)$ be a λ-edge-connected graph, and let $S \subseteq \{0,1\}^E$ be the *cycle space* of G, that is, S consists of the incidence vector of every edge-subset of G where every vertex has even degree. It is known that S is cube-ideal with connectivity λ [BG86]. Observe that S is a $GF(2)$-vector space of $GF(2)$-rank $|E| - |V| + 1$. This implies that S has VC dimension $|E| - |V| + 1$, and $|S| = 2^{|E|-|V|+1}$. As G has minimum degree at least λ, it follows that $|E| - |V| + 1 \geq (1 - 2/\lambda)|E| + 1$, where the inequality holds at equality if, and only if, G is λ-regular. Thus, Conjecture 1.7 holds for this example, and the lower bound can be tight. In the next subsection, we see that this conjecture also holds for another class of examples from graphs.

In §5, we shall use the famous *width-length* inequality for ideal clutters to prove Conjecture 1.7 for up-monotone set-systems, in fact, we will show that they have VC dimension at least $(1 - 1/\lambda)n$.

1.2 Applications to graph theory and combinatorial optimization

Our results have a number of applications to objects that are of importance in graph theory and combinatorial optimization, including perfect matchings, strong orientations, dijoins, and ideal clutters. We shall briefly explain two of these here, and postpone the remaining two to §5 and §6.

Let $G = (V, A, E)$ be a mixed graph with arcs A, which are directed, and edges E, which are not yet directed. A *G-strong orientation* is an orientation $\overrightarrow{E}$ of the edges such that the digraph $(V, A \cup \overrightarrow{E})$ is strongly connected. A blocking notion is that of a *pseudo dicut*, which is a cut of the form $\delta_G(U) \subseteq A \cup E$, where $U \subset V$, $U \neq \emptyset$ and $\delta^-_A(U) = \emptyset$. Observe that an orientation $\overrightarrow{E}$ is G-strong if, and only if, in every pseudo dicut $\delta_G(U)$ at least one edge of E is oriented to enter U.

Suppose every pseudo dicut of $G = (V, A, E)$ contains at least two edges from E. Let $\overrightarrow{E}$ be an arbitrary reference orientation of E, and denote by $\mathrm{SCR}(G; \overrightarrow{E})$ the set of vectors $x \in \{0,1\}^{\overrightarrow{E}}$ such that the re-orientation of $\overrightarrow{E}$ obtained after flipping the arcs in $\{a \in \overrightarrow{E} : x_a = 1\}$ is G-strong. We shall see that $\mathrm{SCR}(G; \overrightarrow{E})$ is a cube-ideal set-system, and thus use our results to obtain the following consequence.

Theorem 1.8 (proved in §6)**.** *Let $G = (V, A, E)$ be a mixed graph where every pseudo dicut contains at least λ edges from E, where $\lambda \geq 3$. Then the number of G-strong orientations is at least $2^{(1-H(1/\lambda))|E|}$.*

This is the first exponential lower bound on the number of G-strong orientations of such a mixed graph, as far as we know. When $A = \emptyset$, the lower bound in Theorem 1.8 is easy to obtain and can in fact be improved to $2^{(1-\frac{2}{\lambda})|E|+1}$ by using the theory of ear decompositions as follows. Every 2-vertex-connected block of G can be constructed from a cycle by successively adding *H-paths* to graphs H already constructed; the cycle and the paths are referred to as *ears* [[Die25], Chapter 3]. It can be readily checked that the number of the ears is exactly $|E| - |V| + k$, where k is the number of 2-vertex-connected blocks of G. Note that by orienting each ear so that it becomes a directed cycle or path, one obtains a strong orientation of G. The number of such orientations is $2^{|E|-|V|+k}$. As G is λ-edge-connected, every vertex has degree at least λ, so $|E| - |V| + k \geq (1 - \frac{2}{\lambda})|E| + 1$, implying in turn that the number of strong orientations of G is at least $2^{(1-\frac{2}{\lambda})|E|+1}$. In fact, we just proved a lower bound of $|E| - |V| + k \geq (1 - \frac{2}{\lambda})|E| + 1$ on the VC dimension of $\mathrm{SCR}(G; \overrightarrow{E})$, thus confirming Conjecture 1.7 for $\mathrm{SCR}(G; \overrightarrow{E})$ when $A = \emptyset$.

Let us exhibit another application. A *bipartite digraph* is a digraph $D = (V, A)$ where every vertex is a source or a sink. A *dicut* is a cut of the form $\delta^+(U) \subseteq A$ where $\delta^-(U) = \emptyset$, $U \subset V$, $U \neq \emptyset$. A blocking notion is that of a *dijoin*, which is a subset of A that intersects every dicut at least once. An important open problem in combinatorial optimization is Woodall's conjecture, which states that the minimum size of a dicut, say τ, is equal to the maximum number of arc-disjoint dijoins [Woo78]. It suffices to prove this conjecture for bipartite digraphs where every sink has degree τ [ACZ23]. We shall prove the following theorem as another consequence of our results.

Theorem 1.9 (proved in §6)**.** *For every integer $\tau \geq 3$, there is a constant $\theta > 0$ such that the following statement holds:*

Let $D = (V, A)$ be a bipartite digraph where every sink has degree τ, and every dicut has size at least τ. Then the number of minimal dijoins of D intersecting every minimum dicut exactly once is at least $e^{\theta|A|}$.

2 3-connected cube-ideal set-systems

In this section, we prove Theorem 1.1, which states that if $S \subseteq \{0,1\}^n$ is a cube-ideal set-system with connectivity $\lambda \geq 3$, then $|S| \geq 2^{(1-H(1/\lambda))n}$. We will then use this result to prove a lower bound on the VC dimension of such set-systems, namely Theorem 1.6. The following observation is crucial.

Lemma 2.1. *Let $S \subseteq \{0,1\}^n$ be a cube-ideal set-system with connectivity λ, where $\lambda \geq 3$. Then $\mathrm{conv}(S) \supseteq \left[\frac{1}{\lambda}, 1 - \frac{1}{\lambda}\right]^n$.*

Proof. As S is cube-ideal, $\operatorname{conv}(S)$ is described by capacity and GSC inequalities. As S has connectivity λ, every GSC inequality in the description must involve at least λ variables. Thus, any point in $\left[\frac{1}{\lambda}, 1-\frac{1}{\lambda}\right]^n$ satisfies the inequality description of $\operatorname{conv}(S)$, implying the desired containment. □

We will show that the containment above alone implies that $|S|$ must be exponentially large. We need the following well-known inequality.

Lemma 2.2 (see [MU17], §10.2, also [Gal14], Theorem 3.1). *For all integers $n \geq 1$ and $\lambda \geq 2$, the number of subsets of $[n]$ of size at most n/λ is at most $2^{H(1/\lambda)n}$.*

Given $S \subseteq \{0,1\}^n$ and $q \in \{0,1\}^n$, to *twist S by q* is to replace S by $S \triangle q := \{p \triangle q : p \in S\}$, where the second $\triangle$ denotes coordinate-wise addition modulo 2. Take a coordinate $i \in [n]$. Denote by e_i the i^{th} unit vector of appropriate dimension. To *twist coordinate i of S* is to replace S by $S \triangle e_i$.

We are now ready to prove the following.

Theorem 2.3. *Take an integer $\lambda \geq 3$. Let $S \subseteq \{0,1\}^n$ be a set-system such that $\operatorname{conv}(S) \supseteq \left[\frac{1}{\lambda}, 1-\frac{1}{\lambda}\right]^n$. Then $|S| \geq 2^{(1-H(1/\lambda))n}$.*

Proof. For each $w \in \{-1,+1\}^n$, let $x[w]$ be a point in S which maximizes $w^\top x$. As $\operatorname{conv}(S) \supseteq \left[\frac{1}{\lambda}, 1-\frac{1}{\lambda}\right]^n$, we have the inequality below,

$$w^\top x[w] = \max\left\{w^\top x : x \in \operatorname{conv}(S)\right\} \geq \frac{\lambda-1}{\lambda}|\mathrm{support}_+(w)| - \frac{1}{\lambda}|\mathrm{support}_-(w)|,$$

where $\mathrm{support}_+(w) = \{i \in [n] : w_i = +1\}$ and $\mathrm{support}_-(w) = \{i \in [n] : w_i = -1\}$. There exists an $x^\star \in S$ such that

$$\left|\left\{w \in \{-1,+1\}^n : w^\top x^\star = w^\top x[w]\right\}\right| \geq \frac{2^n}{|S|}.$$

Let $T := \left\{w \in \{-1,+1\}^n : w^\top x^\star = w^\top x[w]\right\}$. Note that for any $w \in T$, we have

$$w^\top x^\star \geq \frac{\lambda-1}{\lambda}|\mathrm{support}_+(w)| - \frac{1}{\lambda}|\mathrm{support}_-(w)|.$$

After twisting the coordinates, if necessary, we may assume that $x^\star = \mathbf{0}$; observe that twisting coordinate i maps $x_i \mapsto 1 - x_i$ and $w_i \mapsto -w_i$. This means that for any $w \in T$, we have

$$0 \geq \frac{\lambda-1}{\lambda}|\mathrm{support}_+(w)| - \frac{1}{\lambda}|\mathrm{support}_-(w)|,$$

or equivalently, $|\mathrm{support}_+(w)| \leq n/\lambda$. This implies in turn that

$$T \subseteq \{w \in \{-1,1\}^n : |\mathrm{support}_+(w)| \leq n/\lambda\}.$$

Subsequently,

$$|T| \leq \left|\left\{y \in \{0,1\}^n : \mathbf{1}^\top y \leq n/\lambda\right\}\right| \leq 2^{H(1/\lambda)n}$$

where the rightmost inequality follows from Lemma 2.2. Putting the inequalities together, we get

$$2^{H(1/\lambda)n} \geq |T| \geq \frac{2^n}{|S|}$$

implying in turn that $|S| \geq 2^{(1-H(1/\lambda))n}$, as desired. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $S \subseteq \{0,1\}^n$ be a cube-ideal set-system with connectivity λ, where $\lambda \geq 3$. By Lemma 2.1, $\operatorname{conv}(S) \supseteq \left[\frac{1}{\lambda}, 1-\frac{1}{\lambda}\right]^n$, so $|S| \geq 2^{(1-H(1/\lambda))n}$ by Theorem 2.3, as required. □

Next, we use the above to give a lower bound on the VC dimension of cube-ideal set-systems with connectivity at least 3. We shall need the following well-known lemma.

Lemma 2.4 (Sauer-Shelah [Sau72, She72]). *Let $n \geq k \geq 1$ be integers, and let $S \subseteq \{0,1\}^n$ be a set-system of VC dimension at most k. Then $|S|$ is at most the number of subsets of $[n]$ of size at most k.*

We need the following convenient consequence of the lemma.

Corollary 2.5. *Let $\lambda \geq 2$ be an integer, and let $S \subseteq \{0,1\}^n$ be a set-system such that $|S| \geq 2^{H(1/\lambda)n}$. Then S has VC dimension at least $\frac{n}{\lambda}$.*

Proof. Let $k^\circ := \frac{n}{\lambda}$ and $k := \lfloor k^\circ \rfloor$. We have that $|S| \geq 2^{H(1/\lambda)n} \geq 2^{H(k/n)n}$ as H is a strictly increasing function. Furthermore, by Lemma 2.2, $2^{H(k/n)n}$ is greater than or equal to the number of subsets of $[n]$ of size at most k.

If $k^\circ = k$, i.e., k° is an integer, then $|S|$ is strictly larger than the number of subsets of $[n]$ of size at most $k-1$. It therefore follows from (the contrapositive of) Lemma 2.4 that S has VC dimension strictly larger than $k-1$, so S has VC dimension at least k°.

Otherwise, $k^\circ > k$, so $|S| \geq 2^{H(1/\lambda)n} > 2^{H(k/n)n}$. Subsequently, by Lemma 2.2, $|S|$ is strictly larger than the number of subsets of $[n]$ of size at most k. It therefore follows from (the contrapositive of) Lemma 2.4 that S has VC dimension strictly larger than k, so S has VC dimension at least $k+1 > k^\circ$. $\square$

We are ready to prove Theorem 1.6.

Proof of Theorem 1.6. Let $S \subseteq \{0,1\}^n$ be a cube-ideal set-system with connectivity λ, where $\lambda \geq 3$. We know from Theorem 1.1 that $|S| \geq 2^{(1-H(1/\lambda))n}$. It therefore follows from Corollary 2.5 that S has VC dimension at least $H^{-1}(1-H(1/\lambda)) \cdot n$, as desired. $\square$

3 Faces of 3-connected cube-ideal set-systems

In this section, we prove Theorem 1.2 on a certain face of the convex hull of a cube-ideal set-system with connectivity at least 3. Below, $\mathrm{aff}(F)$ refers to the affine hull of F.

Lemma 3.1. *Let $\lambda \geq 3$ be an integer, and let $S \subseteq \{0,1\}^n$ be a cube-ideal set-system with connectivity λ such that if $x_i, i \in [n]$ appears in a valid λ-GSC inequality, then it appears in a valid λ-SC inequality. Let F be the minimal face of $\mathrm{conv}(S)$ containing $\frac{1}{\lambda}\mathbf{1}$. Then $\left[\frac{1}{\lambda+1}, 1-\frac{1}{\lambda+1}\right]^n \cap \mathrm{aff}(F) \subseteq F$.*

Proof. Let $\mathcal{V}$ be the set of all pairs (I,J) such that I, J are disjoint subsets of $[n]$, and $\sum_{i\in I} x_i + \sum_{j \in J}(1-x_j) \geq 1$ is valid for $\mathrm{conv}(S)$. Such an inequality is tight at $\frac{1}{\lambda}\mathbf{1}$ if, and only if, $J = \emptyset$ and $|I| = \lambda$; here we used the inequality $1 - \frac{1}{\lambda} > \frac{1}{\lambda}$ guaranteed by $\lambda \geq 3$. Thus,

$$F = \mathrm{conv}(S) \cap \{x : x(I) = 1,\ \forall\, (I,\emptyset) \in \mathcal{V} \text{ s.t. } |I| = \lambda\}$$
$$\mathrm{aff}(F) = \{x : x(I) = 1,\ \forall\, (I,\emptyset) \in \mathcal{V} \text{ s.t. } |I| = \lambda\}.$$

Let $p \in \left[\frac{1}{\lambda+1}, 1-\frac{1}{\lambda+1}\right]^n \cap \mathrm{aff}(F)$. We need to show that $p \in F$. To this end, let $(I,J) \in \mathcal{V}$. If $|I|+|J| \geq \lambda+1$, then p satisfies the corresponding GSC inequality as $p \in \left[\frac{1}{\lambda+1}, 1-\frac{1}{\lambda+1}\right]^n$. If $|I| = \lambda$ and $J = \emptyset$, then p satisfies the corresponding GSC inequality at equality as $p \in \mathrm{aff}(F)$. Otherwise, $|I|+|J| = \lambda$ and $J \neq \emptyset$. Let $k \in J$. By hypothesis, there exists $(K, \emptyset) \in \mathcal{V}$ such that $|K| = \lambda$ and $k \in K$. Given that p satisfies the corresponding GSC inequality at equality, it follows that $1-p_k = \sum_{\ell \in K\setminus k} p_\ell$, where the right-hand side involves $|K|-1 = \lambda-1 \geq 2$ variables. Subsequently, as $p \in \left[\frac{1}{\lambda+1}, 1-\frac{1}{\lambda+1}\right]^n$, we have

$$\sum_{i\in I} p_i + \sum_{j\in J}(1-p_j) = \sum_{i\in I} p_i + \sum_{\ell\in K\setminus k} p_\ell + \sum_{j\in J\setminus k}(1-p_j) \geq \frac{1}{\lambda+1}\left(|I|+2+|J|-1\right) = 1,$$

so p satisfies the corresponding GSC inequality, as required. $\square$

In what follows, we shall use the inclusion above to lower bound the number of $0, 1$ points inside the face F. To this end, we need the following theorem.

Theorem 3.2 ([Bar13]). *For every $\alpha, \beta \geq 1$ there is $\gamma := \gamma(\alpha, \beta) > 0$ such that the following statement holds:*

Suppose that $P \subset \mathbb{R}^d$ is a polytope containing the set

$$\left\{x \in \mathbb{R}^d : -1 \leq u_i^\top x \leq 1,\, \forall i \in [m]\right\},$$

where $\|u_i\|_2 \leq 1$, $\forall i \in [m]$ and $m \leq \alpha d$. Suppose further that P lies inside the ball

$$\left\{x \in \mathbb{R}^d : \|x\|_2 \leq \beta\sqrt{d}\right\}.$$

Then P has at least $e^{\gamma d}$ vertices.

Digression. We can apply Theorem 3.2 to get an alternate proof of Theorem 2.3 for sufficiently large n but with an inexplicit lower bound. More specifically, we let $P = \frac{2\lambda}{\lambda-2}(\mathrm{conv}(S) - \frac{1}{2}\mathbf{1})$. Then P contains $[-1, 1]^n$ and is contained in a Euclidean ball of radius $\frac{\lambda}{\lambda-2}\sqrt{n}$ around the origin. Thus, we can set $\alpha = 1$ and $\beta = \frac{\lambda}{\lambda-2}$. For sufficiently large n, Barvinok [Bar13] gives an explicit formula for $\gamma := \gamma(\alpha, \beta)$, which for our choices of α, β leads to the following: Choose any $\varepsilon \in (0, 1)$ and $\rho > 0$ such that the following inequality holds:

$$\ln\left(1 - \exp\left\{-\frac{\rho^2}{2}\right\}\right) > -\frac{\varepsilon^2}{4}.$$

Then for all sufficiently large $n > n_0(\alpha, \beta, \varepsilon, \rho)$ we can choose γ as follows:

$$\gamma = \frac{(\lambda-2)^2(1-\varepsilon)^2}{2\lambda^2\rho^2}\left(1 - \exp\left\{-\frac{\rho^2}{2}\right\}\right) + \ln\left(1 - \exp\left\{-\frac{\rho^2}{2}\right\}\right) > 0.$$

In contrast, Theorem 2.3 gives $\gamma := H(1/\lambda)\ln 2$ for all n.

Moving on, we are ready to prove the following lemma, which borrows some ideas from [[Bar13], Corollary 1.3].

Lemma 3.3. *For every integer $\lambda \geq 2$ and constants $\alpha \in \left(\frac{1}{\lambda+1}, \frac{1}{2}\right]$, $\beta \in (0, 1]$, there is a constant $\theta > 0$ such that the following statement holds:*

Let $S \subseteq \{0, 1\}^n$ be a set-system whose convex hull contains $\alpha\mathbf{1}$ (possibly on its boundary), and let F be the minimal face of the convex hull containing $\alpha\mathbf{1}$. Suppose that F has dimension at least βn, and $\left[\frac{1}{\lambda+1}, 1 - \frac{1}{\lambda+1}\right]^n \cap \mathrm{aff}(F) \subseteq F$. Then $|S \cap F| \geq e^{\theta n}$.

Proof. Let $d := \dim(F) \geq \beta n$, let A be the affine hull of F, and let $R := S \cap F \subseteq \{0, 1\}^n \cap A$. By hypothesis, $\mathrm{conv}(R) \supseteq \left[\frac{1}{\lambda+1}, 1 - \frac{1}{\lambda+1}\right]^n \cap A$. Let us shift $\mathrm{conv}(R), A$ so that they both contain the origin. To this end, let $a := \alpha\mathbf{1}$, $P := \mathrm{conv}(R) - a$, and $L := A - a$ which is a linear subspace of dimension d containing P.

First we show that P contains a large hypercube-like polytope with $O(d)$ facets. To this end, let $\varepsilon := \alpha - \frac{1}{\lambda+1} > 0$. We claim that $Q := [-\varepsilon, \varepsilon]^n \cap L \subseteq P$. To see this, observe that $a + y \in \left[\frac{1}{\lambda+1}, 1 - \frac{1}{\lambda+1}\right]^n$ for each $y \in [-\varepsilon, \varepsilon]^n$, so $a + y \in \mathrm{conv}(R)$ for all $y \in [-\varepsilon, \varepsilon]^n$ such that $a + y \in A$, implying in turn that $Q \subseteq P$. Observe that

$$P \supseteq Q = \left\{x \in L : -\varepsilon \leq u_i^\top x \leq \varepsilon,\, \forall i \in [n]\right\},$$

where $u_i \in L$ is the orthogonal projection of the standard unit vector $e_i \in \mathbb{R}^n$ onto L, for $i \in [n]$. In particular, $\|u_i\|_2 \leq 1$ for all $i \in [n]$. Furthermore, we know that $n \leq \frac{1}{\beta}d$.

Secondly, we show that P is contained in a Euclidean ball of radius $O(\sqrt{d})$. For every $p \in S$, we have

$$\|p - a\|_2 \leq \|p\|_2 + \|a\|_2 \leq \sqrt{n} + \alpha\sqrt{n} = (1 + \alpha)\sqrt{n} \leq \frac{1 + \alpha}{\sqrt{\beta}}\sqrt{d}.$$

Thus,

$$P \subset B := \left\{ x \in L : \|x\|_2 \leq \frac{1+\alpha}{\sqrt{\beta}} \sqrt{d} \right\}.$$

Finally, we apply a linear transformation from $L \subset \mathbb{R}^n$ to $\mathbb{R}^d$. To this end, let $\ell_1, \ldots, \ell_d$ be an orthonormal basis for L, and let $M \in \mathbb{R}^{n \times d}$ be the matrix whose columns are $\ell_1, \ldots, \ell_d$. Let $f : L \to \mathbb{R}^d$ be the linear transformation that maps ℓ_i to $e_i \in \mathbb{R}^d$ for $i \in [d]$. Note that $f^{-1}(z) = Mz$.

We now work with $f(P)$ instead of P. We have

$$f(P) \supseteq f(Q) = \left\{ z \in \mathbb{R}^d : -\varepsilon \leq w_i^\top z \leq \varepsilon, \forall i \in [n] \right\},$$

where $w_i = M^\top u_i$ for each $i \in [n]$. Observe that

$$\|w_i\|_2 \leq \|M^\top\|_2 \|u_i\|_2 = \sqrt{\lambda_{\max}(MM^\top)} \cdot \|u_i\|_2 = \|u_i\|_2 \leq 1,$$

where $\|M\|_2$ denotes the spectral norm of M, which is equal to the square root of the largest eigenvalue of $MM^\top$; the latter has the same nonzero spectrum as $M^\top M = I_d$, so $\lambda_{\max}(MM^\top) = 1$.

Next we have

$$f(P) \subset f(B) := \left\{ z \in \mathbb{R}^d : \|Mz\|_2 \leq \frac{1+\alpha}{\sqrt{\beta}} \sqrt{d} \right\} = \left\{ z \in \mathbb{R}^d : \|z\|_2 \leq \frac{1+\alpha}{\sqrt{\beta}} \sqrt{d} \right\},$$

where the second equality follows from $\|Mz\|_2 = \sqrt{z^\top M^\top M z} = \|z\|_2$.

Subsequently, by applying Theorem 3.2 to $\varepsilon^{-1} f(P)$, we obtain that $\varepsilon^{-1} f(P)$ has at least $e^{\gamma d}$ vertices, for some constant $\gamma := \gamma\left(\frac{1}{\beta}, \frac{\varepsilon^{-1}(1+\alpha)}{\sqrt{\beta}}\right) > 0$. Observe that $\varepsilon^{-1} f(P)$ has the same number of vertices as $f(P)$ and also P, namely $|R|$. Thus, $|S \cap F| = |R| \geq e^{\gamma d} \geq e^{\gamma \beta n}$, so $\theta := \gamma\beta > 0$ is the desired constant. $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. It follows from Lemma 3.1 that $\left[\frac{1}{\lambda+1}, 1 - \frac{1}{\lambda+1}\right]^n \cap \operatorname{aff}(F) \subseteq F$. We can therefore apply Lemma 3.3 for $\alpha = \frac{1}{\lambda}$ to obtain the result. $\square$

4 2-connected cube-ideal set-systems

In this section, we study cube-ideal set-systems with connectivity (at least) 2, and prove Theorem 1.4 on the size of their core. We will need to study the 2-cover graph and core of such set-systems.

Take a *preorder* $([n], \succeq)$, i.e., a binary relation $\succeq$ on $[n]$ that satisfies *reflexivity* ($i \succeq i, \forall i \in [n]$) and *transitivity* ($i \succeq j, j \succeq k \Rightarrow i \succeq k, \forall i, j, k \in [n]$). The *comparability graph* of the preorder is the undirected graph on vertex set $[n]$ with an edge between every pair of distinct vertices that are comparable in the preorder.

Let $S \subseteq \{0,1\}^n$ be a set-system with connectivity at least 2. Recall that $G(S)$ denotes the 2-cover graph of S. Note that $G(S)$ is invariant under twisting S.

Lemma 4.1. *Let $S \subseteq \{0,1\}^n$ be a set-system where $\frac{1}{2}\mathbf{1} \in \operatorname{conv}(S)$. Then, after possibly twisting S, every 2-GSC inequality valid for $\operatorname{conv}(S)$ is of the form $x_i \geq x_j$ for distinct $i, j \in [n]$. Furthermore, $G(S)$ is a comparability graph.*

Proof. Let us write $\frac{1}{2}\mathbf{1}$ as a convex combination of the points in S with coefficients $\lambda_p, p \in S$ where $\lambda \geq \mathbf{0}$ and $\sum_{p \in S} \lambda_p = 1$. After twisting the coordinates of S, if necessary, we may assume that $\lambda_{\mathbf{0}} > 0$. Every 2-GSC inequality valid for $\operatorname{conv}(S)$ is satisfied at equality at $\frac{1}{2}\mathbf{1}$, and therefore at $\mathbf{0}$, so it must be of the form $x_i \geq x_j$ for some distinct indices $i, j \in [n]$. Consider now the binary relation $\succeq$ on $[n]$ where $i \succeq j$ if $x_i \geq x_j$ is a valid inequality for $\operatorname{conv}(S)$, for $i, j \in [n]$. It can be readily checked that $([n], \succeq)$ is reflexive and transitive, so it is a preorder, and clearly $G(S)$ is its comparability graph, as required. $\square$

We obtain the following fact about the 2-cover graph of a cube-ideal set-system.

Theorem 4.2. *Let $S \subseteq \{0,1\}^n$ be a cube-ideal set-system with connectivity at least 2. Then $G(S)$ is a comparability graph.*

Proof. As S is cube-ideal with connectivity at least 2, it follows that $\frac{1}{2}\mathbf{1} \in \mathrm{conv}(S)$. Subsequently, $G(S)$ is a comparability graph by Lemma 4.1. □

Recall that $\mathrm{core}(S)$ is the set of points in S that satisfy every 2-GSC inequality valid for $\mathrm{conv}(S)$ at equality. We have the following phenomenon.

Theorem 4.3. *Let $S \subseteq \{0,1\}^n$ be a cube-ideal set-system with connectivity at least 2. Then $\mathrm{core}(S)$ is also cube-ideal.*

Proof. Observe that an inequality description of $\mathrm{conv}(\mathrm{core}(S))$ can be obtained from that of $\mathrm{conv}(S)$ by adding, for each 2-GSC inequality valid for $\mathrm{conv}(S)$, the inequality obtained by flipping the direction of the inequality. For each 2-GSC inequality, however, the reverse inequality is also GSC: for example, $x_i + x_j \geq 1$ reversed is simply $(1 - x_i) + (1 - x_j) \geq 1$. Subsequently, $\mathrm{core}(S)$ remains a cube-ideal set-system. □

We now give a description for the convex hull of the core of a cube-ideal set-system S. Let $\sum_{i\in I} x_i + \sum_{j\in J}(1 - x_j) \geq 1$ be a GSC inequality valid for $\mathrm{conv}(S)$. Recall that the inequality is rainbow if $I \cup J$ intersects every connected component of $G(S)$ at most once. As we see below, capacity and rainbow inequalities are sufficient to describe $\mathrm{conv}(\mathrm{core}(S))$.

Lemma 4.4. *Let $S \subseteq \{0,1\}^n$ be a cube-ideal set-system with connectivity at least 2, and fix an inequality description of $\mathrm{conv}(S)$ comprised of capacity and GSC inequalities. Then the following statements hold:*

1. *after possibly twisting S, we have $x_i = x_j$ for all $x \in \mathrm{conv}(\mathrm{core}(S))$ and indices i, j in the same connected component of $G(S)$,*
2. *every facet-defining inequality for $\mathrm{conv}(\mathrm{core}(S))$ is equivalent to either a capacity or rainbow inequality in the fixed description of $\mathrm{conv}(S)$.*

Proof. **(1)** By Lemma 4.1, we may assume after possibly twisting S that every 2-GSC inequality valid for $\mathrm{conv}(S)$ is of the form $x_i \geq x_j$ for distinct $i, j \in [n]$. In particular, $x_i = x_j$ for all $x \in \mathrm{conv}(\mathrm{core}(S))$ and any pair i, j of indices in the same connected component of $G(S)$.

(2) Take a non-capacity facet-defining inequality $a^\top x \geq \beta$ for $\mathrm{conv}(\mathrm{core}(S))$. A description of the polytope $\mathrm{conv}(\mathrm{core}(S))$ is obtained from the fixed inequality description of $\mathrm{conv}(S)$ after setting all the 2-GSC inequalities to equality. Subsequently, $a^\top x \geq \beta$ is equivalent to an implicit inequality in the fixed description for $\mathrm{conv}(S)$, say of the form

$$\sum_{i\in I} x_i + \sum_{j\in J}(1 - x_j) \geq 1 \tag{1}$$

for disjoint subsets $I, J \subseteq [n]$. Denote by $\mathcal{K}$ the set of connected components of $G(S)$, and for each $K \in \mathcal{K}$, take a representative index $i_K \in K$. By part (1), the inequality $\sum_{i\in I} x_i + \sum_{j\in J}(1 - x_j) \geq 1$ can be written equivalently as the following valid inequality for $\mathrm{conv}(\mathrm{setcore}(S))$:

$$\sum_{K\in\mathcal{K}} |I \cap K| x_{i_K} + \sum_{K\in\mathcal{K}} |J \cap K|(1 - x_{i_K}) \geq 1. \tag{2}$$

Given that $\mathbf{1} \geq x \geq \mathbf{0}$, and the right-hand side value in (2) is 1, the nonzero coefficients $|I \cap K|, |J \cap K|$ on the left-hand side can be truncated to 1 all the while keeping the inequality valid. Since (2) is facet-defining for $\mathrm{conv}(\mathrm{core}(S))$, and therefore not strictly dominated by another valid inequality, we must have that $|I \cap K|, |J \cap K| \leq 1$ for all $K \in \mathcal{K}$. Furthermore, we cannot have $|I \cap K| = |J \cap K| = 1$ for some $K \in \mathcal{K}$, since otherwise (2) will be dominated by the equality $x_{i_K} + (1 - x_{i_K}) = 1$. Subsequently, $|(I \cup J) \cap K| \leq 1$ for all $K \in \mathcal{K}$, implying in turn that (1) is a rainbow inequality valid for $\mathrm{conv}(S)$. □

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. By Lemma 4.4 part (1), after possibly twisting S, we may assume that $x_i = x_j$ for all $x \in \mathrm{core}(S)$ and indices i, j in the same connected component of $G(S)$. Let $\mathrm{setcore}(S) \in \{0,1\}^d$ be the set-system obtained from S after keeping only one representative from each of the d connected components of $G(S)$, and dropping the remaining indices. Clearly, each rainbow inequality in the description for $\mathrm{conv}(S)$ yields a valid GSC inequality for $\mathrm{conv}(\mathrm{setcore}(S))$ with the same number of variables, and by Lemma 4.4 part (2), these inequalities together with the capacity inequalities are sufficient to describe $\mathrm{conv}(\mathrm{setcore}(S))$. In particular, S' is a cube-ideal set-system with connectivity $\kappa \geq 3$. Thus, $|\mathrm{core}(S)| = |\mathrm{setcore}(S)\,| \geq 2^{(1-H(1/\kappa))d}$ by Theorem 1.1. □

5 Ideal clutters

In this section, we introduce ideal clutters, connect them to cube-ideal set-systems in two ways, and deduce what our results mean for them; these will be useful in the next section.

Let V be a finite set of *elements*, and let $\mathcal{C}$ be a family of subsets of V called *members* or *sets*. $\mathcal{C}$ is a *clutter* over *ground set* V if no member contains another [EF70]. $\mathcal{C}$ is *ideal* if the associated set covering polyhedron, namely $Q(\mathcal{C}) := \left\{x \in \mathbb{R}^V_+ : \sum_{v \in C} x_v \geq 1, \forall C \in \mathcal{C}\right\}$, has only integral vertices [CN94].

In integer and linear programming, ideal clutters correspond to set covering linear programs that have an integral optimal solution for any objective vector for which there is a finite optimum. This can be guaranteed when, for instance, the coefficient matrix of the linear system defining $Q(\mathcal{C})$ is *totally unimodular*, i.e., every nonzero sub-determinant is ± 1. In graph and matroid theory, they correspond to multi-commodity flow problems where the *cut condition* necessary for the existence of a flow is also sufficient [Sey81, Gue01, Gue16].

To every set-system $S \subseteq \{0,1\}^n$ we can associate a clutter, and through this correspondence we can see a connection between cube-idealness and idealness. The *cuboid* of S, denoted $\mathrm{cuboid}(S)$, is the clutter over ground set $[2n]$ whose members are $\{2i-1 : p_i = 1\} \cup \{2j : p_j = 0\}$, $\forall p \in S$. It is known S is cube-ideal if, and only if, $\mathrm{cuboid}(S)$ is an ideal clutter [ACGL20]. We can use this to prove the following statement claimed in the introduction.

Example 5.1. *Let $S := \{\mathbf{0}, e_1, e_1+e_2, \ldots, e_1+e_2+\cdots+e_n\} \subseteq \{0,1\}^n$. Then S is a cube-ideal set-system with connectivity* 2 *such that* $\mathrm{conv}(S)$ *is full-dimensional and* $|S| = n+1$.

Proof. It can be readily checked that $\mathrm{conv}(S)$ is full-dimensional, $|S| = n+1$, and that $\frac{1}{2}\mathbf{1}$ is the midpoint of the edge of $\mathrm{conv}(S)$ connecting $\mathbf{0}, \mathbf{1} \in S$. In particular, S must have connectivity 2. It remains to show that S is cube-ideal. Observe that the incidence matrix of $\mathrm{cuboid}(S)$ has the *consecutive* 1*s property*, that is, its columns can be permuted so that the 1s in each row appear consecutively. This implies that the coefficient matrix of $Q(\mathrm{cuboid}(S))$ is totally unimodular [FG64], implying in turn that $\mathrm{cuboid}(S)$ is ideal, thus S is a cube-ideal set-system. □

Let $\mathcal{C}$ be a clutter over ground set V. A *cover* is a subset $B \subseteq V$ such that $B \cap C \neq \emptyset$ for all $C \in \mathcal{C}$. The *covering number* of $\mathcal{C}$, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover. A cover is *minimal* if it does not contain another cover. The *blocker* of $\mathcal{C}$, denoted $b(\mathcal{C})$, is the clutter over ground set V whose members are the minimal covers of $\mathcal{C}$ [EF70]. It is well-known that $b(b(\mathcal{C})) = \mathcal{C}$ [Isb58, EF70]. Observe that if $\mathcal{C}$ is ideal, then the vertices of $Q(\mathcal{C})$ are precisely the indicator vectors of the minimal covers of $\mathcal{C}$. A fascinating feature of idealness is that it is closed under taking the blocker [Ful71, Leh79]. In fact, $\mathcal{C}$ is ideal if, and only if, the *width-length* inequality holds for all $w, \ell \in \mathbb{R}^V_{\geq 0}$, that is, $\min\{w(C) : C \in \mathcal{C}\} \cdot \min\{\ell(B) : B \in b(\mathcal{C})\} \leq w^\top \ell$ [Leh79].

To every clutter we can associate a set-system, and through this correspondence we get yet another connection between idealness and cube-idealness. Let $S(\mathcal{C}) := \{\mathbf{1}_C : C \subseteq V, C \text{ contains a set in } \mathcal{C}\} \subseteq \{0,1\}^V$. The set-system $S(\mathcal{C})$ is *up-monotone*, that is, if $p \geq q$ for some $p, q \in \{0,1\}^V$ where $q \in S(\mathcal{C})$, then $p \in S(\mathcal{C})$. Note further that the points in S of minimal support are precisely the indicator vectors of the sets in $\mathcal{C}$. It is known that $\mathcal{C}$ is ideal if, and only if, $S(\mathcal{C})$ is a cube-ideal set-system [ACGL20]. We have the following.

Theorem 5.2. *Let $S \subseteq \{0,1\}^n$ be a cube-ideal set-system with connectivity $\lambda \geq 1$ that is up-monotone. Then S has VC dimension at least $(1-1/\lambda)n$.*

Proof. Observe that $S = S(\mathcal{C})$ for a clutter $\mathcal{C}$ over ground set $[n]$. It can be readily checked that S has VC dimension exactly $n - \min\{|C| : C \in \mathcal{C}\}$ and connectivity exactly $\tau(\mathcal{C}) =: \lambda$. As S is cube-ideal, it follows that $\mathcal{C}$ is ideal. Thus, by the width-length inequality, $\min\{|C| : C \in \mathcal{C}\} \leq \frac{n}{\min\{B : B \in b(\mathcal{C})\}} = \frac{n}{\lambda}$. Subsequently, S has VC dimension at least $(1-1/\lambda)n$. $\square$

Given an integer $\tau \geq 1$, $\mathcal{C}$ is *τ-cover-minimal* if it has covering number τ, and every element appears in a minimum cover. A clutter is *cover-minimal* if it is τ-cover-minimal for some integer $\tau \geq 1$. The *core* of a cover-minimal clutter $\mathcal{C}$, denoted $\mathrm{core}(\mathcal{C})$, is the clutter of all sets of $\mathcal{C}$ that intersect every minimum cover exactly once.

Lemma 5.3. *Let $\mathcal{C}$ be an ideal τ-cover-minimal clutter, for some integer $\tau \geq 1$. Then $S(\mathcal{C})$ is a cube-ideal set-system with connectivity τ, where every variable appears in a valid τ-SC inequality. Furthermore, for F the minimal face of $\mathrm{conv}(S(\mathcal{C}))$ containing $\frac{1}{\tau}\mathbf{1}$, the dimension of F is $|V| - r$ where r is the rank of $\{\mathbf{1}_B : B \in b(\mathcal{C}), |B| = \tau\}$, and $S(\mathcal{C}) \cap F = \{\mathbf{1}_C : C \in \mathrm{core}(\mathcal{C})\}$.*

Proof. Let $S := S(\mathcal{C})$. As $\mathcal{C}$ is ideal, S is a cube-ideal set-system whose convex hull is described by $\mathbf{0} \leq x \leq \mathbf{1}, x(B) \geq 1, \forall B \in b(\mathcal{C})$. In particular, S has connectivity $\tau(\mathcal{C}) = \tau$, and as $\mathcal{C}$ is τ-cover-minimal, every variable appears in a τ-SC inequality valid for S. Furthermore, the minimal face of $\mathrm{conv}(S)$ containing $\frac{1}{\tau}\mathbf{1}$ is

$$F = \mathrm{conv}(S) \cap \{x : x(B) = 1, \forall B \in b(\mathcal{C}) \text{ s.t. } |B| = \tau\},$$

where all the tight constraints are as described. Subsequently, the dimension of F is $|V| - r$. It remains to prove $S \cap F = \{\mathbf{1}_C : C \in \mathrm{core}(\mathcal{C})\}$. The inclusion $\supseteq$ is clear. For the reverse inclusion, suppose $\widehat{C} \subseteq V$ contains a set $C \in \mathcal{C}$, and $\mathbf{1}_{\widehat{C}} \in S \cap F$. We claim that $\widehat{C} = C$. For if not, pick $v \in \widehat{C} \setminus C$, and let B be a minimum cover of $\mathcal{C}$ that contains v, which exists as $\mathcal{C}$ is cover-minimal. As $B \cap C \neq \emptyset$ and $v \in B$, it follows that $|B \cap \widehat{C}| \geq 2$, a contradiction as $|B \cap \widehat{C}| = 1$. $\square$

We may therefore apply Theorem 1.2 to obtain the following inequality.

Theorem 5.4. *For every integer $\tau \geq 3$ and $\beta \in (0,1)$, there is a constant $\theta > 0$ such that the following statement holds:*

Let $\mathcal{C}$ be an ideal τ-cover-minimal clutter over ground set V. Suppose the rank of $\{\mathbf{1}_B : B \in b(\mathcal{C}), |B| = \tau\}$ is at most $(1-\beta)|V|$. Then $|\mathrm{core}(\mathcal{C})| \geq e^{\theta|V|}$.

Proof. Let $S := S(\mathcal{C})$. By Lemma 5.3, S is a cube-ideal set-system with connectivity $\lambda := \tau$, where every variable appears in a valid λ-SC inequality. Furthermore, for F the minimal face of $\mathrm{conv}(S)$ containing $\frac{1}{\lambda}\mathbf{1}$, the dimension of F is at least $|V| - (1-\beta)|V| = \beta|V|$, and $|\mathrm{core}(\mathcal{C})| = |S \cap F|$. Thus, by Theorem 1.2, $|\mathrm{core}(\mathcal{C})| \geq e^{\theta|V|}$ for $\theta = \theta(\lambda, \beta)$. $\square$

Let $\mathcal{C}$ be a 2-cover-minimal clutter over ground set V – these are also known as *tangled* clutters [ACHL21]. Denote by $G(\mathcal{C})$ the graph over vertex set V whose edges correspond to the minimum covers of $\mathcal{C}$. A cover of $\mathcal{C}$ is *rainbow* if it intersects every connected component of $G(\mathcal{C})$ at most once. The *rainbow covering number of* $\mathcal{C}$, denoted $\mu(\mathcal{C})$, is the minimum size of a rainbow cover of $\mathcal{C}$; if there is no rainbow cover, then $\mu(\mathcal{C}) := +\infty$. Observe that $\mu(\mathcal{C}) \geq 3$. We prove the following exponential lower bound on the size of the core of such clutters.

Theorem 5.5. *Let $\mathcal{C}$ be an ideal 2-cover-minimal clutter, let d be the number of connected components of $G(\mathcal{C})$, and let μ be its rainbow covering number. Then $|\mathrm{core}(\mathcal{C})| \geq 2^{(1-H(1/\mu))d}$.*

Proof. Let $S := S(\mathcal{C})$. We know from Lemma 5.3 that S is a cube-ideal set-system with connectivity 2. Furthermore, every valid 2-GSC inequality is in fact a 2-SC inequality, so $G(S) = G(\mathcal{C})$ and $\mathrm{core}(S) = \{\mathbf{1}_C : C \in \mathrm{core}(\mathcal{C})\}$. The claimed inequality now follows from Theorem 1.4. $\square$

In a similar fashion, Conjecture 1.5 implies the following (the two conjectures are in fact equivalent).

Conjecture 5.6. *For every integer $\tau \geq 3$, there is a constant $\theta > 0$ such that the following statement holds:*

Let $\mathcal{C}$ be an ideal τ-cover-minimal clutter over ground set V. Let d be the dimension of the minimal face of $Q(b(\mathcal{C}))$ containing $\frac{1}{\tau}\mathbf{1}$. Then $|\mathrm{core}(\mathcal{C})| \geq e^{\theta d}$.

6 Applications to combinatorial optimization

In this section, we present three further applications of our results to combinatorial optimization.

6.1 Strong orientations of mixed graphs

Here we prove Theorem 1.8. We first show that strong orientations of a mixed graph correspond to a cube-ideal set-system. Let us elaborate.

Let $\mathcal{F}$ be a family of subsets of a finite set V. $\mathcal{F}$ is a *crossing family* if $U \cap W, U \cup W \in \mathcal{F}$ for any two sets $U, W \in \mathcal{F}$ such that $U \cap W \neq \emptyset$ and $U \cup W \neq V$. For a crossing family $\mathcal{F}$, a function $f : \mathcal{F} \to \mathbb{Z}$ is *crossing supermodular* if $f(U \cap W) + f(U \cup W) \geq f(U) + f(W)$ for any two sets $U, W \in \mathcal{F}$ such that $U \cap W \neq \emptyset$ and $U \cup W \neq V$.

Let $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$. The system $Ax \leq b$ is *totally dual integral (TDI)* if for every cost vector $c \in \mathbb{Z}^n$ for which $\max\{c^\top x : Ax \leq b\}$ has an optimum solution, its dual $\min\{b^\top y : A^\top y = c, y \geq 0\}$ has an integral optimum solution. If the right-hand side b is integral and the system $Ax \leq b$ is TDI, then the polyhedron $P := \{x : Ax \leq b\}$ is integral [EG77]. A system $Ax \leq b$ is called *box-TDI* if for every pair of integral vectors $\ell \leq u$, the system $Ax \leq b, \ell \leq x \leq u$ is TDI. In particular, every box-TDI system is also TDI.

Let $G = (V, A, E)$ be a mixed graph where every pseudo dicut contains at least two edges from E. Let $\overrightarrow{E}$ be an arbitrary reference orientation of E. Denote by $\mathrm{SCR}(G; \overrightarrow{E})$ the set of vectors $x \in \{0,1\}^{\overrightarrow{E}}$ such that the re-orientation of $\overrightarrow{E}$ obtained after flipping the arcs in $\{a \in \overrightarrow{E} : x_a = 1\}$ is G-strong, i.e., turns G into a strongly connected digraph.

Theorem 6.1. *For a mixed graph $G = (V, A, E)$, and a reference orientation $\overrightarrow{E}$, the convex hull of $\mathrm{SCR}(G; \overrightarrow{E})$ is described by*

$$\sum_{a\in\delta^+_{\overrightarrow{E}}(U)} x_a + \sum_{b\in\delta^-_{\overrightarrow{E}}(U)} (1 - x_b) \geq 1 \quad \forall U \subset V, U \neq \emptyset, \delta^-_A(U) = \emptyset, \tag{3}$$

$$\mathbf{1} \geq x \geq \mathbf{0}. \tag{4}$$

In particular, the set-system $\mathrm{SCR}(G; \overrightarrow{E})$ is cube-ideal.

Proof. It can be readily checked that the integral solutions to (3)-(4) are precisely the points in $\mathrm{SCR}(G; \overrightarrow{E})$. Thus, it remains to show that the system is integral. To this end, let $\mathcal{F} := \{U \subset V : U \neq \emptyset, \delta^-_A(U) = \emptyset\}$, which is a crossing family. Note that (3) can be rewritten as

$$x(\delta^+_{\overrightarrow{E}}(U)) - x(\delta^-_{\overrightarrow{E}}(U)) \geq 1 - |\delta^-_{\overrightarrow{E}}(U)| \quad \forall U \in \mathcal{F}. \tag{5}$$

Given that $U \in \mathcal{F} \mapsto 1 - |\delta^-_{\overrightarrow{E}}(U)| \in \mathbb{Z}$ is a crossing supermodular function, we conclude from a well-known result [EG77] that (5) is box-TDI. In particular, (3)-(4) is TDI, and is therefore integral, as required. $\square$

Suppose every pseudo dicut of G contains at least two edges from E. The *2-pseudo-dicut graph of G* is the graph H on vertex set E with an edge between distinct e, f for every pseudo dicut of G that contains only e, f from E. A pseudo dicut of G is *rainbow* if it contains at most one element of E from every connected component of the 2-pseudo-dicut graph. In particular, every rainbow pseudo dicut contains at least 3 elements from E. We are now ready to prove the following theorem. Note that $\kappa := +\infty$ if there is no rainbow pseudo dicut.

Theorem 6.2. *Let $G=(V,A,E)$ be a mixed graph where every pseudo dicut contains at least 2 edges from E. Let d be the number of connected components of the 2-pseudo-dicut graph of G, and let κ be the minimum number of elements from E in a rainbow pseudo dicut. Then the number of G-strong orientations is at least $2^{(1-H(1/\kappa))d}$.*

Proof. Let $\overrightarrow{E}$ be an arbitrary reference orientation of E, and let $S := \mathrm{SCR}(G;\overrightarrow{E})$, which is cube-ideal by Theorem 6.1. Fix the inequality description (3)-(4) for $\mathrm{conv}(S)$. It therefore follows from Theorem 1.4 that $|\mathrm{core}(S)| \geq 2^{(1-H(1/\kappa'))d'}$, where d' is the number of connected components of $G(S)$, and κ' is the minimum number of variables used in a rainbow inequality in (3). Observe that $G(S)$ is precisely the 2-pseudo-dicut graph of G. Furthermore, the rainbow inequalities in (3) correspond to the rainbow pseudo dicuts of G, and the number of variables in the inequality is equal to the number of edges from E is the pseudo dicut. Subsequently, $d'=d$ and $\kappa'=\kappa$, so $|S| \geq |\mathrm{core}(S)| \geq 2^{(1-H(1/\kappa))d}$, as required. $\square$

We obtain Theorem 1.8 as a consequence of this theorem.

Proof of Theorem 1.8. Let $G=(V,A,E)$ be a mixed graph where every pseudo dicut contains at least λ edges from E, where $\lambda \geq 3$. Then the 2-pseudo-dicut graph of G has no edge, so it has exactly $|E|$ connected components, and every pseudo dicut of G is rainbow. Subsequently, the number of G-strong orientations is at least $2^{(1-H(1/\lambda))|E|}$. $\square$

6.2 Dijoins in τ-sink-regular bipartite digraphs

Proof of Theorem 1.9. Let $\mathcal{C}$ be the clutter of minimal dijoins of D. It is known that $\mathcal{C}$ is an ideal clutter, and that $b(\mathcal{C})$ consists of the minimal dicuts of D [LY78]. Thus, $\mathcal{C}$ is τ-cover-minimal, and $\mathrm{core}(\mathcal{C})$ is precisely the clutter of minimal dijoins of D intersecting every minimum dicut exactly once. We shall therefore apply Theorem 5.4 to argue that $|\mathrm{core}(\mathcal{C})|$ is exponentially large. To this end, let $\mathcal{F} := \{U \subset V : U \neq \emptyset, |\delta^+(U)| = \tau, \delta^-(U) = \emptyset\}$, and let M be the matrix whose rows correspond to $\{\mathbf{1}_{\delta^+(U)} : U \in \mathcal{F}\}$. We claim that M has row rank at most $(1-\beta)|A|$ for $\beta = \frac{1}{3}$, thus allowing us to apply Theorem 5.4 to finish the proof. Let S,T be the sets of sources and sinks of D, respectively. Then we have

$$\mathbf{1}_{\delta^+(U\cap S)} - \mathbf{1}_{\delta^-(U\cap T)} = \mathbf{1}_{\delta^+(U)} - \mathbf{1}_{\delta^-(U)} = \mathbf{1}_{\delta^+(U)} \qquad \forall U \in \mathcal{F}.$$

Note that $\mathbf{1}_{\delta^-(v)}, \forall v \in T$ are rows of M. Thus, after applying elementary row operations to the rows of M corresponding to $\mathbf{1}_{\delta^+(U)}, U \in \mathcal{F}, U \not\supseteq S$, we obtain a matrix N whose rows are $\mathbf{1}_{\delta^-(v)}$, $\forall v \in T$; $\mathbf{1}_{\delta^+(U\cap S)}$, $\forall U \in \mathcal{F}, U \not\supseteq S$. The first set of rows of N clearly has rank at most $|T|$, while the second set has rank at most $|S|$, implying in turn that N, and therefore M has rank at most $|S|+|T|=|V|$. Given that every source has degree at least τ (as the arcs incident with it form a dicut), it follows that $|S| \leq |T|$, so $|V| \leq 2|T| = \frac{2|A|}{\tau} \leq \frac{2}{3}|A|$, thus proving the claim. $\square$

6.3 Perfect matchings in r-graphs

Theorem 5.4 has a consequence for the number of perfect matchings in certain graphs. To elaborate, let $r \geq 3$ be an integer. An *r-graph* is an r-regular graph $G=(V,E)$ where $|V|$ is even, and every *odd cut* has size at least r, that is, $|\delta(U)| \geq r$ for all $U \subset V$ where $|U|$ is odd. It is known that every r-graph is *matching-covered*, that is, every edge belongs to a perfect matching [Sey79]. Perfect matchings intersecting every minimum odd cut exactly once correspond to the core of a certain ideal r-cover-minimal clutter. Through this connection, we shall prove the following as a consequence of Theorem 5.4.

Theorem 6.3. *For every integer $r \geq 3$ and $\beta \in (0,1)$, there is a constant $\theta > 0$ such that the following statement holds:*

Let $G=(V,E)$ be an r-graph where the rank of $\{\mathbf{1}_{\delta(U)} : |\delta(U)| = r, |U| \text{ is odd}\}$ is at most $(1-\beta)|E|$. Then the number of perfect matchings of G that intersect every minimum odd cut exactly once is at least $e^{\theta|E|}$.

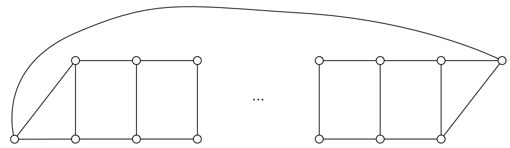

Figure 2: A staircase on at least 4 vertices. This graph has exactly 3 perfect matchings that intersect every minimum odd cut exactly once.

Proof. A *postman set* is a subset $J \subseteq E$ where every vertex of G is incident to an odd number of edges from J. Let $\mathcal{C}$ be the clutter over ground set E of the minimal postman sets of G. It is known that $\mathcal{C}$ is an ideal clutter, and that $b(\mathcal{C})$ consists of the minimal odd cuts of G [EJ73]. In particular, as G is an r-graph, $\mathcal{C}$ is an r-cover-minimal clutter, and $\mathrm{core}(\mathcal{C})$ is precisely the clutter of perfect matchings of G that intersect every minimum odd cut exactly once. The claim now follows from Theorem 5.4. $\square$

The case $r = 3$ is of particular interest, in which case the corollary implies an existing result that every essentially 4-edge-connected cubic graph has exponentially many perfect matchings [Bar13]. There exists 3-graphs with exactly 3 perfect matchings that intersect every minimum odd cut exactly once, namely *staircases* as depicted in Figure 2. For such graphs, the rank of $\{\mathbf{1}_{\delta(U)} : |\delta(U)| = 3,\ |U| \text{ is odd}\}$ is $|E| - 2$. In particular, the condition on the rank cannot be dropped in Theorem 6.3. That said, the number of perfect matchings in any 3-graph $G = (V, E)$ is at least $2^{|V|/3656}$ [EKK$^+$11]. This result implies that for every $r \geq 4$, every $(r-1)$-edge-connected r-graph has at least $2^{f(r)\cdot|V|}$ perfect matchings for $f(r) = \frac{1}{3656}\left(1 - \frac{1}{r}\right)\left(1 - \frac{2}{r}\right)$ [EKK$^+$11]. It has been conjectured by Lovász and Plummer that for every $r \geq 3$, there exist constants $c_1(r), c_2(r)$ such that every r-regular matching-covered graph $G = (V, E)$ contains at least $c_2(r) \cdot c_1(r)^{|V|}$ perfect matchings; furthermore that, $c_1(r) \to \infty$ as $r \to \infty$ [[LP86], Conjecture 8.1.8].

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