PACKING ODD T-JOINS WITH AT MOST TWO TERMINALS

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Abstract. Take a graph G, an edge subset $\Sigma \subseteq E(G)$, and a set of terminals $T \subseteq V(G)$ where $|T|$ is even. The triple (G, Σ, T) is called a *signed graft*. A T-join is *odd* if it contains an odd number of edges from Σ. Let ν be the maximum number of edge-disjoint odd T-joins. A *signature* is a set of the form $\Sigma \triangle \delta(U)$ where $U \subseteq V(G)$ and $|U \cap T|$ is even. Let τ be the minimum cardinality a T-cut or a signature can achieve. Then $\nu \leq \tau$ and we say that (G, Σ, T) *packs* if equality holds here.

We prove that (G, Σ, T) packs if the signed graft is Eulerian and it excludes two special non-packing minors. Our result confirms the Cycling Conjecture for the class of clutters of odd T-joins with at most two terminals. Corollaries of this result include, the characterizations of weakly and evenly bipartite graphs, packing two-commodity paths, packing T-joins with at most four terminals, and a new result on covering edges with cuts.

1. The main result

A *signed graph* is a pair (G, Σ) where G is a graph and $\Sigma \subseteq E(G)$. A subset S of the edges is *odd* (resp. *even*) in (G, Σ) if $|S \cap \Sigma|$ is odd (resp. even). In particular, an edge e is odd if $e \in \Sigma$ and it is even otherwise. A *graft* is a pair (G, T) where G is a graph, $T \subseteq V(G)$ and $|T|$ is even. Vertices in T are *terminal* vertices. A *T-join* is an edge subset that induces a subgraph of G with the odd degree vertices equal to T. A *T-cut* is a cut $\delta(U) = \{uv \in E : u \in U, v \notin U\}$ where $|U \cap T|$ is odd. A *signed graft* is a triple (G, Σ, T) where (G, Σ) is a signed graph and (G, T) is a graft. Thus an *odd T-join* of (G, Σ, T) is a T-join of G that contains an odd number of edges of Σ. Take an edge subset $C \subseteq E(G)$. Then C is a *circuit* if it induces a connected subgraph where every vertex has degree two, and C is a *cycle* if it induces a subgraph where every vertex has even degree. When $T = \emptyset$ an (inclusion-wise) minimal odd T-join is an odd circuit. When $T = \{s, t\}$ a minimal odd T-join is either an odd st-path, or it is the union of an even st-path P and an odd circuit C where P and C share at most one vertex. When $T = \{s, t\}$ we say that a set $B \subseteq E(G)$ is an st-cut (resp. an st-join) if it is a T-cut (resp. a T-join).

A *signature* of the signed graft (G, Σ, T) is a set of the form $\Sigma \triangle \delta(U)$, where $U \subseteq V(G)$ and $|U \cap T|$ is even.[1] Observe that if Γ is a signature, then (G, Σ, T) and (G, Γ, T) have the same collection of odd T-joins. We will need the following basic result:

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[1]Given sets A, B the set $A - B = \{a \in A : a \notin B\}$, and the set $A \triangle B = (A \cup B) - (A \cap B)$.

Theorem 1.1. *Let (G, Σ, T) be a signed graft, and let $F \subseteq E(G)$. Then the following statements hold:*

- *(Zaslavsky [18]) Assume that $T = \emptyset$. If F contains no odd cycle, then there is a signature disjoint from F. If F contains no signature, then there is an odd cycle disjoint from F.*
- *If F does not contain a T-join, then there is a T-cut disjoint from F. If F does not contain a T-cut, then there is a T-join disjoint from F.*

This theorem is very useful and will be applied many times without reference throughout this paper. The first application is the following:

Proposition 1.2. *Let (G, Σ, T) be a signed graft. Let B be a minimal set of edges that intersects every odd T-join. Then B is either a T-cut or a signature. In particular, B intersects every odd T-join with odd parity.*

Proof. By the minimality of B, it suffices to show that B contains a T-cut or a signature, as T-cuts and signatures intersect every odd T-join. To this end, let us assume that B does not contain a T-cut. Then there is a T-join J disjoint from B. Since B intersects every odd T-join, it follows that J is an even T-join. It also follows that B intersects every odd cycle C, for if not, then $J \triangle C$ would be an odd T-join disjoint from B, which is not the case. Hence, B contains a signature of $(G, \Sigma, \emptyset)$. That is, there is a cut $\delta(U)$ such that $\Sigma \triangle \delta(U) \subseteq B$. It suffices to show that $|U \cap T|$ is even. Since $B \cap J = \emptyset$, we get that $(\Sigma \triangle \delta(U)) \cap J = \emptyset$, so in particular, $|(\Sigma \triangle \delta(U)) \cap J|$ is even. Since $|\Sigma \cap J|$ is even, it follows that $\delta(U) \cap J$ is even, implying in turn that $|U \cap T|$ is even, as required. □

Given a signed graft, a *cover* is a set of edges that intersects every odd T-join *with odd parity*.[2] Then by proposition 1.2 every minimal set of edges that intersects every odd T-join is a cover.

The maximum number of pairwise (edge) disjoint odd T-joins in (G, Σ, T) is denoted $\nu(G, \Sigma, T)$. The cardinality of a minimum cover is denoted $\tau(G, \Sigma, T)$. Clearly, $\tau(G, \Sigma, T) \geq \nu(G, \Sigma, T)$. We say that (G, Σ, T) *packs* if equality holds. $\widetilde{K_5}$ is the signed graft $(K_5, E(K_5), \emptyset)$ and F_7 is the signed graft (G, Σ, T) in figure 1. Note, $4 = \tau(\widetilde{K_5}) > \nu(\widetilde{K_5}) = 2$ and $3 = \tau(F_7) > \nu(F_7) = 1$. Thus $\widetilde{K_5}$ and F_7 do not pack.

Let (G, Σ, T) be a signed graft. (G, Γ, T) is obtained by *resigning* (G, Σ, T) if Γ is a signature of (G, Σ, T). For $e \in E(G)$, we say that $(G \backslash e, \Sigma - \{e\}, T)$ is obtained by *deleting* e. For $e = uv \in E(G) - \Sigma$, we say that $(G/e, \Sigma, T')$ is obtained by *contracting* e where $T' = T - \{u, v\}$ if both or none of u, v are in T and $T' = T - \{u, v\} \cup \{w\}$ if exactly one of u, v is in T where w is the vertex obtained from e by contracting e. A signed graft is a *minor* of (G, Σ, T) if it is obtained by sequentially

[2] This definition is not standard!

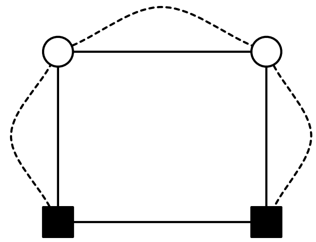


FIGURE 1. Signed graft F_7. Dashed edges form the signature, square vertices are terminals.

deleting/contracting edges and resigning. Note, we can always do all deletions first, resign, and then do all contractions. We often do not distinguish between signed grafts related by resigning. In particular we denote by $(G, \Sigma, T)/I \setminus J$ the signed graft obtained from (G, Σ) by contracting edge set I and deleting edge set J. Observe that this is only well defined if I does not contain an odd circuit or an odd T-join.

We say that a signed graft (G, Σ, T) is *Eulerian* if every non-terminal vertex has even degree and either: every terminal has odd degree and the signature has an odd number of edges; or every terminal has even degree and the signature has an even number of edges. So $(G, \Sigma, \emptyset)$ is Eulerian if every vertex has even degree. Notice that resigning preserves the Eulerian property.

We can now state the main result of the paper,

Theorem 1.3. *If an Eulerian signed graft has at most two terminals and it does not contain either of* $\widetilde{K_5}$ *or* F_7 *as a minor then it packs.*

Observe that the Eulerian condition cannot be omitted. For instance $(K_4, E(K_4), \emptyset)$ does not pack and does not contain either of $\widetilde{K_5}$ or F_7 as a minor. Similarly, the signed graft obtained from F_7 by deleting the unique edge between the two terminal vertices does not pack and does not contain either $\widetilde{K_5}$ or F_7 as a minor.

1.1. **Special cases.** We say that a graph H is an *odd-minor* of a graph G if H is obtained from G by first deleting edges and then contracting *all* edges on a cut. Theorem 1.3 implies,

Corollary 1.4 (Geelen and Guenin [3])**.** *Let G be a graph that does not contain K_5 as an odd minor and where every vertex has even degree. Then the minimum number of edges needed to intersect all odd circuits is equal to the maximum number of pairwise disjoint odd circuits.*

Proof. Consider the signed graft $(G, E(G), T)$ where $T = \emptyset$. Since $T = \emptyset$, F_7 is not a minor of $(G, E(G), T)$. We claim that $\widetilde{K_5}$ is not a minor of $(G, E(G), T)$ either. Suppose for a contradiction that $\widetilde{K_5} = (G, E(G), \emptyset)/I \setminus J$. Let $(H, E(H), \emptyset) = (G, E(G), \emptyset) \setminus J$. We may assume that we resign $(H, E(H), \emptyset)$ to obtain $(H, E(H) - B, \emptyset)$ where B is a cut of $E(H)$, $I \subseteq B$ and that $\widetilde{K_5} = (H, E(H) -$

$B, \emptyset)/I$. As $\widetilde{K_5}$ has no even edge, $I = B$. But then K_5 is an odd-minor of G, a contradiction. Since all vertices of G have even degree and since $T = \emptyset$, $(G, E(G), \emptyset)$ is Eulerian. Thus $\tau(G, \Sigma, \emptyset) = \nu(G, \Sigma, \emptyset)$ by theorem 1.3. Since $T = \emptyset$ each odd T-joins contains an odd circuit and the result follows. □

A *blocking vertex* (resp. *blocking pair*) in a signed graft is a vertex (resp. pair of vertices) that intersects every odd circuit.

Proposition 1.5. *Consider a signed graft (G, Σ, T) where $T = \{s, t\}$. If any of (1)-(6) hold, then $(G, \Sigma, \{s, t\})$ does not contain $\widetilde{K_5}$ or F_7 as a minor:*

(1) there exists a blocking vertex,
(2) s, t is a blocking pair,
(3) every minimal odd st-join is connected,
(4) G is a plane graph with at most two odd faces,
(5) G is a plane graph and u, v is a blocking pair where s, u, t, v appear on a facial cycle in this order,
(6) G has an embedding on the projective plane where every face is even and s, t are connected by an odd edge.

Proof sketch. Observe that (3) contains (2) and (6). Thus it suffices to show the result for (1), (3), (4) and (5). Suppose that (G, Σ, T) with $T = \{s, t\}$ belongs to one of these classes, and let (G', Σ', T') be a minor of it. Then,

- if (G, Σ, T) belongs to one of (1), (4), then so does (G', Σ', T),
- if (G, Σ, T) belongs to (3) and $T' = T$, then (G', Σ', T') belongs to (3),
- if (G, Σ, T) belongs to (5) and $T' = T$, then (G', Σ', T') belongs to (5),
- if (G, Σ, T) belongs to (3) and $T' = \emptyset$, then (G', Σ', T') belongs to (1),
- if (G, Σ, T) belongs to (5) and $T' = \emptyset$, then (G', Σ', T') has a blocking pair.

In all of the aforementioned cases, (G', Σ', T') is not equal to either of $\widetilde{K_5}$ or F_7 (we leave this as a simple exercise), finishing the proof. □

Theorem 1.3 implies that an Eulerian signed graft with two terminals that is in any of classes (1)-(6) packs. We will now show that some of these cases lead to classical results.

Proposition 1.5(1) and theorem 1.3 imply,

Corollary 1.6. *Let (H, T) be a graft with $|T| \le 4$. Suppose that every vertex of H not in T has even degree and that all the vertices in T have degrees of the same parity. Then the maximum number of pairwise disjoint T-joins is equal to the minimum size of a T-cut.*

Proof. Suppose that $T = \{s, t, s', t'\}$. Let $\Sigma = \delta_H(s')$ and identify s', t' to obtain G. Denote by v the vertex corresponding to s', t' in G. Then the signed graft $(G, \Sigma, \{s, t\})$ contains a blocking vertex v, so by proposition 1.5(1) it has no F_7 or $\widetilde{K}_5$ minor. By construction $(G, \Sigma, \{s, t\})$ is Eulerian. Hence, theorem 1.3 implies that $\tau(G, \Sigma, \{s, t\}) = \nu(G, \Sigma, \{s, t\})$. Observe that an odd st-join of $(G, \Sigma, \{s, t\})$ is a T-join of H, and that an st-cut or a signature of (G, Σ) is a T-cut of H. The result now follows. □

In fact this result holds as long as $|T| \leq 8$ [1].

Proposition 1.5(2) and theorem 1.3 imply,

Corollary 1.7 (Hu [7], Rothschild and Whinston [10]). *Let H be a graph and choose two pairs (s_1, t_1) and (s_2, t_2) of vertices, where $s_1 \neq t_1$, $s_2 \neq t_2$, the degrees of s_1, t_1, s_2, t_2 have the same parity, and all the other vertices have even degree. Then the maximum number of pairwise disjoint paths that are between s_i and t_i for some $i = 1, 2$, is equal to the minimum size of an edge subset whose deletion removes all s_1t_1- and s_2t_2-paths.*

Proof. Let $\Sigma = \delta_H(s_1) \triangle \delta_H(t_2)$ and identify s_1, s_2 as well as t_1, t_2 to obtain G. (So all the edges between s_1 and s_2 and between t_1 and t_2 have turned into loops.) Denote by s (resp. t) the vertex of G corresponding to s_1, s_2 (resp. t_1, t_2) in H. The signed graft $(G, \Sigma, \{s, t\})$ has $\{s, t\}$ as a blocking pair, so by proposition 1.5(2) it has no F_7 or $\widetilde{K}_5$ minor. By construction (G, Σ) is Eulerian. Thus, theorem 1.3 implies that $\tau(G, \Sigma, \{s, t\}) = \nu(G, \Sigma, \{s, t\})$. Observe that a minimal odd st-join of $(G, \Sigma, \{s, t\})$ is an s_it_i-path of H, for some $i = 1, 2$. The result now follows. □

Next we shall derive corollaries using duals of plane graphs.

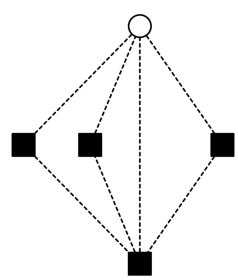


FIGURE 2. Signed graft. All edges are in the signature and square vertices are terminals.

Note, in the next theorem, the *length* of a circuit, resp. T-join, is the number of its edges, and a circuit, resp. T-join, is odd, if it contains an odd number of edges in Σ.

Corollary 1.8. *Let (G, Σ, T) be a signed graft where G is a plane graph with exactly two odd faces. Suppose that $\Sigma = E(G)$ or that all T-joins have even length. If (G, Σ, T) does not contain the signed* *graft in figure 2 as a minor, then the maximum number of pairwise disjoint signatures is equal to the minimum of the following two quantities:*

- *the length of the shortest odd circuit,*
- *the length of the shortest odd T-join.*

Proof. Denote by s and t the two odd faces of G. Let G^* be the plane dual of G and let Γ be an odd T-join of (G, Σ, T). Then $(G^*, \Gamma, \{s,t\})$ is a signed graft. Notice that if (G, Σ, T) is the signed graft in figure 2, then $(G^*, \Gamma, \{s,t\})$ is F_7. Recall that a *bond* is an inclusion-wise minimal cut.

Claim 1. *Let $B \subseteq E(G) = E(G^*)$.*

(i) If B is an st-cut of $(G^*, \Gamma, \{s,t\})$ then B is an odd cycle of (G, Σ, T).
(ii) If B is a signature of $(G^*, \Gamma, \{s,t\})$ then B is an odd T-join of (G, Σ, T).
(iii) If B is an odd st-join of $(G^*, \Gamma, \{s,t\})$ then B is a signature of (G, Σ, T).

Proof. **(i)** $B = B_1 \triangle \ldots \triangle B_k$ where B_k are bonds of G^*. Since B is an st-cut, an odd number of these bonds are st-bonds. Thus an odd number of $B_1, \ldots, B_k$ are circuits of G separating faces s and t and the remainder are circuits of G with faces s and t on the same side. It follows that B is an odd cycle of (G, Σ, T). **(ii)** As B is a signature of $(G^*, \Gamma, \{s,t\})$, $B \triangle \Gamma = \delta_{G^*}(U)$ where $s, t \notin U$. Denote by $u_1, \ldots, u_k$ the elements of U, then $B \triangle \Gamma = \delta_{G^*}(u_1) \triangle \ldots \triangle \delta_{G^*}(u_k)$. For $i \in [k]$[3], $\delta_{G^*}(u_i)$ is a facial even circuit of (G, Σ) and thus $B \triangle \Gamma$ is an even cycle of (G, Σ). As Γ is an odd T-join of (G, Σ, T) so is B. **(iii)** Since B is an st-join of G^*, $|\delta_{G^*}(u) \cap B|$ is odd if $u = s, t$ and even otherwise. Thus the facial circuits of G that intersect B with odd parity are the ones separating faces s and t. As the facial circuits span the cycle space of G, for every cycle C of G, $|C \cap B|$ and $|C \cap \Sigma|$ have the same parity. Hence, $B \triangle \Sigma = \delta_G(U)$ for some $U \subseteq V(G)$. $|B \cap \Gamma|$ is odd as B is an *odd* st-join of $(G^*, \Gamma, \{s,t\})$. $|\Sigma \cap \Gamma|$ is odd as Γ is an *odd* T-join of (G, Σ, T). Thus $|\delta_G(U) \cap \Gamma| = |(B \triangle \Sigma) \cap \Gamma|$ is even. It follows that $|U \cap T|$ is even, thus B is a signature of (G, Σ, T). ◇

Claim 2. $(G^*, \Gamma, \{s,t\})$ *is Eulerian.*

Proof. Suppose all T-joins of G have even length. Then any circuit of G has even length. Thus all vertices of G^* have even degree. We chose Γ to be a T-join of G, thus $|\Gamma|$ is even. It follows by definition that the signed graft $(G^*, \Gamma, \{s,t\})$ is Eulerian. Suppose that $\Sigma = E(G)$. As s and t are the only two odd faces of G, s and t are the only vertices of G^* of odd degree. We chose Γ to be an odd T-join of $(G, \Sigma = E(G), T)$, thus $|\Gamma|$ is odd. It follows by definition that the signed graft $(G^*, \Gamma, \{s,t\})$ is Eulerian. ◇

[3] $[k] := \{1, 2, \ldots, k\}$

Suppose now that (G, Σ, T) does not contain the signed graft in figure 2 as a minor.

Claim 3. $(G^*, \Gamma, \{s,t\})$ *does not contain either of* $\widetilde{K_5}$ *or* F_7 *as a minor.*

Proof. Since G^* is planar, $(G^*, \Gamma, \{s,t\})$ does not contain $\widetilde{K_5}$ as a minor. Suppose for a contradiction that $(G^*, \Gamma, \{s,t\})/I \setminus J = F_7$. Denote by $e_1, \ldots, e_k$ the elements of J and let (G', Σ', T') be obtained from (G, Σ, T) by deleting edges in I and contracting $e_1, \ldots, e_r$ for some $r \leq k$ as large as possible. If $r = k$ then (G', Σ', T') is given in figure 2, a contradiction. Otherwise, since we could not resign and contract e_{r+1}, e_{r+1} must be in every signature of (G', Σ', T'). Thus, by claim 1 (iii), every odd st-join of $(G^*, \Gamma, \{s,t\})/I \setminus \{e_1, \ldots, e_r\}$ uses e_{r+1} and $(G^*, \Gamma, \{s,t\})/I \setminus J$ has no odd st-join, a contradiction. $\diamond$

By claim 2, claim 3 and theorem 1.3, $\tau = \tau(G^*, \Gamma, \{s,t\}) = \nu(G^*, \Gamma, \{s,t\})$. Thus there is a minimal cover B of $(G^*, \Gamma, \{s,t\})$ with $|B| = \tau$ and pairwise disjoint odd st-joins $L_1, \ldots, L_\tau$ of $(G^*, \Gamma, \{s,t\})$. By proposition 1.2 and claim 1, B is either an odd circuit of (G, Σ, T) or an odd T-join of (G, Σ, T). By claim 1, for all $i \in [\tau]$, L_i is a signature of (G, Σ, T). $\square$

Next we will show that in the previous result, the case where T consists of two vertices is of independent interest. Consider H obtained as follows:

($\star$) start from a plane graph with exactly two faces of odd length and distinct vertices s and t, and identify s and t.

Corollary 1.9. *Let* H *be a graph as in* ($\star$) *and suppose that the length of the shortest odd circuit is* k*. Then there exist cuts* $B_1, \ldots, B_k$ *such that every edge* e *is in at least* $k-1$ *of* $B_1, \ldots, B_k$*.*

Proof. H is obtained as in ($\star$) from a plane graph G with exactly two faces of odd length and distinct vertices s, t. The signed graft $(G, E(G), T)$ where $T = \{s,t\}$ does not contain the signed graft in figure 2 as $|T| < 4$. By corollary 1.8 there exists pairwise disjoint signatures $\Sigma_1, \ldots, \Sigma_p$ and $C \subseteq E(G)$ with $|C| = p$ where C is an odd circuit or an odd T-join of G. In either case C is an odd circuit of H, thus $p \geq k$. Since $\Sigma_1, \ldots, \Sigma_p$ are signatures of $(G, E(G), \{s,t\})$ for all $i \in [p]$, $\Sigma_i = E(G) \triangle \delta_G(U_i) = E(G) - \delta_G(U_i)$ where $s, t \notin U_i$. Since $\Sigma_1, \ldots, \Sigma_p$ are pairwise disjoint, every edge of G (resp. H) is in at least $p - 1 \geq k - 1$ of $B_i = \delta_G(U_i) = \delta_H(U_i)$. $\square$

The attentive reader may have noticed that we can also derive corollary 1.9 directly from theorem 1.3 and proposition 1.5(4). Suppose that H is as in ($\star$) and is loopless. Then by corollary 1.9, there exists cuts $\delta(U_1), \delta(U_2)$ such that every edge is in $\delta(U_1) \cup \delta(U_2)$. It follows that $U_1 \cap U_2, U_1 \cap (V(H) - U_2), (V(H) - U_1) \cap U_2, (V(H) - U_1) \cap (V(H) - U_2)$ are stable sets. Hence, H is 4-colourable.

The following conjecture would generalize the 4-colour theorem,

Conjecture 1.10. *Let H be a graph that does not contain K_5 as an odd minor and suppose that the length of the shortest odd circuit is k. Then there exist cuts $B_1,\dots,B_k$ such that every edge e is in at least $k-1$ of $B_1,\dots,B_k$.*

Graphs in ($\star$) do not contain K_5 as an odd minor [4] and corollary 1.9 implies the previous conjecture for these graphs. We close this section with a sharper version of theorem 1.3.

Theorem 1.11. *Let $(G,\Sigma,\{s,t\})$ be an Eulerian signed graft that does not contain $\widetilde{K_5}$ or F_7 as a minor. Let k be the size of the smallest st-cut and let ℓ be the size of the smallest signature. When $k\geq\ell$ one can in fact find a collection of k pairwise disjoint sets, ℓ of which are odd st-join and $k-\ell$ are even st-paths.*

Proof. Let (G',Σ') be obtained from (G,Σ) by adding $k-\ell$ odd loops. As F_7 and $\widetilde{K_5}$ have no loops, $(G',\Sigma',\{s,t\})$ does not contain $\widetilde{K_5}$ or F_7 as a minor. Since $(G,\Sigma,\{s,t\})$ is Eulerian, so is $(G',\Sigma',\{s,t\})$. It follows from theorem 1.3 that $k=\tau(G',\Sigma',\{s,t\})=\nu(G',\Sigma',\{s,t\})$. Thus there exists k pairwise disjoint odd st-join in $(G',\Sigma',\{s,t\})$ and exactly $k-\ell$ must contain an odd loop that is in (G',Σ') but not in (G,Σ). The result now follows. □

1.2. **Cycling and idealness.** A *clutter* $\mathcal{C}$ is a finite collection of sets, over some finite set $E(\mathcal{C})$, with the property that no set in $\mathcal{C}$ is contained in another set of $\mathcal{C}$. $\mathcal{C}$ is *binary* if for every $S_1,S_2,S_3\in\mathcal{C}$, $S_1\triangle S_2\triangle S_3$ is contained in a set of $\mathcal{C}$. A *cover* of a binary clutter $\mathcal{C}$ is a subset of $E(\mathcal{C})$ that intersects every set in $\mathcal{C}$ with odd parity.[4] An inclusion-wise minimal set of edges that intersects all sets in $\mathcal{C}$, is a cover [8]. The maximum number of pairwise disjoint sets in $\mathcal{C}$ is denoted $\nu(\mathcal{C})$. The minimum size of a cover of $\mathcal{C}$ is $\tau(\mathcal{C})$. $\mathcal{C}$ *packs* if $\tau(\mathcal{C})=\nu(\mathcal{C})$. A binary clutter is *Eulerian* if all minimal covers have the same parity.

Let $\mathcal{C}$ be a clutter and $e\in E(\mathcal{C})$. The *contraction* $\mathcal{C}/e$ and *deletion* $\mathcal{C}\setminus e$ are clutters with $E(\mathcal{C}/e)=E(\mathcal{C}\setminus e)=E(\mathcal{C})-\{e\}$ where $\mathcal{C}/e$ is the collection of inclusion-wise minimal sets in $\{C-\{e\}:C\in\mathcal{C}\}$ and $\mathcal{C}\setminus e:=\{C:e\notin C\in\mathcal{C}\}$. A clutter obtained from $\mathcal{C}$ by a sequence of deletions and contractions is a *minor* of $\mathcal{C}$. Denote by $\mathcal{L}_7$ the clutter of odd T-joins of F_7, by $\mathcal{O}_5$ the clutter of odd circuits of K_5, by $b(\mathcal{O}_5)$ the clutter of complements of cuts of K_5, and by $\mathcal{P}_{10}$ the clutter of T-joins of the Petersen graph where T is the set of all vertices.

Conjecture 1.12 (Cycling Conjecture. Seymour [14], see also Schrijver [12]). *Eulerian binary clutters that do not contain $\mathcal{L}_7$, $\mathcal{O}_5$, $b(\mathcal{O}_5)$, or $\mathcal{P}_{10}$ as a minor, pack.*

Let $(G,\Sigma,\{s,t\})$ be a signed graft and let $\mathcal{H}$ be the clutter of minimal odd st-joins. Note that $\mathcal{H}$ is binary, and it can be readily checked that $\mathcal{H}$ is Eulerian if and only if $(G,\Sigma,\{s,t\})$ is Eulerian.

[4]This is not standard!

Observe also that $\mathcal{L}_7$ (resp. $\mathcal{O}_5$) is a minor of $\mathcal{H}$ if and only if F_7 (resp. $\widetilde{K_5}$) is a minor of $(G, \Sigma, \{s, t\})$. Thus theorem 1.3 can be restated as,

Theorem 1.13. *The Cycling Conjecture holds for Eulerian clutters of minimal odd st-joins.*

Let $\mathcal{H}$ be a clutter. We define,

$$\nu^*(\mathcal{H}) = \max\left\{\sum_{S\in\mathcal{H}}\lambda_S : \sum_{S\in\mathcal{H}:e\in S}\lambda_S \leq 1, \text{ for all } e \in E(\mathcal{H}), \lambda_S \geq 0 \text{ for all } S \in \mathcal{H}\right\}. \tag{1}$$

$\mathcal{H}$ *fractionally packs* if $\tau(\mathcal{H}) = \nu^*(\mathcal{H})$.

Conjecture 1.14 (Flowing Conjecture. Seymour [14, 15])**.** *Binary clutters that do not contain $\mathcal{L}_7$, $\mathcal{O}_5$, or $b(\mathcal{O}_5)$ as a minor, fractionally pack.*

Corollary 1.15 (Guenin [6])**.** *The Idealness Conjecture holds for clutters of minimal odd st-joins.*

Proof. Let $\mathcal{H}$ be the clutter of minimal odd st-joins of the signed graft $(G, \Sigma, \{s, t\})$. Assume that $\mathcal{H}$ has no minor $\mathcal{L}_7$ or $\mathcal{O}_5$. Then $(G, \Sigma, \{s, t\})$ has no minor F_7 or $\widetilde{K_5}$. Let $(G', \Sigma', \{s, t\})$ be obtained from $(G, \Sigma, \{s, t\})$ by replacing every even (resp. odd) edge by two parallel even (resp. odd) edges. Note that $(G', \Sigma', \{s, t\})$ also has no minor F_7 or $\widetilde{K_5}$. It follows by theorem 1.3 that $\tau(G', \Sigma', \{s, t\}) = \nu(G', \Sigma', \{s, t\})$. It can now be readily checked that it implies that $\tau(\mathcal{H}) = \nu^*(\mathcal{H})$ as required, where in equation (1), $\lambda_S \in \{0, \frac{1}{2}, 1\}$ for all $S \in \mathcal{H}$. □

Applying the previous result to the case where $s = t$ we obtain,

Theorem 1.16 (Weakly bipartite graph theorem, Guenin [5])**.** *The Idealness Conjecture holds for clutters of odd circuits of graphs.*

2. Organization of the proof

2.1. **Extremal counterexample.** We start with the following basic result:

Remark 2.1. *Let (G, Σ, T) be an Eulerian signed graft. Then the following statements hold:*

(1) The cardinality of every signature and every T-cut has the same parity as $\tau(G, \Sigma, T)$.

(2) Take an integer $k \geq 0$ such that $k, \tau(G, \Sigma, T)$ have different parities. If $J_1, \ldots, J_k$ are disjoint odd T-joins, then $E(G) - \left(\cup_{i=1}^k J_i\right)$ is also an odd T-join.

Proof. **(1)** We leave this as an exercise. **(2)** Let $J := E(G) - \left(\cup_{i=1}^k J_i\right)$. For every vertex $v \in V(G) - T$, $|\delta(v)|$ is even as the signed graft is Eulerian, so

$$|\delta(v) \cap J| \equiv |\delta(v)| - \sum_{i=1}^{k} |\delta(v) \cap J_i| \equiv 0 - 0 \equiv 0 \pmod 2.$$

Moreover, for every terminal $v \in T$, $|\delta(v)|$ and $\tau(G, \Sigma, T)$ have the same parity by (1), so

$$|\delta(v) \cap J| \equiv |\delta(v)| - \sum_{i=1}^{k} |\delta(v) \cap J_i| \equiv \tau(G, \Sigma, T) - k \equiv 1 \pmod 2.$$

Thus, J is a T-join. By (1), $|\Sigma|, \tau(G, \Sigma, T)$ have the same parity, so

$$|\Sigma \cap J| \equiv \tau(G, \Sigma, T) - \sum_{i=1}^{k} |\Sigma \cap J_i| \equiv \tau(G, \Sigma, T) - k \equiv 1 \pmod 2,$$

it follows that J is an odd T-join, as required. $\square$

A *counterexample* is an Eulerian signed graft with at most two terminals that does not pack and that does not contain $\widetilde{K_5}$ or F_7 as a minor. By remark 2.1 (2), $\tau(G, \Sigma, T) \geq 3$ for every counterexample (G, Σ, T). A counterexample (G, Σ, T) is *extremal* if it satisfies the following properties (in this order):

(M1) it minimizes $\tau(G, \Sigma, T)$,

(M2) it minimizes $|V(G)|$, and

(M3) it maximizes $|E(G)|$.

Remark 2.2. *If there exists a counterexample then there exists an extremal counterexample.*

Proof. Clearly there exists a counterexample (G, Σ, T) that minimizes (M1) and (M2) in that order. It suffices to show that G cannot have an arbitrarily large number of edges. For otherwise some edge $e \in E(G)$ has at least $\tau(G, \Sigma, T)$ parallel edges (all of the same parity). But then $\tau\big((G, \Sigma, T)/e\big) = \tau(G, \Sigma, T)$, $(G, \Sigma, T)/e$ does not pack, it does not contain $\widetilde{K_5}$ or F_7 as a minor and $|V(G/e)| = |V(G)| - 1$, contradicting our choice of (G, Σ, T). $\square$

Let G be a graph, $U \subseteq V(G)$ and $B \subseteq E(G)$. We denote by $G[U]$ the graph with vertices U and edges of G whose ends[5] are in U. We denote by $V_G(B)$ the set of ends of B and we shall omit the subindex G when there is no ambiguity. We write $G[B]$, for the graph with edges B and vertices $V(B)$. We say B is *connected* if $G[B]$ is a connected graph. Let (G, Σ, T) be a signed graft such that $\tau(G, \Sigma, T) \geq 3$, and let $\Omega \in E(G)$. Choose $k \in [\tau(G, \Sigma, T)] - [2]$ of the same parity as $\tau(G, \Sigma, T)$. An *(Ω, k)-packing* is a sequence $(L_1, \ldots, L_k)$ of odd T-joins where, $\Omega \in L_1 \cap L_2 \cap L_3$ and $\Omega \notin L_4 \cup \cdots \cup L_k$, and $L_1, \ldots, L_k$ are pairwise Ω-disjoint[6]. For a subset $L \subseteq E(G)$, we say that a cover B is a *k-mate* of L if $|B - L| \leq k - 3$ and if B is either a signature or a T-cut. Moreover, B is an *extremal k-mate for* L if, for every other k-mate B' of L, $B' \cap L$ is not a proper subset of $B \cap L$.

Proposition 2.3. *Let (G, Σ, T) be an extremal counterexample with $\tau := \tau(G, \Sigma, T)$. Then we may assume*

[5]An *end* of an edge is a vertex incident to the edge.
[6]Two sets A and B are Ω-disjoint if $A \cap B \subseteq \{\Omega\}$.

(1) G is connected,

(2) there exists $\Omega \in E(G)$ that is not in at least one minimum cover,
if $T \neq \emptyset$ we can choose $\Omega \in \delta(v)$ for some $v \in T$,

(3) there do not exist $\tau - 1$ pairwise disjoint odd T-joins,

(4) for every Ω as in (2), there exists an (Ω, τ)-packing,

(5) every odd T-join has a τ-mate.

Proof. **(1)** Identify a vertex of each (connected) component with an arbitrary vertex. (Neither of the obstructions $\widetilde{K_5}$, F_7 has a cut-vertex.)

(2) Let B be a minimum cover. Note $B \neq E(G)$, for otherwise every edge of B is an odd T-join and so (G, Σ, T) packs, which is not the case. If $T = \emptyset$ then let $\Omega \in E - B$. Otherwise, $T = \{s, t\}$. Then we can pick $\Omega \in (\delta(s) \cup \delta(t)) - B$. For otherwise, $\delta(s) \cup \delta(t) \subseteq B$ and thus $\delta(s) \cup \delta(t) = \delta(s) = \delta(t)$, which by (1) implies that $E(G) = \delta(s)$, a contradiction.

(3) Suppose otherwise. Remove some $\tau - 1$ pairwise disjoint odd T-join in (G, Σ, T). By remark 2.1 (2), what is left is an odd T-join. Hence, one can actually find τ pairwise disjoint odd T-joins in (G, Σ, T), contradicting the fact that (G, Σ, T) does not pack.

(4) Add two parallel edges Ω_1, Ω_2 to Ω of the same parity as Ω to obtain Eulerian (G', Σ', T). By the choice of Ω, B remains a minimum cover for (G', Σ', T), so $\tau(G', \Sigma', T) = \tau$. Since $|V(G')| = |V(G)|$ and $|E(G')| > |E(G)|$ and since (G, Σ, T) is an extremal counterexample, (G', Σ', T) packs. Hence, (G', Σ', T) contains a set $L_1, L_2, \ldots, L_\tau$ of pairwise disjoint odd T-joins. All of Ω, Ω_1 and Ω_2 must be used by the odd T-joins in $L_1, L_2, \ldots, L_\tau$, say by L_1, L_2, L_3, since otherwise one finds at least $\tau - 1$ disjoint odd T-joins in (G, Σ, T), contradicting (3). Then $(L_1, (L_2 \cup \{\Omega\}) - \{\Omega_1\}, (L_3 \cup \{\Omega\}) - \{\Omega_2\}, L_4, \ldots, L_\tau)$ is the required (Ω, τ)-packing.

(5) Let L be an odd T-join. Then the signed graft $(G, \Sigma, T) \setminus L$ packs, since (G, Σ, T) is an extremal counterexample and $\tau\big((G, \Sigma, T) \setminus L\big) < \tau$. Let B' be a minimum cover of $(G, \Sigma, T) \setminus L$. Since both (G, Σ, T) and $(G, \Sigma, T) \setminus L$ are Eulerian, it follows that $\tau((G, \Sigma, T) \setminus L)$ and τ have different parities, and so either $\tau((G, \Sigma, T) \setminus L) \leq \tau - 3$ or $\tau((G, \Sigma, T) \setminus L) = \tau - 1$. However, observe that the latter is not possible, because (G, Σ, T) does not pack and $(G, \Sigma, T) \setminus L$ packs. As a result $|B'| = \tau((G, \Sigma, T) \setminus L) \leq \tau - 3$. Let B be a minimal cover contained in $B' \cup L$. Then $|B - L| \leq |B'| \leq \tau - 3$. Moreover, since B is a minimal cover, proposition 1.2 implies that B is either a signature or a T-cut. Thus B is a τ-mate for L. □

2.2. **Ω-systems.** An edge subset of a signed graph or a signed graft is *bipartite* if all circuits contained in it are even. From proposition 2.3 it follows that an extremal counterexample (G, Σ, T) has an (Ω, τ)-packing $(L_1, \ldots, L_\tau)$. We distinguish between the cases where $(L_1 \cup L_2 \cup L_3) - \{\Omega\}$ is bipartite or non-bipartite and define the appropriate data structure in each case.

A *non-bipartite* Ω*-system* consists of a pair $\big((G, \Sigma, T), (L_1, \ldots, L_k)\big)$ where $\tau(G, \Sigma, T) \geq 3$, $k \in \{3, \ldots, \tau(G, \Sigma, T)\}$, k has the same parity as $\tau(G, \Sigma, T)$, and

(N1) (G, Σ, T) is an Eulerian signed graft with $|T| \leq 2$, and if $T = \{s, t\}$, then $\Omega \in \delta(s)$,
(N2) $(L_1, \ldots, L_k)$ is an (Ω, k)-packing where $L_1, \ldots, L_k$ are minimal odd T-joins,
(N3) $(L_1 \cup L_2 \cup L_3) - \{\Omega\}$ is non-bipartite, and
(N4) every odd T-join $L \subseteq L_1 \cup L_2 \cup L_3$ has a k-mate.

To define the other data structures, we need some terminology. Let (G, Σ, T) be a signed graft where $|T| \leq 2$ and let L be a minimal odd T-join. Define $C(L)$ and $P(L)$ as follows:

if $T = \emptyset$, then L is an odd circuit and we define $P(L) := \emptyset$ and $C(L) := L$,
if $T = \{s, t\}$ and L is an odd st-path, we define $P(L) := L$ and $C(L) := \emptyset$,
otherwise, $T = \{s, t\}$ and L is the disjoint union of an even st-path, denoted $P(L)$, and an odd circuit, denoted $C(L)$.

We say that L is *simple* if $C(L) = \emptyset$ (see figure 3) and it is *non-simple* otherwise (see figure 4).

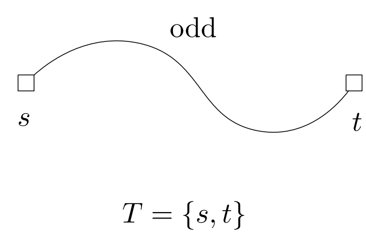


FIGURE 3. An illustration of simple odd T-joins.

A cycle (in a directed graph) is *directed* if it is the disjoint union of directed circuits. An st-join is *directed* if it is the disjoint union of some st-dipaths and some directed circuits.

A *bipartite* Ω*-system* consists of a tuple $\big((G, \Sigma, T), (L_1, \ldots, L_k), m\big)$ where $\tau(G, \Sigma, T) \geq 3$, $k \in \{3, \ldots, \tau(G, \Sigma, T)\}$, k has the same parity as $\tau(G, \Sigma, T)$, and

(B1) (G, Σ, T) is an Eulerian signed graft with $|T| \leq 2$, and if $T = \{s, t\}$, then $\Omega \in \delta(s)$,
(B2) $(L_1, \ldots, L_k)$ is an (Ω, k)-packing and $m \in [k] - [2]$ where

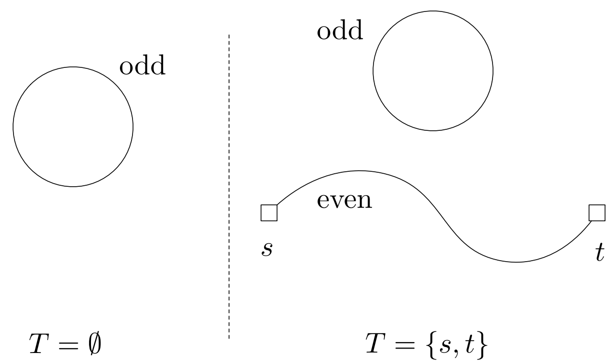


FIGURE 4. An illustration of non-simple odd T-joins.

if $T = \emptyset$, then $m = 3$,

if $T = \{s,t\}$, then for each $j \in [m] - [3]$, L_j contains an even st-path P_j and an odd circuit C_j that are (edge-)disjoint,

if $T = \{s,t\}$, then for each $j \in [k] - [m]$, L_j is connected,

(B3) $\Sigma \cap (L_1 \cup L_2 \cup L_3 \cup P_4 \cup \ldots \cup P_m) = \{\Omega\}$.

A *non-simple bipartite* Ω*-system* consists of a tuple $\big((G,\Sigma,T),(L_1,\ldots,L_k),m,\vec{H}\big)$ where

(NS1) $\big((G,\Sigma,T),(L_1,\ldots,L_k),m\big)$ is a bipartite Ω-system,

(NS2) L_1, L_2, L_3 are minimal odd T-joins, and at least one of them is non-simple,

(NS3) $H = G[L_1 \cup L_2 \cup L_3 \cup P_4 \cup \ldots \cup P_m]$,

L_1, L_2, L_3 are directed T-joins in $\vec{H}$ (if $T = \{s,t\}$ then they are directed st-joins),

if $T = \{s,t\}$, $P_4, \ldots, P_m$ are st-dipaths in $\vec{H}$,

$\vec{H} \setminus \Omega$ is acyclic,

(NS4) in $\vec{H}$, every odd directed T-join that is Ω-disjoint from some odd directed circuit, has a k-mate.

A *simple bipartite* Ω*-system* consists of a tuple $\big((G,\Sigma,\{s,t\}),(L_1,\ldots,L_k),m,\vec{H}\big)$ where

(S1) $\big((G,\Sigma,\{s,t\}),(L_1,\ldots,L_k),m\big)$ is a bipartite Ω-system,

(S2) $H = G[L_1 \cup L_2 \cup L_3 \cup P_4 \cup \ldots \cup P_m]$,

L_1, L_2, L_3 are odd st-dipaths in $\vec{H}$,

$P_4, \ldots, P_m$ are st-dipaths in $\vec{H}$,

$\vec{H}$ is acyclic,

(S3) in $\vec{H}$, every odd st-dipath has a k-mate.

Proposition 2.4. *An extremal counterexample has a non-bipartite, non-simple bipartite, or simple bipartite Ω-system.*

The proof of this proposition is provided in §4.

Given a bipartite Ω-system $((G,\Sigma,\{s,t\}),(L_1,\ldots,L_k),m)$, we define two cut structures.

A *primary cut structure* is a sequence $(U_1,\ldots,U_n)$ where

(PC1) L_2, L_3 are odd st-paths,

(PC2) $n \in [m-2]$ and $s \in U_1 \subset \cdots \subset U_n \subseteq V(G) - \{t\}$,

(PC3) for each $i \in [n-1]$, there exist $q_i \in U_i$, *base* Q_{3+i} and *residue* R_{3+i}, where $Q_{3+i} \subset L_{3+i} - C_{3+i}$ is a q_it-path such that $V(Q_{3+i}) \cap U_i = \{q_i\}$, $R_{3+i} \subset L_{3+i} - C_{3+i}$ is a connected sq_i-join, and $Q_{3+i} \cap R_{3+i} = \emptyset$ (see figure 5),

(PC4) for each $i \in [n-1]$, $\delta(U_i)$ is a k-mate of $R_{3+i} \cup Q_{3+i}$, and for every proper subset W of U_i with $s \in W$, $\delta(W)$ is not a k-mate of $R_{3+i} \cup Q_{3+i}$,

(PC5) $\delta(U_n)$ is a k-mate of L_1, and for every proper subset W of U_n with $s \in W$, $\delta(W)$ is not a k-mate of L_1,

(PC6) there exist $d, q \in U_n$ and a partition of L_1 into *base* Q, *brace* D and *residue* R, where Q is a qt-path with $V(Q) \cap U_n = \{q\}$, D is an sd-path containing Ω with $V(D) \cap U_n = \{s,d\}$ that is vertex-disjoint from Q outside U_n, and R is a connected dq-join (see figure 6).

For $i \in [m]-[n+2]$, set $Q_i := P_i$, $R_i := \emptyset$, and call Q_i the *base* of L_i, and for $i = 2,3$, set $Q_i := P_i = L_i$ and call Q_i the *base* of L_i.

A *secondary cut structure* is a sequence $(U_1,\ldots,U_n)$ where

(SC1) L_1, L_2, L_3 are odd st-paths,

(SC2) $m \geq 4$, $n \in [m-3]$ and $s \in U_1 \subset \cdots \subset U_n \subseteq V(G) - \{t\}$,

(SC3) for each $i \in [n]$, there exist $q_i \in U_i$, *base* Q_{3+i} and *residue* R_{3+i}, where $Q_{3+i} \subset L_{3+i} - C_{3+i}$ is a q_it-path such that $V(Q_{3+i}) \cap U_i = \{q_i\}$, $R_{3+i} \subset L_{3+i} - C_{3+i}$ is a connected sq_i-join, and $Q_{3+i} \cap R_{3+i} = \emptyset$ (see figure 5),

(SC4) for each $i \in [n]$, $\delta(U_i)$ is a k-mate of $R_{3+i} \cup Q_{3+i}$, and for every proper subset W of U_i with $s \in W$, $\delta(W)$ is not a k-mate of $R_{3+i} \cup Q_{3+i}$.

For $i \in [m]-[n+3]$, set $Q_i := P_i$, $R_i := \emptyset$, and call Q_i the *base* of L_i, and for $i \in [3]$, set $Q_i := P_i = L_i$ and call Q_i the *base* of L_i.

A *cut Ω-system* consists of a tuple $\big((G,\Sigma,\{s,t\}),(L_1,\ldots,L_k),m,(U_1,\ldots,U_n),\vec{H}\big)$ where

(C1) $\big((G,\Sigma,\{s,t\}),(L_1,\ldots,L_k),m\big)$ is a bipartite Ω-system,

(C2) $(U_1,\ldots,U_n)$ is a primary or a secondary cut structure,

(C3) H is the union of all bases and, if it exists, the brace,

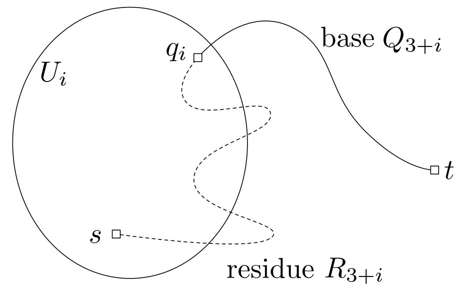


FIGURE 5. Bases and residues of primary ($i \in [n-1]$) and secondary cut structures ($i \in [n]$).

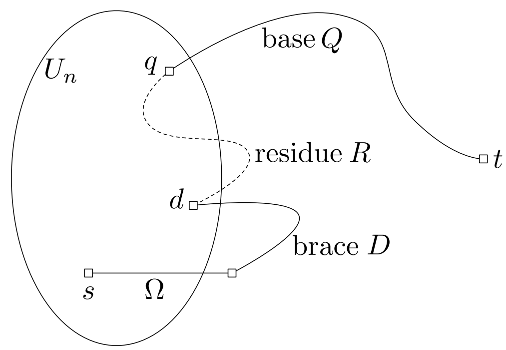


FIGURE 6. The base, residue and brace of U_n for the primary cut structure.

the brace, if it exists, is an sd-dipath in $\vec{H}$,

the bases are directed paths in $\vec{H}$ rooted towards t,

the following digraph $\vec{H}^+$ is acyclic: start from $\vec{H}$, for each q_i add arc (s, q_i), and if d, q existed and $d \neq q$, add arc (d, q),

$\Sigma \cap E(H) = \{\Omega\}$ and Σ has no edge in common with any of the residues.

(C4) for every odd st-dipath P in $\vec{H}$ such that $V(P) \cap U_n = \{s\}$, there is a k-mate for P.

Consider a non-bipartite Ω-system $\big((G, \Sigma, T), \mathcal{L}\big)$. Then $\mathcal{L}$ is the (Ω, k)-packing *associated with* the Ω-system and (G, Σ, T) is the signed graft *associated with* the Ω-system. Similarly, one defines the associated (Ω, k)-packing and the associated signed graft for bipartite and cut Ω-systems. We say that an Ω-system has a particular minor when the associated signed graft does. Theorem 1.3 follows from proposition 2.4 and the following three results,

Proposition 2.5. *A non-bipartite Ω-system has an F_7 minor.*

Proposition 2.6. *A non-simple bipartite Ω-system has an F_7 or a $\widetilde{K_5}$ minor.*

Proposition 2.7. *A simple bipartite Ω-system has an F_7 minor.*

2.3. **Outline of the proof.** In this section we discuss the outline of the proofs of propositions 2.5, 2.6 and 2.7.

A non-bipartite Ω-system $((G,\Sigma,T),(L_1,\dots,L_k))$ comes in the following *flavours*:

(NF1) at least two of L_1,L_2,L_3 are non-simple, and for $i\in[3]$, if L_i is non-simple then $\Omega\in P(L_i)$.

(NF2) at most one of L_1,L_2,L_3 is non-simple, and for $i\in[3]$, if L_i is non-simple then $\Omega\in C(L_i)$.

Note that $T\neq\emptyset$ for both flavours (NF1) and (NF2). We will postpone the proof of the next result to Section 5.

Proposition 2.8. *Every non-bipartite Ω-system is of flavour (NF1) or (NF2).*

A non-bipartite Ω-system $((G,\Sigma,\{s,t\}),(L_1,\dots,L_k))$ is *minimal* if (a) there is no non-bipartite Ω-system whose associated signed graft is a proper minor of $(G,\Sigma,\{s,t\})$, and (b) among all non-bipartite Ω-systems with the same associated signed graft, $|L_1\cup L_2\cup L_3|$ is minimized. Note that every non-bipartite Ω-system contains as a minor a minimal non-bipartite Ω-system. Proposition 2.5 will follow from the following results,

Proposition 2.9. *A minimal non-bipartite Ω-system of flavour (NF1) has an F_7 minor.*

Proposition 2.10. *Consider a minimal non-bipartite Ω-system of flavour (NF2) and assume that there is no non-bipartite Ω-system of flavour (NF1) with the same associated signed graft. Then the Ω-system has an F_7 minor.*

A non-simple bipartite Ω-system $((G,\Sigma,T),(L_1,\dots,L_k),m,\vec{H})$ is *minimal* if there is no non-simple bipartite Ω-system whose associated signed graft is a proper minor of (G,Σ,T). Proposition 2.6 is proved for minimal non-simple bipartite Ω-systems, which clearly is sufficient.

A simple bipartite Ω-system $((G,\Sigma,T),(L_1,\dots,L_k),m,\vec{H})$ comes in the following *flavours*:

(SF1) no odd st-dipath of $\vec{H}$ has an st-cut k-mate,

(SF2) some odd st-dipath of $\vec{H}$ has an st-cut k-mate.

A simple bipartite Ω-system $((G,\Sigma,\{s,t\}),(L_1,\dots,L_k),m,\vec{H})$ is *minimal* if there is no simple bipartite Ω-system whose associated signed graft is a proper minor of $(G,\Sigma,\{s,t\})$. Proposition 2.7 will follow from the following results,

Proposition 2.11. *Let $((G,\Sigma,\{s,t\}),(L_1,\dots,L_k),m,\vec{H})$ be a minimal simple bipartite Ω-system of flavour (SF1) and assume that there is no non-simple bipartite Ω-system whose associated signed graft is a minor of $(G,\Sigma,\{s,t\})$. Then the Ω-system has an F_7 minor.*

Proposition 2.12. *Let $((G,\Sigma,\{s,t\}),(L_1,\ldots,L_k),m,\vec{H})$ be a minimal simple bipartite Ω-system of flavour (SF2) and assume that there is no non-simple bipartite Ω-system whose associated signed graft is a minor of $(G,\Sigma,\{s,t\})$. Then the Ω-system has an F_7 minor.*

Our proof of proposition 2.12 is more involved.

Proposition 2.13. *A simple bipartite Ω-system of flavour (SF2) has a cut Ω-system.*

Proof. Let $((G,\Sigma,T),(L_1,\ldots,L_k),m,\vec{H})$ be a simple bipartite Ω-system of flavour (SF2). After redefining $\mathcal{L}$, if necessary, we may assume that L_1 has an st-cut k-mate. Choose $U_1\subseteq V(G)-\{t\}$ with $s\in U_1$ such that $\delta(U_1)$ is a k-mate of L_1, and for every proper subset W of U_1 with $s\in W$, $\delta(W)$ is not a k-mate of L_1. It is easily seen that (U_1) is a primary cut structure. Let R be the residue for L_1, and update $\vec{H}:=\vec{H}\setminus R$. It is easily seen that $((G,\Sigma,T),(L_1,\ldots,L_k),m,(U_1),\vec{H})$ is a cut Ω-system. □

Let $\big((G,\Sigma,\{s,t\}),(L_1,\ldots,L_k),m,(U_1,\ldots,U_n),\vec{H}\big)$ be a cut Ω-system. The Ω-system is *minimal* if, among all cut Ω-systems whose associated signed graft is a minor of $(G,\Sigma,\{s,t\})$, $|E(\vec{H})|$ is minimized, and the size n of the cut structure is maximized, in this order of priority. The Ω-system is *primary* (resp. *secondary*) if $(U_1,\ldots,U_n)$ is a primary (resp. secondary) cut structure. Proposition 2.12 will follow from proposition 2.13 and the following results,

Proposition 2.14. *Let $\big((G,\Sigma,\{s,t\}),(L_1,\ldots,L_k),m,(U_1,\ldots,U_{n-1},U),\vec{H}\big)$ be a minimal cut Ω-system that is primary and assume there is no non-simple bipartite Ω-system whose associated signed graft is a minor of $(G,\Sigma,\{s,t\})$. Then the Ω-system has an F_7 minor.*

Proposition 2.15. *Let $\big((G,\Sigma,\{s,t\}),(L_1,\ldots,L_k),m,(U_1,\ldots,U_n),\vec{H}\big)$ be a minimal cut Ω-system that is secondary and assume there is no non-simple bipartite Ω-system whose associated signed graft is a minor of $(G,\Sigma,\{s,t\})$. Then the Ω-system has an F_7 minor.*

2.4. **Organization of the paper.** Section 5 develops some preliminary results for non-bipartite Ω-systems. The proof of proposition 2.9 for Ω-systems of flavour (NF1) is given in §6. The proof of proposition 2.10 for Ω-systems of flavour (NF2) is given in §7. Section 8 develops some preliminary results for bipartite Ω-systems. The proof of proposition 2.6, along with preliminaries, is given in §9, §10, §11 and §12. Section 13 describes another preliminary and the proof of proposition 2.11 can be found in §14. Section 15 develops our last preliminary and the proofs of propositions 2.14 and 2.15 can be found in §16, §17, respectively. The outline is summarized in figure 7.

3. Covers

In this section, we develop tools that will be helpful in dealing with covers.

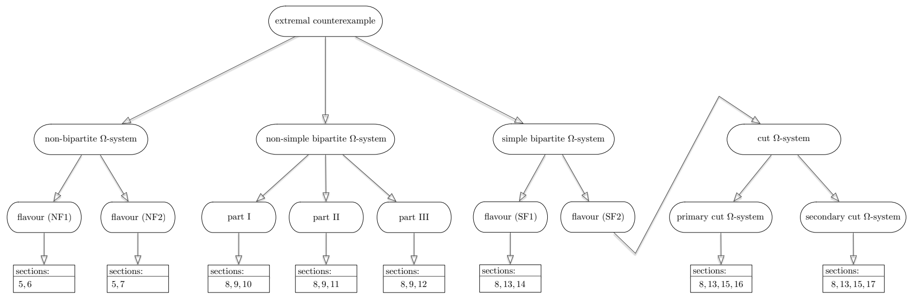


FIGURE 7. Outline of the proof.

3.1. **Caps and mates.** Let (G, Σ, T) be a signed graft and let $\mathcal{L} = (L_1, \ldots, L_k)$ be an (Ω, k)-packing. We say that for $\ell \in [k]$ a set $B \subseteq E(G)$ is a *cap of* L_ℓ *in* $\mathcal{L}$ if the following hold,

(T1) B is either a signature or a T-cut,

(T2) $\Omega \in B$,

(T3) $B \subseteq L_1 \cup \ldots \cup L_k$, and

(T4) for all $i \in [k] - \{\ell\}$, $|B \cap L_i| = 1$, and $|B \cap L_\ell| \geq 3$.

The next result characterizes k-mates of sets in an (Ω, k)-packing.

Proposition 3.1. *Let (G, Σ, T) be a signed graft and $\mathcal{L} = (L_1, \ldots, L_k)$ be an (Ω, k)-packing. Then for $\ell \in [k]$, B is a k-mate of L_ℓ if and only if B is a cap of L_ℓ in $\mathcal{L}$.*

Proof. Suppose first that B is a k-mate of L_ℓ. By definition of k-mates, (T1) holds and $|B - L_\ell| \leq k-3$. (T2) holds for otherwise, $B \cap L \neq \emptyset$ for all $L \in \mathcal{L}$ which implies $|B - L_\ell| \geq |\mathcal{L}| - 1 = k-1$, a contradiction. If $\ell \in [3]$, then $B - L_\ell$ intersects the $k-3$ pairwise disjoint sets $L_4, \ldots, L_k$. If $\ell \in [k] - [3]$, then $B - L_\ell$ intersects the $k-3$ pairwise disjoint sets in $\{L_3, L_4, \ldots, L_k\} - \{L_\ell\}$. In either cases $|B - L_\ell| = k-3$ and (T3) and (T4) hold.

Suppose (T1)-(T4) hold. Suppose $\ell \in [3]$ say $\ell = 1$. Then $B - L_1 \subseteq L_4 \cup \ldots \cup L_k$. Moreover, $|B \cap L_i| = 1$ for all $i \in \{4, \ldots, k\}$. Thus $|B - L_1| \leq k-3$, so B is a k-mate of L_1. Suppose $\ell \notin [3]$ say $\ell = 4$. Then $B - L_4 \subseteq \{\Omega\} \cup L_5 \cup \ldots \cup L_k$. Thus $|B - L_4| \leq k-3$, so B is a k-mate of L_4. □

Proposition 3.2. *Let (G, Σ, T) be a signed graft and let $L_4, \ldots, L_k$ be pairwise disjoint odd T-joins. Let L be a subset of $E(G) - (L_4 \cup \ldots \cup L_k)$ that has a k-mate B. Then $B \subseteq L \cup L_4 \cup \ldots \cup L_k$.*

Proof. We have

$$k - 3 \leq \sum_{i=4}^{k} |B \cap L_i| \leq |B - L| \leq k - 3,$$

where the first inequality follows from $B \cap L_i \neq \emptyset$, the second as $L \cap (L_4 \cup \ldots \cup L_k) = \emptyset$ and the third because B is a k-mate of L. Hence, equality holds throughout, so $|B - L| = k - 3$ and the result follows. ☐

Proposition 3.3. *Let (G, Σ, T) be a signed graft and take two (Ω, k)-packings*

$$\mathcal{L} = (L_1, L_2, L_3, L_4, \ldots, L_k) \quad \textit{and} \quad \mathcal{L}' = (L'_1, L'_2, L_3, L_4, \ldots, L_k).$$

Let B_1, B'_1 be k-mates of L_1, L'_1, respectively. Let $B \subseteq B_1 \cup B'_1$ be a cover that is either a signature or a T-cut. Then,

(1) $\Omega \in B$,
(2) $B \subseteq L_1 \cup L'_1 \cup L_4 \cup \ldots \cup L_k$,
(3) $|B \cap L_i| = 1$ for all $i \in \{3, \ldots, k\}$,
(4) B is a k-mate of $L_1 \cup L'_1$,
(5) $|B \cap L_1| \geq 3$ or $|B \cap L'_1| \geq 3$,
(6) if $B \cap (L'_1 - L_1) = \emptyset$ then B is a k-mate of L_1,
(7) if $B \cap (L'_1 - L_1) = B \cap (L_1 - L'_1) = \emptyset$ then B is a k-mate of $L_1 \cap L'_1$.

Proof. By proposition 3.1 B_1 (resp. B'_1) is a cap of L_1 (resp. L'_1) in $\mathcal{L}$ (resp. $\mathcal{L}'$). Thus,

(a) $$B_1 \cup B'_1 \subseteq L_1 \cup L'_1 \cup L_4 \cup \ldots \cup L_k,$$

(b) $$|B_1 \cap L_i| = |B'_1 \cap L_i| = 1 \quad \text{for all} \quad i \in \{4, \ldots, k\},$$

Since $B \subseteq B_1 \cup B'_1$, (a) implies that (2) holds. As B is a cover and $B \cap L_3 \neq \emptyset$, (1) must hold as well. Let $i \in \{4, \ldots, k\}$. Then by (b)

$$|B \cap L_i| \leq |B_1 \cap L_i| + |B'_1 \cap L_i| \leq 2.$$

Hence, as B is a cover, $|B \cap L_i| = 1$ so (3) holds. Combining this with (a) yields

$$|B - (L_1 \cup L'_1)| \leq \sum_{i=4}^{k} |B \cap L_i| = k - 3$$

and so B is a k-mate of $L_1 \cup L'_1$ so (4) holds. It follows (as every cover has cardinality at least $\tau(G, \Sigma) \geq k$) that $|B \cap (L_1 \cup L'_1)| \geq 3$. Hence, for some $L \in \{L_1, L'_1\}$, $|B \cap L| > 1$ and so $|B \cap L| \geq 3$ thus (5) holds. (6) and (7) trivially follow from (4). ☐

The following are immediate corollaries.

Proposition 3.4. *Let (G,Σ,T) be a signed graft and $\mathcal{L}=(L_1,\dots,L_k)$ be an (Ω,k)-packing. Suppose for $i=1,2$, B_i is a k-mate of L_i and let $B\subseteq B_1\cup B_2$ be a cover that is either a signature or a T-cut. Then*

(1) $\Omega\in B$,
(2) $B\subseteq L_1\cup L_2\cup L_4\cup\ldots\cup L_k$,
(3) $|B\cap L_i|=1$ *for all* $i\in\{3,\dots,k\}$,
(4) $|B\cap L_1|\geq 3$ *or* $|B\cap L_2|\geq 3$,
(5) for $i=1,2$, *if* $|B\cap L_i|=1$ *then* B *is a* k*-mate of* L_{3-i}.

Proof. Choose $\mathcal{L}'=(L_2,L_1,L_3,\dots,L_k)$ and apply proposition 3.3 parts (5) and (6). □

Proposition 3.5. *Let (G,Σ,T) be a signed graft and $\mathcal{L}=(L_1,\dots,L_k)$ be an (Ω,k)-packing. Suppose B_1 and B_1' are k-mates of L_1 and let $B\subseteq B_1\cup B_1'$ be a cover that is either a signature or a T-cut. Then B is also a k-mate of L_1.*

Proof. Choose $\mathcal{L}'=\mathcal{L}$ and apply proposition 3.3(6). □

3.2. **Signatures versus T-cuts.**

Proposition 3.6. *Let (G,Σ,T) be a signed graft with $|T|\leq 2$ and let $(L_1,\dots,L_k)$ be an (Ω,k)-packing. Suppose that L_1,L_2 are minimal odd T-joins and, for $i=1,2$ L_i is simple or $\Omega\in C(L_i)$. Suppose further that for $i=1,2$ there exists a k-mate B_i of L_i. Then one of B_1,B_2 is a signature.*

Proof. By proposition 3.1, for each $i=1,2$, B_i is a cap of L_i in $\mathcal{L}$. Thus, $B_1\cap L_2=B_2\cap L_1=\{\Omega\}$. Hence, if $\Omega\in C(L_1)$ then $B_2\cap C(L_1)=\{\Omega\}$, implying that B_2 is a signature. Similarly, if $\Omega\in C(L_2)$ then B_1 is a signature. Otherwise, $T=\{s,t\}$ and L_1,L_2 are simple. Suppose for a contradiction that for $i=1,2$, $B_i=\delta(U_i)$ where $U_i\subseteq V(G)-\{t\}$. Let $B=\delta(U_1\cap U_2)\subseteq B_1\cup B_2$. By proposition 3.1 $\{\Omega\}=L_2\cap B_1=L_2\cap\delta(U_1)$. Since L_2 is simple and since $U_1\cap U_2\subset U_1$, $\delta(U_1\cap U_2)\cap L_2=\{\Omega\}$, it follows that $L_2\cap B=\{\Omega\}$ (recall $\omega\in\delta(s)$). Similarly, we have $L_1\cap B=\{\Omega\}$, contradicting proposition 3.4 part (4). □

Proposition 3.7. *Let (G,Σ,T) be a signed graft with $T=\{s,t\}$ and let $(L_1,\dots,L_k)$ be an (Ω,k)-packing, where L_1,L_2,L_3 are minimal odd T-joins. Suppose that L_1 is non-simple and that L_2,L_3 are simple. Suppose that for $i=2,3$ there exists a k-mate B_i of L_i. Then $\Omega\in C(L_1)$.*

Proof. By proposition 3.6 one of B_2,B_3 is a signature, say B_2. Thus $B_2\cap C(L_1)\neq\emptyset$. But proposition 3.1 implies that $B_2\cap L_1=\{\Omega\}$ and the result follows. □

Proposition 3.8. *Let (G,Σ,T) be a signed graft with $|T|\leq 2$ and let $(L_1,\dots,L_k)$ be an (Ω,k)-packing. Suppose that L_2 is a non-simple minimal odd T-join and that there exists a k-mate B_1 of L_1. Then,*

(1) if $\Omega\in P(L_2)$ then B_1 is a T-cut,

(2) if $\Omega\in C(L_2)$ then B_1 is a signature.

Proof. **(1)** By proposition 3.1, $B_1\cap L_2=\{\Omega\}$. Since $\Omega\in P(L_2)$, $B_1\cap C(L_2)=\emptyset$. Since $C(L_2)$ is an odd circuit, B_1 is not a signature. It follows from the definition of k-mate that B_1 is a T-cut. **(2)** Proceeding as above we have $B_1\cap P(L_2)=\emptyset$. If $T=\emptyset$, then we are done. Otherwise, $T=\{s,t\}$ and $P(L_2)$ is an st-path, so B_1 is not an st-cut. It follows that B_1 is a signature. □

4. Non-bipartite, non-simple and simple bipartite Ω-systems

In this section, we prove proposition 2.4, stating that every extremal counterexample has a non-bipartite, non-simple bipartite, or simple bipartite Ω-system.

Proof of proposition 2.4. Let (G,Σ,T) be an extremal counterexample with $\tau:=\tau(G,\Sigma,T)$. By proposition 2.3 parts (2) and (4) there exists an (Ω,τ)-packing $\mathcal{L}=(L_1,\dots,L_\tau)$ of odd T-joins. By proposition 2.3 part (5) every odd T-join has a τ-mate. If $(L_1\cup L_2\cup L_3)-\{\Omega\}$ is non-bipartite, then $\big((G,\Sigma,T),\mathcal{L}\big)$ is a non-bipartite Ω-system. Otherwise, $(L_1\cup L_2\cup L_3)-\{\Omega\}$ is bipartite. We will show that (G,Σ,T) has a non-simple bipartite or simple bipartite Ω-system.

We can rearrange the elements of the sequence $\mathcal{L}$ to ensure (B2) is satisfied for some $m\in[\tau]-[2]$. For each $i\in[3]$, let B_i be a τ-mate of L_i. Since $(L_1\cup L_2\cup L_3)-\{\Omega\}$ is bipartite, it follows that, for each $i\in[3]$, either L_i is simple or $\Omega\in C(L_i)$. Therefore, by proposition 3.6, at least two of B_1,B_2,B_3, say B_1 and B_2, are signatures. By proposition 3.1, B_1 (resp. B_2) is a cap of L_1 (resp. L_2) in $\mathcal{L}$. Let U be the subset of $V(L_1)-T$ for which $L_1\cap\delta(U)=(L_1\cap B_1)-\{\Omega\}$, and let $\Gamma:=B_1\triangle\delta(U)$. It is clear that Γ is a signature for (G,Σ,T). We will show that $((G,\Gamma,T),\mathcal{L},m)$ is a bipartite Ω-system. It is clear that (B1) and (B2) hold. To prove (B3), we need to show that, for $i\in[3]$, $\Gamma\cap L_i=\{\Omega\}$, and for $i\in[m]-[3]$, $\Gamma\cap P_i=\emptyset$. By definition, $\Gamma\cap L_1=\{\Omega\}$.

Claim 1. *For $i=2,3$, $B_1\cap P_i=\emptyset$ and $\delta(U)\cap L_i=\emptyset$.*

Proof. Since $B_1\cap L_i=\{\Omega\}$ and $\Omega\notin P_i$, it follows that $B_1\cap P_i=\emptyset$. To prove the next equation, choose vertices s,s',t as follows: Ω has ends s,s', if $T\neq\emptyset$ then $T=\{s,t\}$, and if $T=\emptyset$ then $t:=s$. Notice that $s,s',t\notin U$ and $Q_i:=L_i-\{\Omega\},Q_1:=L_1-\{\Omega\}$ are $s't$-paths. Suppose for a contradiction that $\delta(U)\cap L_i\neq\emptyset$. Then our choice of U implies that L_i and L_1 have a vertex $u\in U$ in common. Consider the cycle $C:=Q_i[u,t]\cup Q_1[u,t]$.[7] Since $B_1\cap L_i=\{\Omega\}$ and $(B_1\cap L_1)-\{\Omega\}=\delta(U)\cap L_1$, it

[7]Given a path P and vertices $a,b\in V(P)$, $P[a,b]$ denotes the subpath between a and b.

follows that $B_1 \cap C = \delta(U) \cap Q_1[u,t]$, implying in turn that $|B_1 \cap C|$ is odd. As B_1 is a signature, it follows that $C \subseteq (L_1 \cup L_2) - \{\Omega\}$ is an odd cycle, a contradiction as $(L_1 \cup L_i) - \{\Omega\}$ is bipartite. $\diamond$

Thus, for $i = 2, 3$

$$\Gamma \cap L_i = (B_1 \triangle \delta(U)) \cap L_i = (B_1 \cap L_i) \triangle (\delta(U) \cap L_i) = \{\Omega\}.$$

Claim 2. *For $i \in [m] - [3]$, $\delta(U) \cap P_i = \emptyset$.*

Proof. As B_1, B_2 are signatures and $|B_1 \cap L_i| = |B_2 \cap L_i| = 1$, it follows that $B_1 \cap P_i = B_2 \cap P_i = \emptyset$. Hence, $B_2 \cap (L_1 \cup P_i) = \{\Omega\}$, implying that $(L_1 \cup P_i) - \{\Omega\}$ is bipartite. Suppose for a contradiction that $\delta(U) \cap P_i \neq \emptyset$. Then L_1 and P_i have a vertex of U in common, and so $(L_1 \cup P_i) - \{\Omega\}$ is non-bipartite, a contradiction. $\diamond$

Hence, for $i \in [m] - [3]$

$$\Gamma \cap P_i = (B_1 \triangle \delta(U)) \cap P_i = (B_1 \cap P_i) \triangle (\delta(U) \cap P_i) = \emptyset.$$

Therefore, (B3) holds and $((G, \Gamma, T), \mathcal{L}, m)$ is a bipartite Ω-system. Among all bipartite Ω-systems whose associated signed graft is (G, Γ, T), we may assume that the (Ω, τ)-packing $\mathcal{L}$ of odd T-joins has the smallest total number of edges.

Let $H := G[L_1 \cup L_2 \cup L_3 \cup P_4 \cup \cdots \cup P_m]$. Orient the edges of H so that each of L_1, L_2, L_3 is a directed T-join, and if $T = \{s, t\}$ and $\Omega \in \delta(s)$, each of $P_4, \ldots, P_m$ is an st-dipath; call this digraph $\vec{H}$.

Claim 3. *$\vec{H} \setminus \Omega$ is acyclic.*

Proof. Suppose otherwise. Let C be a directed circuit in $\vec{H} \setminus \Omega$. We assume that $\Omega = (s, s')$ and that either $T = \emptyset$ or $T = \{s, t\}$. When $T = \emptyset$, set $t := s$. Create $m - 3$ copies $\overline{\Omega}_4, \ldots, \overline{\Omega}_m$ of the arc (s', s). For each $i \in [3]$, let $Q_i := L_i - \{\Omega\}$ and for each $i \in [m] - [3]$, let $Q_i := \{\overline{\Omega}_i\} \cup P_i$. Notice that $Q_1, \ldots, Q_m$ are pairwise arc-disjoint directed $s't$-joins, and Q_1, Q_2, Q_3 are $s't$-dipaths. We can now decompose $(Q_1 \cup \cdots \cup Q_m) - C$ into pairwise arc-disjoint directed $s't$-joins $Q'_1 \cup \cdots \cup Q'_m$, where

- Q'_1, Q'_2, Q'_3 are $s't$-dipaths, and
- for $i \in [m] - [3]$, $\overline{\Omega}_i \in Q'_i$.

For $i \in [3]$, let $L'_i := Q'_i \cup \{\Omega\}$, and for $i \in [m] - [3]$, let P'_i be an st-dipath contained in $Q'_i - \{\overline{\Omega}_i\}$. Then L'_1, L'_2, L'_3 are directed odd st-joins and $P'_4, \ldots, P'_m$ are even st-dipaths in $\vec{H}$. Let $\mathcal{L}' = (L'_1, L'_2, L'_3, C_4 \cup P'_4, \ldots, C_m \cup P'_m, L_{m+1}, \ldots, L_\tau)$. It can now be readily checked that $((G, \Gamma, T), \mathcal{L}', m)$ is a bipartite Ω-system, a contradiction as $\mathcal{L}'$ has fewer edges than $\mathcal{L}$. $\diamond$

It is now easily seen that $((G,\Gamma,T),\mathcal{L},m,\vec{H})$ is either a non-simple bipartite or simple bipartite Ω-system, finishing the proof. □

5. Preliminaries for non-bipartite Ω-systems

In this section we prove results required for the proofs of propositions 2.9 and 2.10. We also prove proposition 2.8, namely, that every non-bipartite Ω-system is of flavour (NF1) or (NF2).

5.1. **The two flavours (NF1) and (NF2).** Let us start with the following:

Proposition 5.1. *Let (G,Σ) be a signed graph whose edges can be partitioned for some distinct vertices x,y into xy-paths $Q_1,Q_2,\ldots,Q_n$. If, for every distinct $i,j\in[n]$, $Q_i\cup Q_j$ is bipartite, then (G,Σ) is bipartite.*

Proof. We will proceed by induction on n. For $n=1$ this is obvious. Suppose $n>1$. By the induction hypothesis, $Q_1\cup\ldots\cup Q_{n-1}$ is bipartite, and so by theorem 1.1, there is a signature Γ of $(G,\Sigma,\emptyset)$ disjoint from $Q_1\cup\cdots\cup Q_{n-1}$, so $\Gamma\subseteq Q_n$. As $Q_1\cup Q_n$ is an even cycle, it follows that $|\Gamma|$ is even. Let U be the vertex subset of $V(Q_n)-\{x,y\}$ for which $\delta(U)\cap Q_n=\Gamma\cap Q_n$. We claim that $\delta(U)=\Gamma$, and this will imply that $(G,\Sigma,\emptyset)$, and therefore (G,Σ), is bipartite.

Suppose, for a contradiction, that $\Gamma\subsetneq\delta(U)$. Take an edge $\{v,u\}\in\delta(U)-\Gamma$ with $u\in U$. Then $\{v,u\}$ belongs to some $Q_j\in\{Q_1,\ldots,Q_{n-1}\}$. We may assume that $\{v,u\}\in Q_j[x,u]$. Let $C=Q_1[x,u]\cup Q_j[x,u]$. Then $|C\cap\Gamma|=|Q_1[x,u]\cap\delta(U)|$, which is odd as $x\notin U$ and $u\in U$. Hence, C is an odd cycle, but $C\subseteq Q_1\cup Q_j$, which is a contradiction. Therefore, $\Gamma=\delta(U)$, and this completes the proof. □

Next we prove that every non-bipartite Ω-system is of flavour (NF1) or (NF2).

Proof of proposition 2.8. Let $((G,\Sigma,T),(L_1,\ldots,L_k))$ be a non-bipartite Ω-system that is not of flavour (NF2). We will show (NF1) holds.

Proposition 3.7 implies that at least two of L_1,L_2,L_3 are non-simple. It remains to show that $\Omega\in P(L_1)\cap P(L_2)\cap P(L_3)$. Suppose otherwise. Then, for some $i\in[3]$, L_i is non-simple and $\Omega\in C(L_i)$. By proposition 3.8, B_1,B_2,B_3 are signatures, and whenever $L_i\in\{L_1,L_2,L_3\}$ is non-simple, $\Omega\in C(L_i)$.

For each $j\in[3]$, let $Q_j=L_j-\{\Omega\}$. Suppose s,s' are the ends of Ω. When $T=\emptyset$, Q_1,Q_2 and Q_3 are $s's$-paths, and when $T=\{s,t\}$, Q_1,Q_2 and Q_3 are all $s't$-paths. Moreover, for every permutation i,j,k of $1,2,3$, $(Q_i\cup Q_j)\cap B_k=\emptyset$, implying that $Q_i\cup Q_j$ is bipartite. Therefore, from proposition 5.1 we conclude that $Q_1\cup Q_2\cup Q_3=(L_1\cup L_2\cup L_3)-\{\Omega\}$ is bipartite, which is a contradiction. □

5.2. **A disentangling lemma.**

Lemma 5.2. *Let $((G,\Sigma,\{s,t\}),(L_1,\ldots,L_k))$ be a minimal non-bipartite Ω-system. For $i=1,2$, let $R_i\cup Q_i$ be a non-trivial partition of L_i such that $\Omega\in Q_1\cap Q_2$, $R_1\cup Q_2$ is a minimal odd st-join and $R_1\cup R_2$ is an even cycle. Let Q_3 be a minimal subset of L_3 such that $Q_3\cup R_1$ contains a minimal odd st-join. Then one of the following does not hold:*

(i) $(L_1\cup Q_2\cup Q_3)-\{\Omega\}$ *is non-bipartite,*

(ii) $R_2\cup\{\Omega\}$ *does not have a k-mate,*

(iii) R_1 *is a path whose internal vertices all have degree two in* $G[L_1\cup Q_2\cup Q_3]$.

Proof. Suppose otherwise. We will show that $((G,\Sigma,\{s,t\}),(L_1,\ldots,L_k))$ is not a minimal non-bipartite Ω-system, which will yield a contradiction. Let $(G',\Sigma'):=(G,\Sigma)\setminus R_2/R_1$ and define $L'_1,\ldots,L'_k$ as follows: for $i\in[3]$ $L'_i:=Q_i$, and for $i\in\{4,\ldots,k\}$ L'_i is a minimal odd st-join of (G',Σ') contained in L_i. We claim that $((G',\Sigma',\{s,t\}),(L'_1,\ldots,L'_k))$ is a non-bipartite Ω-system.

(N1) Since $R_1\cup R_2$ is an even cycle, every minimal cover of $(G,\Sigma,\{s,t\})$ disjoint from R_1 has an even number of edges in common with R_2. Hence, $(G',\Sigma',\{s,t\})$ is Eulerian and $\tau(G',\Sigma',\{s,t\})$, $\tau(G,\Sigma,\{s,t\})$ have the same parity. **(N3)** Observe that (i) implies $(L'_1\cup L'_2\cup L'_3)-\{\Omega\}$ is non-bipartite. **(N4)** Let $L'\subseteq L'_1\cup L'_2\cup L'_3$ be a minimal odd st-join of $(G',\Sigma',\{s,t\})$. By (iii) one of $L', L'\cup R_1$ is a minimal odd st-join of $(G,\Sigma,\{s,t\})$. In the former case, let B' be a k-mate of L' in $(G,\Sigma,\{s,t\})$. By definition, $|B'-L'|\le k-3$ and so $B'-L'\subseteq L_4\cup\cdots\cup L_k$, implying that $B'\cap R_1=\emptyset$. Thus B' is still a k-mate for L' in $(G',\Sigma',\{s,t\})$. In the latter case, when $L'\cup R_1$ is a minimal odd st-join of $(G,\Sigma,\{s,t\})$, $L'\cup R_2$ also contains a minimal odd st-join L. Let B be a k-mate of L in $(G,\Sigma,\{s,t\})$. Once again, $|B-L|\le k-3$ and so $B-L\subseteq L_4\cup\cdots\cup L_k$, implying that $B\cap R_1=\emptyset$. As a result, $B-R_2$ is a k-mate for L' in $(G',\Sigma',\{s,t\})$. **(N2)** As $\tau(G',\Sigma',\{s,t\}),\tau(G,\Sigma,\{s,t\})$ have the same parity, $\tau(G',\Sigma',\{s,t\}),k$ have the same parity. We need to show $\Omega\in L'_3$ and $\tau(G',\Sigma',\{s,t\})\ge k$. By (N4) L'_1 has a k-mate B' in $(G',\Sigma',\{s,t\})$. Then $|B'-L'_1|\le k-3$ and so $B'-L'_1\subseteq L'_4\cup\cdots\cup L'_k$. Since $B'\cap L'_3\neq\emptyset$, $B'\cap L'_3=\{\Omega\}$, and so $\Omega\in L'_3$. Suppose for a contradiction that $\tau(G',\Sigma',\{s,t\})<k$. The parity condition implies that $\tau(G',\Sigma',\{s,t\})\le k-2$. Let B' be a minimum cover in $(G',\Sigma',\{s,t\})$. For $|B'|\le k-2$ and $L'_1,L'_4,\ldots,L'_k$ are $k-2$ pairwise disjoint odd st-joins, we have $|B'|=k-2$, and as $B'\cap L'_2\neq\emptyset$, $\Omega\in B'$. Let B be a minimal cover of $(G,\Sigma,\{s,t\})$ contained in $B'\cup R_2$ and containing B'. By proposition 1.2, B is either a signature or an st-cut. However, $|B-(R_2\cup\{\Omega\})|=|B'-\{\Omega\}|=k-3$, implying that B is a k-mate of $R_2\cup\{\Omega\}$ in $(G,\Sigma,\{s,t\})$, contradicting (ii). □

5.3. **Mates and connectivity.** Recall that if $(G,\Sigma,\{s,t\})$ is a signed graft with signatures Σ_1,Σ_2 then by definition $\Sigma_1\triangle\Sigma_2$ is a cut where both s,t are on the same shore. We will require the following easy remark,

Remark 5.3. *Let G be a graph with distinct vertices s, t. For $i = 1, 2$ let $W_i \subseteq V(G) - \{t\}$ where $s \in W_1 \subseteq W_2$. Let P be an st-path and let Ω be the edge of P incident to s. If $P \cap \delta(W_2) = \{\Omega\}$ then $P \cap \delta(W_1) = \{\Omega\}$.*

Proposition 5.4. *Let $(G, \Sigma, \{s, t\})$ be a signed graft and $(L_1, \ldots, L_k)$ be an (Ω, k)-packing, where L_2 is an odd st-path. Suppose there exist an st-cut B_1 that is a k-mate of L_1 and a signature B_2 that is a k-mate of L_2. Choose $U_1 \subseteq V(G) - \{t\}$ such that $B_1 = \delta(U_1)$ and let $W = (V(L_1) \cap U_1) - \{s\}$. Then there exists a path in $G[U_1]$ between s and W that is disjoint from B_2.*

Proof. Suppose for a contradiction there is no such path. Then there exists $U' \subset U_1$ such that $s \in U'$ and $W \subseteq U_1 - U'$ and all edges with one end in U' and one end in $U_1 - U'$ are in B_2. Then the st-cut $B = \delta(U') \subseteq B_1 \cup B_2$ and by construction $L_1 \cap B = \{\Omega\}$. By proposition 3.1 $L_2 \cap B_1 = L_2 \cap \delta(U_1) = \{\Omega\}$. Since L_2 is an odd st-path, and since $U' \subset U_1$ by remark 5.3, $\delta(U') \cap L_2 = \{\Omega\}$. But then $|B \cap L_1| = |B \cap L_2| = 1$, contradicting proposition 3.4 part (4). □

6. Non-bipartite Ω-system of flavour (NF1)

In this section we prove proposition 2.9, namely that a minimal non-bipartite Ω-system of flavour (NF1) has an F_7 minor. For convenience, whenever L_i is non-simple, we write $P_i := P(L_i)$ and $C_i := C(L_i)$. Let $(G, \Sigma, \{s, t\})$ be a signed graft and let $\delta(U)$ be an st-cut that is a k-mate of a minimal odd st-join L. We say that $U \subseteq V(G) - \{t\}$ is *shore-wise minimal* if among all k-mates of L of the form $\delta(U')$ where $U' \subseteq V(G) - \{t\}$, U' is not a proper subset of U.

Proposition 6.1. *Let $((G, \Sigma, \{s, t\}), \mathcal{L} = (L_1, \ldots, L_k))$ be a non-bipartite Ω-system of flavour (NF1), where $\Omega \in \delta(s)$. Then,*

(1) for $i \in [3]$, every k-mate of L_i is an st-cut.

Furthermore, for $i \in [3]$, let $\delta(U_i)$ be a k-mate of L_i where U_i is shore-wise minimal. Then

(2) for $i \in [3]$, if L_i is non-simple, then $P_i \cap \delta(U_i) = \{\Omega\}$ and $C_i \cap \delta(U_i) \neq \emptyset$,

(3) for distinct $i, j \in [3]$, if L_i, L_j are non-simple, then $U_i \subset U_j$ or $U_j \subset U_i$,

(4) for distinct $i, j \in [3]$, if L_i is non-simple and L_j is simple, then $U_i \subset U_j$.

Proof. **(1)** Let $i \in [3]$ and let B be a k-mate of L_i. By (NF1) one of $\{L_1, L_2, L_3\} - \{L_i\}$, say L_j, is non-simple and $\Omega \in P_j$. Proposition 3.8 then implies that B is an st-cut.

Now for $i \in [3]$, let $B_i = \delta(U_i)$ be a k-mate of L_i where $U_i \subseteq V(G) - \{t\}$ is shore-wise minimal. We need to prove (2)-(4). We may assume L_1 and L_2 are non-simple. By proposition 3.1, for $i \in [3]$, B_i is a cap of L_i in $\mathcal{L}$.

(2) We may assume $i = 1$. Consider the (Ω, k)-packing

$$\mathcal{L}' = (C_1 \cup P_2, C_2 \cup P_1, L_3, \ldots, L_k).$$

As $((G, \Sigma, \{s,t\}), \mathcal{L})$ is a non-bipartite Ω-system, $C_1 \cup P_2$ has a k-mate B_1'. By proposition 3.1, B_1' is a cap of $C_1 \cup P_2$ in $\mathcal{L}'$, implying that $B_1' \cap (C_2 \cup P_1) = \{\Omega\}$ and so $B_1' \cap C_2 = \emptyset$. Thus, by proposition 3.8, B_1' is an st-cut $\delta(U)$ where $U \subseteq V(G) - \{t\}$. Consider the st-cut $B = \delta(U_1 \cap U) \subseteq B_1 \cup B_1'$. Since B_1 is a cap of L_1 in $\mathcal{L}$, and B_1' is a cap of $C_1 \cup P_2$ in $\mathcal{L}'$, $P_2 \cap \delta(U_1) = P_1 \cap \delta(U) = \{\Omega\}$. Thus $B \cap P_2 = \delta(U_1 \cap U) \cap P_2 = \{\Omega\}$ and $B \cap P_1 = \delta(U_1 \cap U) \cap P_1 = \{\Omega\}$ (see remark 5.3). It follows by proposition 3.3 that B is a k-mate of $L_1 \cap (C_1 \cup P_2) = C_1 \cup \{\Omega\}$. In particular, B is a k-mate of L_1. Since U_1 is shore-wise minimal, $U_1 \subseteq U$. Hence, as $P_1 \cap \delta(U) = \{\Omega\}$, we have $P_1 \cap \delta(U_1) = \{\Omega\}$. Also, since B_1 is a cap of L_1 in $\mathcal{L}$, $C_1 \cap \delta(U_1) \neq \emptyset$.

(3) Since $\delta(U_i), \delta(U_j)$ are, respectively, caps of L_i, L_j in $\mathcal{L}$,

$$\delta(U_i) \cap C_j = \emptyset \quad \text{and} \quad \delta(U_j) \cap C_i = \emptyset.$$

Thus, either $V(C_i) \subseteq U_j$ or $V(C_i) \cap U_j = \emptyset$, and either $V(C_j) \subseteq U_i$ or $V(C_j) \cap U_i = \emptyset$. By (2), $P_i \cap \delta(U_i) = P_j \cap \delta(U_j) = \{\Omega\}$, and so $\delta(U_i) \cap C_i \neq \emptyset$ and $\delta(U_j) \cap C_j \neq \emptyset$. By proposition 3.4 (4),

$$\delta(U_i \cap U_j) \cap (C_i \cup C_j) \neq \emptyset \quad \text{and} \quad \delta(U_i \cup U_j) \cap (C_i \cup C_j) \neq \emptyset.$$

It therefore follows that, after possibly interchanging the role of i, j, we have that $V(C_i) \subseteq U_j$ and $V(C_j) \cap U_i = \emptyset$. But then proposition 3.4 (5) implies that $\delta(U_i \cap U_j)$ is a k-mate of L_i. Hence, as U_i is shore-wise minimal, $U_i \subset U_j$ as required.

(4) Since $\delta(U_i)$ is a cap of L_i in $\mathcal{L}$, $\delta(U_i) \cap L_j = \{\Omega\}$, and as L_j is simple, $L_j \cap \delta(U_i \cap U_j) = \{\Omega\}$ (see remark 5.3). Therefore, by proposition 3.4 (5), $\delta(U_i \cap U_j)$ is a k-mate of L_i. Since U_i is shore-wise minimal, $U_i \subset U_j$ as required. □

Lemma 6.2. *Let $((G, \Sigma, \{s,t\}), \mathcal{L} = (L_1, \ldots, L_k))$ be a minimal non-bipartite Ω-system of flavour (NF1), where $\Omega \in \delta(s)$ and among all non-bipartite Ω-systems with the same associated signed graft, the number of non-simple minimal odd st-joins among L_1, L_2, L_3 is maximum. Suppose, for $i \in [3]$, $B_i = \delta(U_i)$ is a k-mate of L_i where U_i is shore-wise minimal and where $U_1 \subset U_2 \subset U_3$. Then the following hold:*

1. *For distinct $i, j \in [3]$, if L_i and L_j are non-simple, then C_i and C_j have at most one vertex in common.*
2. *For distinct $i, j \in [3]$, if L_i is non-simple and L_j is simple, then C_i and L_j have at most one vertex in common.*

(3) Suppose L_3 is simple. If L is a minimal odd st-join contained in $P_2 \cup L_3$, then $L \cap \delta(U_3) = L_3 \cap \delta(U_3)$.

(4) Let L_0 be the path with a single vertex s and let $U_0 := \emptyset$. For some $j \in [3]$, take $v \in V(L_j) \cap (U_j - U_{j-1})$. Let U be the component of $G[U_j - U_{j-1}]$ containing v. Then $V(L_{j-1}) \cap U \neq \emptyset$.

(5) Suppose L_3 is non-simple. Then there is a path in $G[\overline{U_3}]$ between $V(C_3)$ and t, where $\overline{U_3} = V(G) - U_3$.

Proof. Observe that L_1 and L_2 are non-simple. By proposition 6.1, for each $i \in [3]$, if L_i is non-simple then $P_i \cap \delta(U_i) = \{\Omega\}$ and $C_i \cap \delta(U_i) \neq \emptyset$. Thus $V(C_1) \subseteq U_2$, $V(C_2) \subseteq U_3 - U_1$, and if L_3 is non-simple, $V(C_3) \cap U_2 = \emptyset$. Moreover, for $i = 1, 2$, $V(P_i) \cap U_3 = \{s\}$, and if L_3 is non-simple, $V(P_3) \cap U_3 = \{s\}$.

(1) We will first prove that C_1 and C_2 have at most one vertex in common. Suppose otherwise. We will obtain a contradiction by proving that $((G, \Sigma, \{s, t\}), \mathcal{L})$ is not a minimal non-bipartite Ω-system.

Choose distinct vertices $u, v \in V(C_1) \cap V(C_2)$. Notice that $u, v \in U_2 - U_1$. Let R_1 be a uv-path contained in C_1 that avoids vertex s. Let R_2 be the uv-path contained in C_2 such that $R_1 \cup R_2$ is an even cycle (notice that C_2 is an odd circuit). For $i = 1, 2$, let $Q_i = L_i - R_i$, and let $Q_3 = L_3$. Observe that $V(R_1) \subset V(C_1) \subseteq U_2$, that R_1 is internally vertex-disjoint from $C_1 - R_1$ as C_1 is a circuit, and that R_1 is vertex-disjoint from $P_1 \cup P_2 \cup Q_3$ as $V(P_1) \cap U_2 = V(P_2) \cap U_2 = V(Q_3) \cap U_2 = \{s\}$. Notice further that R_1 is internally vertex-disjoint from $C_2 - R_2$. For if not, $C_2 \triangle (R_1 \cup R_2)$ can be partitioned into non-empty parts C_2', X where C_2' is an odd circuit and X is an even cycle. But then $((G, \Sigma, \{s, t\}) \backslash X, (L_1 \triangle (R_1 \cup R_2), C_2' \cup P_2, L_3, \ldots, L_k))$ is another non-bipartite Ω-system, contradicting the minimality of the Ω-system $((G, \Sigma, \{s, t\}), \mathcal{L})$. It therefore follows that the internal vertices of R_1 all have degree two in $G[L_1 \cup Q_2 \cup Q_3]$. Observe that $(Q_1 \cup Q_2 \cup Q_3 \cup R_1) - \{\Omega\}$ is non-bipartite as it contains the odd cycle C_1. Lemma 5.2 therefore implies $R_2 \cup \{\Omega\}$ has a k-mate B. Observe that B is also a k-mate of L_2 and of $L_1 \triangle (R_1 \cup R_2)$, as $R_2 \cup \{\Omega\} \subset L_2$ and $R_2 \cup \{\Omega\} \subset L_1 \triangle (R_1 \cup R_2)$. Thus by proposition 6.1 B is an st-cut, so $B = \delta(U)$ for some $U \subseteq V(G) - \{t\}$. Then $\delta(U_2 \cap U)$ is a cover contained in $B_2 \cup B$, and so by proposition 3.5, it is a k-mate of L_2. Thus the shore-wise minimality of U_2 implies that $U_2 \subseteq U$. As $\delta(U)$ is a k-mate of $L_1 \triangle (R_1 \cup R_2)$, it follows that $\delta(U) \cap (L_2 \triangle (R_1 \cup R_2)) = \{\Omega\}$. In particular, $\delta(U) \cap (C_2 - R_2) = \emptyset$ and as $u, v \in U_2 \subseteq U$, we get that $V(C - R_2) \subseteq U$.

We claim that $s \in V(C_1 - R_1)$. For if not, similarly as above, $(C_2 - R_2) \cup \{\Omega\}$ also has a k-mate $\delta(W)$, where $W \subseteq V(G) - \{t\}$ and $U_2 \subseteq W$ and $V(R_2) \subseteq W$. Since $\delta(U \cup W)$ is contained in $\delta(U) \cup \delta(W)$, and $\delta(U), \delta(W)$ are k-mates for L_2, proposition 3.5 implies that $\delta(U \cup W)$ is also a k-mate for L_2. Hence, $\delta(U \cup W) \cap C_2 \neq \emptyset$ and so $V(C_2) \not\subseteq U \cup W$. However, $V(C_2 - R_2) \subseteq U$ and $V(R_2) \subseteq W$, and so $V(C_2) \subseteq U \cup W$, which is not the case.

Hence, $s \in V(C_1 - R_1)$. Let $\tilde{C}_1 = (C_1 - R_1) \cup R_2$ and $\tilde{C}_2 = (C_2 - R_2) \cup R_1$. Consider the (Ω, k)-packing

$$\tilde{\mathcal{L}} = (\tilde{L}_1 = \tilde{C}_1 \cup P_1, \tilde{L}_2 = \tilde{C}_2 \cup P_2, L_3, \ldots, L_k).$$

The minimality of the non-bipartite Ω-system $((G, \Sigma, \{s,t\}), \mathcal{L})$ implies that $\tilde{C}_1$ and $\tilde{C}_2$ are odd circuits, and since $V(\tilde{C}_1 \cup \tilde{C}_2) \subseteq U_3$ and $V(P_1 \cup P_2) \cap U_3 = \{s\}$, it follows that $\tilde{\mathcal{L}}$ is an (Ω, k)-packing. By proposition 6.1, for $i = 1, 2$, there is a k-mate $\delta(\tilde{U}_i)$ for $\tilde{L}_i$, where $\tilde{U}_i \subseteq V(G) - \{t\}$ is shore-wise minimal. Since $s \in V(\tilde{C}_1)$ and $u, v \in V(\tilde{C}_1) \cap V(\tilde{C}_2)$, it follows from proposition 6.1 that $\tilde{U}_1 \subset \tilde{U}_2 \subset U_3$. Hence, in particular, $V(\tilde{C}_1) \subseteq \tilde{U}_2$ and in turn $V(R_2) \subset \tilde{U}_2$, so R_2 is vertex-disjoint from C_3. Thus, similarly as above, $R_1 \cup \{\Omega\}$ has a k-mate $\delta(U'), U' \subseteq V(G) - \{t\}$.

Note that $\delta(U)$ is a k-mate of $\tilde{L}_1$ and $\delta(U')$ is a k-mate of $\tilde{L}_2$. Since $s \in V(C_1 - R_1)$ and $(C_1 - R_1) \cap \delta(U) = (C_1 - R_1) \cap \delta(U') = \emptyset$, we have $V(C_1 - R_1) \subseteq U \cap U'$ and in particular, $u, v \in U \cap U'$. Consider $\delta(U \cup U')$ which is contained in $\delta(U) \cup \delta(U')$. Since $R_1 \cap \delta(U) = \emptyset$, it follows that $R_1 \cap \delta(U \cup U') = \emptyset$, and so by proposition 3.4, $\delta(U \cup U')$ is a k-mate of L_2, implying that $R_2 \cap \delta(U \cup U') \neq \emptyset$, a contradiction as $R_2 \cap \delta(U') = \emptyset$. Hence, C_1 and C_2 have at most one vertex in common.

Suppose now that L_3 is non-simple. Notice first that by proposition 6.1 (2), $P_3 \cap \delta(U_3) = \{\Omega\}$, so $V(P_3) \cap U_3 = \{s\}$. Since $V(C_1) \subseteq U_2$ and $V(C_3) \cap U_2 = \emptyset$, it follows that C_1 and C_3 are vertex-disjoint. It remains to show that C_2 and C_3 have at most one vertex in common. Suppose otherwise. We will once again obtain a contradiction by proving that $((G, \Sigma, \{s,t\}), \mathcal{L})$ is not a minimal non-bipartite Ω-system. As we just showed, C_1 and C_2 have at most one vertex in common. Choose distinct vertices $u, v \in V(C_2) \cap V(C_3)$ and let R_2 be a uv-path contained in C_2 that is vertex-disjoint from C_1. Let R_3 be the uv-path contained in C_3 such that $R_2 \cup R_3$ is an even cycle. As before, the minimality of the Ω-system implies that the internal vertices of R_2 all have degree two in $G[L_1 \cup L_2 \cup (L_3 - R_3)]$ (recall that $V(P_3) \cap U_3 = \{s\}$). Lemma 5.2 therefore implies $R_3 \cup \{\Omega\}$ has a k-mate B. As B is also a k-mate of L_3, proposition 6.1 implies that B is an st-cut, so $B = \delta(U)$ for some $U \subseteq V(G) - \{t\}$. Then $\delta(U_3 \cap U)$ is a cover contained in $B_3 \cup B$, and so by proposition 3.5, it is k-mate of L_3. Thus the shore-wise minimality of U_3 implies that $U_3 \subseteq U$.

We claim $C_2 - R_2$ has a vertex in common with C_1. For if not, similarly as above, $(C_3 - R_3) \cup \{\Omega\}$ also has a k-mate $\delta(W)$, where $W \subseteq V(G) - \{t\}$ and $U_3 \subseteq W$. Since $\delta(U \cup W)$ is contained in $\delta(U) \cup \delta(W)$, and $\delta(U), \delta(W)$ are k-mates for L_3, proposition 3.5 implies that $\delta(U \cup W)$ is also a k-mate for L_3. Hence, $\delta(U \cup W) \cap C_3 \neq \emptyset$ and so $V(C_3) \not\subseteq U \cup W$. However, $u, v \in U_3 \subseteq U \cup W$, forcing $V(R_3) \subseteq W$ and $V(C_3 - R_3) \subseteq U$, and so $V(C_3) \subseteq U \cup W$, which is not the case.

Hence, $C_2 - R_2$ has a vertex in common with C_1. Let $\tilde{C}_2 = (C_2 - R_2) \cup R_3$ and $\tilde{C}_3 = (C_3 - R_3) \cup R_2$. Consider the (Ω, k)-packing

$$\tilde{\mathcal{L}} = (L_1, \tilde{L}_2 = \tilde{C}_2 \cup P_2, \tilde{L}_3 = \tilde{C}_3 \cup P_3, L_4, \ldots, L_k).$$

The minimality of the non-bipartite Ω-system $((G, \Sigma, \{s,t\}), \mathcal{L})$ implies that $\tilde{\mathcal{L}}$ is an (Ω, k)-packing. By proposition 6.1, for $i = 2, 3$, there is a k-mate $\delta(\tilde{U}_i)$ for $\tilde{L}_i$, where $\tilde{U}_i \subseteq V(G) - \{t\}$ is shore-wise minimal. Since $\tilde{C}_2$ has vertices in common with the both of $C_1, \tilde{C}_3$, it follows from proposition 6.1 that either $U_1 \subset \tilde{U}_2 \subset \tilde{U}_3$ or $\tilde{U}_3 \subset \tilde{U}_2 \subset U_1$. Hence, in particular, $V(R_3) \subset U_1 \cup \tilde{U}_3$ and so the internal vertices of R_3 have degree two in $G[L_1 \cup (L_2 - R_2) \cup L_3]$. Thus, similarly as above, $R_2 \cup \{\Omega\}$ has a k-mate $\delta(U'), U' \subseteq V(G) - \{t\}$. Note $\delta(U_2 \cap U')$ is a cover contained in $\delta(U_2) \cup \delta(U')$, and so by proposition 3.5, it is a k-mate of L_2. Thus the shore-wise minimality of U_2 implies that $U_2 \subseteq U'$.

Note that $\delta(U)$ is a k-mate of L_3 and $\delta(U')$ is a k-mate of L_2. Since $C_2 - R_2$ has a vertex x in common with C_1, $(C_2 - R_2) \cap \delta(U) = (C_2 - R_2) \cap \delta(U') = \emptyset$, and $x \in U_2 \subset U \cap U'$, we must have $V(C_2 - R_2) \subseteq U \cap U'$ and in particular, $u, v \in U \cap U'$. Consider $\delta(U \cup U')$ which is contained in $\delta(U) \cup \delta(U')$. Since $R_2 \cap \delta(U') = \emptyset$, it follows that $R_2 \cap \delta(U \cup U') = \emptyset$, and so by proposition 3.4, $\delta(U \cup U')$ is a k-mate of L_3, implying that $R_3 \cap \delta(U \cup U') \neq \emptyset$, a contradiction as $R_3 \cap \delta(U) = \emptyset$. Hence, C_2 and C_3 have at most one vertex in common, thereby finishing the proof.

(2) Suppose that L_3 is simple. It is clear that C_1 and L_3 have at most one vertex (in particular, s) in common. We will show that C_2 and L_3 have at most one vertex in common. Suppose otherwise. Choose distinct $u, v \in V(C_2) \cap V(L_3)$, and let R_3 be the uv-path contained in L_3. Let R_2 be the uv-path contained in C_2 such that $R_2 \cup R_3$ is an even cycle.

We claim that R_2 is vertex-disjoint from C_1. Let $\tilde{C}_2 := (C_2 - R_2) \cup R_3$ and $\tilde{L}_3 := (L_3 - R_3) \cup R_2$. The minimality of our non-bipartite Ω-system implies $\tilde{L}_3$ is still simple. Consider the (Ω, k)-packing

$$\tilde{\mathcal{L}} := (L_1, \tilde{L}_2 = \tilde{C}_2 \cup P_2, \tilde{L}_3, L_4, \ldots, L_k).$$

The minimality of the non-bipartite Ω-system $((G, \Sigma, \{s,t\}), \mathcal{L})$ implies that $\tilde{\mathcal{L}}$ is an (Ω, k)-packing. By proposition 6.1, for $i = 2, 3$, there exists a k-mate $\delta(\tilde{U}_i)$ of $\tilde{L}_i$, where $\tilde{U}_i \subseteq V(G) - \{t\}$ is shore-wise minimal, and $U_1 \subset \tilde{U}_2 \subset \tilde{U}_3$. In particular, $V(R_2) \cap \tilde{U}_2 = \emptyset$ and $V(C_1) \subseteq \tilde{U}_2$, so R_2 is vertex-disjoint from C_1.

As a result, the internal vertices of R_2 all have degree two in $G[L_1 \cup L_2 \cup (L_3 - R_3)]$. Thus lemma 5.2 implies that $R_3 \cup \{\Omega\}$ has a k-mate B. As B is also a k-mate of L_3, proposition 6.1 implies that $B = \delta(U)$ for some $U \subseteq V(G) - \{t\}$. However, since $\delta(U) \cap C_2 = \emptyset$ and $u, v \notin U$, it follows that $V(C_2) \cap U = \emptyset$. Consider $\delta(U_2 \cap U)$, which is contained in $\delta(U_2) \cup \delta(U)$. Since $\delta(U_2)$ is a k-mate of L_2, $\delta(U)$ is a k-mate of L_3, and $C_2 \cap \delta(U_2 \cap U) = \emptyset$, it follows from proposition 3.4 that $\delta(U_2 \cap U) \cap L_3 \neq \emptyset$, implying in turn that $\delta(U_2)\cap L_3\neq\{\Omega\}$, a contradiction. Thus, C_2 and L_3 have at most one vertex in common.

(3) Among all non-bipartite Ω-systems with the same associated signed graft, the number of non-simple minimal odd st-joins among L_1,L_2,L_3 is maximum. Hence, L must be a simple minimal odd st-join, and the minimality of the Ω-system implies that $P:=L\triangle P_2\triangle L_3$ is an even st-path. Consider the (Ω,k)-packing

$$\tilde{\mathcal{L}}=(L_1,\tilde{L}_2:=C_2\cup P,\tilde{L}_3:=L,L_4,\dots,L_k).$$

The minimality of the non-bipartite Ω-system $((G,\Sigma,\{s,t\}),\mathcal{L})$ implies that $\tilde{\mathcal{L}}$ is an (Ω,k)-packing. By proposition 6.1, for $i=2,3$, there exists a k-mate $\delta(\tilde{U}_i)$ of $\tilde{L}_i$ where $\tilde{U}_i$ is shore-wise minimal, and $\tilde{U}_2\subset\tilde{U}_3$. We claim that $U_3=\tilde{U}_3$, thereby finishing the proof of (3). Let $B:=\delta(U_3\cap\tilde{U}_3)$. Since $L_3,\tilde{L}_3$ are simple, $\delta(U_3)\cap(\tilde{L}_3-L_3)=\emptyset$ and $\delta(\tilde{U}_3)\cap(L_3-\tilde{L}_3)=\emptyset$, it follows that $B\cap(\tilde{L}_3-L_3)=B\cap(L_3-\tilde{L}_3)=\emptyset$. Therefore, proposition 3.3 implies that B is a k-mate for the both of L_3 and $\tilde{L}_3$, and so the shore-wise minimality of $U_3,\tilde{U}_3$ implies that $U_3\subset U_3\cap\tilde{U}_3$ and $\tilde{U}_3\subset U_3\cap\tilde{U}_3$. Hence, $U_3=\tilde{U}_3$, as claimed.

(4) Suppose otherwise. Assume first that $j=1$. Observe that $\delta(U)\subseteq\delta(U_1)$. Since $\delta(U_1-U)=\delta(U_1)\triangle\delta(U)$, it follows that $\delta(U_1-U)\subseteq\delta(U_1)$, implying in turn that $\delta(U_1-U)$ is also a k-mate of L_1, contradicting the shore-wise minimality of U_1. Assume next that $j\neq 1$. Observe that $\delta(U)\subseteq\delta(U_{j-1})\cup\delta(U_j)$ and $\delta(U)\cap L_{j-1}=\emptyset$. However, since $\delta(U_j-U)=\delta(U_j)\triangle\delta(U)$ and $\delta(U_j)\cap L_{j-1}=\{\Omega\}$,

$$\delta(U_j-U)\subseteq\delta(U_{j-1})\cup\delta(U_j)\quad\text{and}\quad\delta(U_j-U)\cap L_{j-1}=\{\Omega\}.$$

Hence, proposition 3.4 implies that $\delta(U_j-U)$ is a k-mate of L_j, contradicting the shore-wise minimality of U_j.

(5) By proposition 6.1 (2), $P_3\cap\delta(U_3)=\{\Omega\}$, so $V(P_3)\cap U_3=\{s\}$. Suppose for a contradiction that (5) does not hold. Then there is a subset $U\subset\overline{U_3}$ containing t such that $U\cap V(C_3)=\emptyset$, and such that there is no edge of $G[\overline{U_3}]$ with one end in U and one end not in U. Let $\overline{U}=V(G)-U$. Then $\delta(\overline{U})\subset\delta(U_3)$ and so $|\delta(\overline{U})-L_3|\leq k-3$. However, $\delta(\overline{U})\cap L_3=\{\Omega\}$, and so $|\delta(\overline{U})|\leq k-2$, a contradiction as $k\leq\tau(G,\Sigma)$. □

We are now ready to prove proposition 2.9.

Proof of proposition 2.9. Let $((G,\Sigma,\{s,t\}),\mathcal{L}=(L_1,\dots,L_k))$ be a minimal non-bipartite Ω-system of flavour (NF1), where Ω has ends s,s'. Recall that at least two, say L_1 and L_2, of L_1,L_2,L_3 are non-simple, and $\Omega\in P_1\cap P_2\cap P_3$.

By proposition 6.1, for each $i \in [3]$, there exists a k-mate $B_i = \delta(U_i)$ where $U_i \subseteq V(G) - \{t\}$ is shore-wise minimal, and we may assume $U_1 \subset U_2 \subset U_3$. Moreover, for $i \in [3]$, if L_i is non-simple then $B_i \cap P_i = \{\Omega\}$ and $B_i \cap C_i \neq \emptyset$. Let $U_0 = \emptyset$.

In the first case, assume that L_3 is non-simple. Let $U_4 := V(G)$ and let C_0 (resp. C_4) be the path of single vertex s (resp. t). Then by lemma 6.2,

(a) for $j \in [4]$, there exists a shortest path Q_j in $G[U_j - U_{j-1}]$ between $V(C_{j-1})$ and $V(C_j)$, and
(b) for $j \in [2]$, C_j and C_{j+1} have at most one vertex in common.

Moreover, let P_3' be the shortest path contained in P_3 connecting s' to $V(C_3 \cup Q_4) - U_3$. It is now clear that $C_1 \cup C_2 \cup C_3 \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup P_3'$ has an F_7 minor.

In the remaining case, L_3 is simple. As above, lemma 6.2 implies

(a') for $j \in [2]$, there exists a shortest path Q_j in $G[U_j - U_{j-1}]$ between $V(C_{j-1})$ and $V(C_j)$,
(b') there exists a shortest path Q_3 in $G[U_3 - U_2]$ between $V(C_2)$ and $V(L_3)$,
(c') C_1 and C_2 have at most one vertex in common, and C_2 and L_3 have at most one vertex in common, and
(d') if P_2 and L_3 share a vertex w other than s, s', t, then either (a) $V(L_3[s', w]) \subseteq V(G) - U_3$ and $L_3[s', w] \cup P_2[s', w]$ is an even cycle, or (b) $V(L_3[w, t]) \subseteq V(G) - U_3$ and $L_3[w, t] \cup P_2[w, t]$ is an even cycle.

It is now clear that $C_1 \cup C_2 \cup L_3 \cup Q_1 \cup Q_2 \cup Q_3 \cup P_2$ has an F_7 minor. □

7. Non-bipartite Ω-system of flavour (NF2)

In this section we prove proposition 2.10, namely that a minimal non-bipartite Ω-system of flavour (NF2) has an F_7 minor, as long as there is no non-bipartite Ω-system of flavour (NF1) with the same associated signed graft. Observe that L_1, L_2 and L_3 are connected. For convenience, whenever L_i is non-simple, we write $P_i := P(L_i)$ and $C_i := C(L_i)$.

Proposition 7.1. *Let $(G, \Sigma, \{x, y\})$ be a non-bipartite signed graft whose edges can be partitioned into odd xy-paths Q_1, Q_2. For each $i = 1, 2$, direct the edges of Q_i from x to y, and assume that every directed circuit in $Q_1 \cup Q_2$ is even. Let $\vec{H}$ be the directed signed graft obtained by contracting all arcs that belong to at least one directed circuit. Then $\vec{H}$ is a non-bipartite and acyclic directed signed graft whose edges can be partitioned into two odd xy-dipaths.*

Proof. Let A be the set of all arcs that belong to at least one directed circuit. It is clear by construction that $\vec{H}$ is acyclic and can be partitioned as the union of two xy-dipaths Q_1', Q_2' where for $i = 1, 2$, $Q_i' = Q_i - A$ (Q_i' is equal to Q_i/A). Since every directed circuit is even, it follows that Q_1', Q_2' are odd xy-dipaths. To show $\vec{H}$ is non-bipartite, let C be an odd circuit of $Q_1 \cup Q_2$. Clearly, $C - A$ is a cycle of $\vec{H}$, and again, since every directed circuit is even, it follows that $C - A$ is an odd cycle of $\vec{H}$. In particular, $\vec{H}$ is non-bipartite. $\square$

Proposition 7.2. *Let $((G,\Sigma,\{s,t\}),\mathcal{L}=(L_1,\ldots,L_k))$ be a non-bipartite Ω-system of flavour (NF2), where Ω has ends s,s'. For $i\in[3]$, let B_i be a k-mate of L_i. Then,*

(1) exactly one of B_1,B_2,B_3, say B_3, is an st-cut,

(2) L_1 and L_2 are simple,

(3) $(L_1\cup L_2)-\{\Omega\}$ is non-bipartite and $(L_1\cup L_3)-\{\Omega\},(L_2\cup L_3)-\{\Omega\}$ are bipartite.

Furthermore, choose $U\subseteq V(G)-\{s,t\}$ such that $B_1\triangle B_2=\delta(U)$. Then,

(4) for every $L\subseteq L_1\cup L_2\cup L_3$, $(L\cap B_1)-\{\Omega\}=(L\cap L_1)\cap\delta(U)$,

(5) L_1 and L_2 have at least one vertex of U in common.

Proof. **(1)** Proposition 3.6 implies that at least two of B_1,B_2,B_3 are signatures. Suppose for a contradiction that each of B_1,B_2,B_3 is a signature. For $i\in[3]$, note that $L_i-\{\Omega\}$ is an $s't$-path (recall that if $C(L_i)\neq\emptyset$, then $\Omega\in C(L_i)$ and the only vertex common to $C(L_i),P(L_i)$ is s), so let $Q_i:=L_i-\{\Omega\}$. Since

$$B_1\cap(Q_2\cup Q_3)=B_2\cap(Q_3\cup Q_1)=B_3\cap(Q_1\cup Q_2)=\emptyset$$

and B_1,B_2,B_3 are signatures, it follows that $Q_1\cup Q_2, Q_2\cup Q_3$ and $Q_3\cup Q_1$ are bipartite. Thus by proposition 5.1, $Q_1\cup Q_2\cup Q_3=(L_1\cup L_2\cup L_3)-\{\Omega\}$ is bipartite, a contradiction. **(2)** Suppose, for $j\in[3]$, L_j is non-simple. Then $\Omega\in C_j$ and so by proposition 3.8, the covers in $\{B_1,B_2,B_3\}-\{B_j\}$ are signatures, and so by (1), $j=3$. **(3)** Since B_1 and B_2 are signatures, it follows that $Q_2\cup Q_3$ and $Q_1\cup Q_3$ are bipartite. Then by proposition 5.1, $Q_1\cup Q_2$ must be non-bipartite. **(4)** By proposition 3.4, $B_1\subseteq L_1\cup L_4\cup\ldots\cup L_k$. Thus, $L\cap B_1\subseteq L_1\cap B_1$, and so $L\cap B_1=L\cap(L_1\cap B_1)$. Hence, it suffices to show that $(L_1\cap B_1)-\{\Omega\}=L_1\cap\delta(U)$. Again, by proposition 3.4, $L_1\cap B_2=\{\Omega\}$ and $\Omega\in L_1\cap B_1$, so

$$L_1\cap\delta(U)=L_1\cap(B_1\triangle B_2)=(L_1\cap B_1)\triangle(L_1\cap B_2)=(L_1\cap B_1)-\{\Omega\},$$

as required. **(5)** By (3) $(L_1\cup L_2)-\{\Omega\}$ contains an odd circuit C. Since B_1 is a signature, $|B_1\cap C|$ is odd. By (4) $C\cap B_1=(C\cap L_1)\cap\delta(U)$. Decompose $C\cap L_1$ into pairwise vertex-disjoint paths $Q_1,\ldots,Q_\ell$. Then, for some $i\in[\ell]$, $|Q_i\cap\delta(U)|$ is odd, and so Q_i has one end, say y, in U and the other in $V(G)-U$. Since $y\in V(L_1)\cap V(L_2)$, the result follows. $\square$

Proposition 7.3. *Let $((G,\Sigma,\{s,t\}),\mathcal{L}=(L_1,\ldots,L_k))$ be a non-bipartite Ω-system of flavour (NF2), where $\Omega\in\delta(s)$. Suppose there exist C_1',P_1',L_2' and L_3' such that*

(1) $C_1'\cup P_1'\cup L_2'\cup L_3'\subseteq L_1\cup L_2\cup L_3$,

(2) C_1' is an odd cycle, P_1' is an even st-join, and L_2', L_3' are odd st-joins,

(3) $\Omega \in P_1' \cap L_2' \cap L_3'$ and $\Omega \notin C_1'$,

(4) the four sets C_1', P_1', L_2', L_3' are pairwise Ω-disjoint.

Let $L_1' := C_1' \cup P_1'$, and for each $j \in [3]$, let $\tilde{L}_j$ be a minimal odd st-join contained in L_j'. Then $((G, \Sigma, \{s,t\}), \tilde{\mathcal{L}} = (\tilde{L}_1, \tilde{L}_2, \tilde{L}_3, L_4, \ldots, L_k))$ is a non-bipartite Ω-system of flavour (NF1).

Proof. We will first show that $\Omega \in \tilde{L}_1 \cap \tilde{L}_2 \cap \tilde{L}_3$. For $j \in [3]$, let $\tilde{B}_j$ be a k-mate of $\tilde{L}_j$. By proposition 3.2, for $j \in [3]$, $\tilde{B}_j \subseteq \tilde{L}_j \cup L_4 \cup \ldots \cup L_k$. Hence, for distinct $i, j \in [3]$, $\tilde{L}_i \cap \tilde{B}_j \subseteq \{\Omega\}$ and so $\tilde{L}_i \cap \tilde{B}_j = \{\Omega\}$, implying that $\Omega \in \tilde{L}_1 \cap \tilde{L}_2 \cap \tilde{L}_3$.

As $C_1' \cap (\tilde{L}_2 \cup \tilde{L}_3 \cup L_4 \cup \cdots \cup L_k) = \emptyset$, we have $\tilde{B}_2 \cap C_1' = \tilde{B}_3 \cap C_1' = \emptyset$. Since C_1' is an odd cycle, $\tilde{B}_2, \tilde{B}_3$ are st-cuts. So by proposition 3.6 one of $\tilde{L}_2, \tilde{L}_3$, say $\tilde{L}_2$, is non-simple and $\Omega \in P(\tilde{L}_2)$. Hence, $((G, \Sigma, \{s,t\}), \tilde{\mathcal{L}})$ is a non-bipartite Ω-system of flavour (NF1) (because it is not of flavour (NF2)). $\square$

Lemma 7.4. *Let $((G, \Sigma, \{s,t\}), \mathcal{L} = (L_1, \ldots, L_k))$ be a minimal non-bipartite Ω-system of flavour (NF2), where Ω has ends s, s', and assume there is no non-bipartite Ω-system of flavour (NF1) with the same associated signed graft. Suppose that L_1, L_2 are simple and $(L_1 \cup L_2) - \{\Omega\}$ is non-bipartite. Then the following hold:*

(1) For $i = 1, 2$, the only vertices L_i and L_3 have in common are s, s', t.

(2) For $i = 1, 2$, direct the edges of L_i from s to t. Then every directed circuit in $L_1 \cup L_2$ is even.

Proof. For $i \in [3]$, let B_i be a k-mate of L_i. Since $(L_1 \cup L_2) - \{\Omega\}$ is non-bipartite, proposition 7.2 implies that for $i = 1, 2$, $(L_i \cup L_3) - \{\Omega\}$ is bipartite, B_3 is an st-cut and B_1, B_2 are signatures. Thus there exists $U \subseteq V(G) - \{s,t\}$ such that $B_1 \triangle B_2 = \delta(U)$. By proposition 7.2, L_1 and L_2 have a vertex y in common in U, and the two cycles $L_1[s', y] \cup L_2[s', y]$, $L_1[y, t] \cup L_2[y, t]$ are odd.

(1) In the first case, assume L_3 is simple. Suppose for a contradiction that L_3 has a vertex other than s, s', t in common with one of L_1, L_2. Let v_1 (resp. v_2) be the closest vertex to s (resp. furthest vertex from s) of L_3 different from s, s', t that also belongs to one of L_1, L_2. We may assume that $v_2 \in V(L_2) \cap V(L_3)$, and choose $j \in \{1, 2\}$ so that $v_1 \in V(L_j) \cap V(L_3)$.

Claim 1. *There exists an odd cycle C in $(L_1 \cup L_2) - \{\Omega\}$ that is disjoint from either $L_j[s', v_1]$ or $L_2[v_2, t]$.*

Proof. Suppose otherwise. Then $j=1$ and y must belong to the interior of the both of $L_1[s',v_1], L_2[v_2,t]$. Let

$$P_1' = L_1[s,y] \cup L_2[y,t]$$
$$C_1' = L_1[y,v_1] \cup L_3[v_1,v_2] \cup L_2[v_2,y]$$
$$L_1' = C_1' \cup P_1'$$
$$L_2' = L_3[s,v_1] \cup L_1[v_1,t]$$
$$L_3' = L_2[s,v_2] \cup L_2[v_2,t].$$

By proposition 7.2, P_1' is an even st-join, C_1' is an odd cycle, and for $j \in [3]$, L_j' is an odd st-join. Therefore, for $j \in [3]$, there is a minimal odd st-join $\tilde{L}_j$ contained in L_j'. Proposition 7.3 implies that $((G,\Sigma,\{s,t\}),(\tilde{L}_1,\tilde{L}_2,\tilde{L}_3,L_4,\ldots,L_k))$ is a non-bipartite Ω-system of flavour (NF1), contrary to our hypothesis. $\diamond$

Observe that $L_3[s',v_1]$ and $L_3[v_2,t]$ are paths whose internal vertices by definition have degree two in $G[L_1 \cup L_2 \cup L_3]$, and the two cycles $L_3[s',v_1] \cup L_j[s',v_1]$, $L_3[v_2,t] \cup L_2[v_2,t]$ are even. Lemma 5.2 implies that either $L_j[s',v_1] \cup \{\Omega\}$ or $L_2[v_2,t] \cup \{\Omega\}$ has a k-mate B. Since $B \cap C = \emptyset$, it follows that B is an st-cut. However, B is also a k-mate for one of L_j, L_2. Hence, since B_3 is also an st-cut, proposition 3.6 implies that one of L_j, L_2, L_3 is non-simple and Ω lies in its even st-path, a contradiction.

In the remaining case, L_3 is non-simple and $\Omega \in C_3$. We will first show that C_3 has no vertex other than s, s' in common with either of L_1, L_2. Suppose otherwise. Choose a vertex $v \in V(C_3) - \{s,s'\}$ that also belongs to one of L_1, L_2, and such that all the internal vertices of the subpath $C_3[s',v]$ in $C_3 - \{\Omega\}$ have degree two in $G[L_1 \cup L_2 \cup L_3]$. Let $C_3[s,v] := \{\Omega\} \cup C_3[s',v]$ and $C_3[v,s] := C_3 - C_3[s,v]$. By symmetry between L_1 and L_2, we may assume that $v \in V(L_1) \cap V(C_3)$.

Claim 2. *There exists an odd cycle C in $(L_1 \cup L_2) - \{\Omega\}$ that is disjoint from $L_1[s',v]$.*

Proof. Suppose otherwise. Then y must belong to the interior of $L_1[s',v]$. Let

$$C_3' = L_1[s,v] \cup C_3[v,s]$$
$$L_3' = C_3' \cup P_3$$
$$L_1' = C_1[s,v] \cup L_1[v,t]$$
$$C' = L_1[s',y] \cup L_2[s',y].$$

By proposition 7.2, C_3', C' are odd cycles and L_1', L_3' are odd st-joins. Therefore, L_1' has a k-mate B. Since $L_1' \cap C' = \emptyset$, it follows that $B \cap C' = \emptyset$ and so B is an st-cut. However, $B \cap L_3' = \{\Omega\}$, implying that $B \cap C_3' = \{\Omega\}$, a contradiction. ◇

Recall that $C_3[s', v]$ is a path whose internal vertices have degree two in $G[L_1 \cup L_2 \cup L_3]$, and the cycle $C_3[s', v] \cup L_1[s', v]$ is even. Lemma 5.2 therefore implies that $L_1[s, v] = L_1[s', v] \cup \{\Omega\}$ has a k-mate B. Since $B \cap C = \emptyset$, it follows that B is an st-cut. However, $B \cap (L_1[s, v] \cup C_3[v, s]) = \{\Omega\}$, a contradiction (as $L_1[s, v] \cup C_3[v, s]$ is an odd cycle).

We next show that P_3 has no vertex other than s, t in common with either of L_1, L_2. Suppose otherwise. Choose a vertex $v \in V(P_3) - \{s, t\}$ that also belongs to one of L_1, L_2, and such that all the internal vertices of the subpath $P_3[v, t]$ have degree two in $G[L_1 \cup L_2 \cup L_3]$. By symmetry between L_1 and L_2, we may assume that $v \in V(L_1) \cap V(P_3)$.

Claim 3. *There exists an odd cycle C in $(L_1 \cup L_2) - \{\Omega\}$ that is disjoint from $L_1[v, t]$.*

Proof. Suppose otherwise. Then y must belong to the interior of $L_1[v, t]$. Let

$$L_1' = L_1[s, v] \cup P_3[v, t]$$
$$C' = L_1[y, t] \cup L_2[y, t].$$

By proposition 7.2, C' is an odd cycle, and L_1' is an odd st-join. Therefore, L_1' has a k-mate B. Since $L_1' \cap C' = \emptyset$, it follows that $B \cap C' = \emptyset$ and so B is an st-cut. However, $B \cap C_3 = \{\Omega\}$, a contradiction. ◇

Recall that $P_3[v, t]$ is a path whose internal vertices have degree two in $G[L_1 \cup L_2 \cup L_3]$, and the cycle $P_3[v, t] \cup L_1[v, t]$ is even. Lemma 5.2 therefore implies that $L_1[v, t] \cup \{\Omega\} = L_1 - L_1[s', v]$ has a k-mate B. Since $B \cap C = \emptyset$, it follows that B is an st-cut. However, $B \cap C_3 = \{\Omega\}$, a contradiction.

(2) Suppose otherwise. Let C be a directed odd circuit contained in $L_1 \cup L_2$, and let $P_1' \cup P_2'$ be two st-joins in $(L_1 \cup L_2) - C$ such that $P_1' \cup P_2' = (L_1 \cup L_2) - C$ and $P_1' \cap P_2' = \{\Omega\}$. Then one of P_1', P_2' is odd and the other is even, say P_1' is even and P_2' is odd. Let $L_1' := C \cup P_1'$, $L_2' := P_2'$ and $L_3' := L_3$. For $j \in [3]$, let $\tilde{L}_j$ be a minimal odd st-join contained in L_i'. Then proposition 7.3 implies that $((G, \Sigma, \{s, t\}), (\tilde{L}_1, \tilde{L}_2, \tilde{L}_3, L_4, \ldots, L_k))$ is a non-bipartite Ω-system of flavour (NF1), contrary to our hypothesis. □

We are now ready to prove proposition 2.10.

Proof of proposition 2.10. Let $((G, \Sigma, \{s, t\}), \mathcal{L} = (L_1, \ldots, L_k))$ be a minimal non-bipartite Ω-system of flavour (NF2), where Ω has ends s, s', and assume there is no non-bipartite Ω-system of flavour

(NF1) with the same associated signed graft. Proposition 7.2 allows us to assume L_1, L_2 are simple and $(L_1 \cup L_2) - \{\Omega\}$ is non-bipartite, and in turn, lemma 7.4 implies that, for $i = 1, 2$, the only vertices L_i and L_3 have in common are s, s', t. For $i \in \{2, 3\}$, let B_i be a k-mate of L_i. By proposition 7.2, L_3 is an st-cut $\delta(U), U \subseteq V(G) - \{t\}$.

If L_3 is non-simple, then it is easily follows from proposition 7.1 and lemma 7.4 that $L_1 \cup L_2 \cup L_3$ has an F_7 minor. Otherwise, when L_3 is simple, proposition 5.4 implies the existence of a shortest path P in $G[U]$ between s and some vertex, say v, of $(V(L_3) \cap U) - \{s\}$ that is disjoint from B_2. Note that $L_3[s, v] \cup P$ is an odd cycle. It now easily follows from proposition 7.1 and lemma 7.4 that $L_1 \cup L_2 \cup L_3 \cup P$ has an F_7 minor. □

8. Preliminaries for bipartite Ω-systems

8.1. **Basic properties.**

Remark 8.1. *Let $((G, \Sigma, T), (L_1, \ldots, L_k), m)$ be a bipartite Ω-system, where L_1, L_2, L_3 are minimal odd T-joins. Since $(L_1 \cup L_2 \cup L_3) - \{\Omega\}$ is bipartite, for each $i \in [3]$, either L_i is simple or $\Omega \in C(L_i)$.*

Proposition 8.2. *Let $((G, \Sigma, T), (L_1, \ldots, L_k), m)$ be a bipartite Ω-system, where L_1, L_2, L_3 are minimal odd T-joins. For $i \in [3]$, let B_i be a k-mate of L_i. Then at least two of B_1, B_2, B_3 are signatures.*

Proof. By remark 8.1, for every $i \in [3]$, L_i is either simple or $\Omega \in C(L_i)$. The result now follows immediately from proposition 3.6. □

Proposition 8.3. *Let $((G, \Sigma, T), (L_1, \ldots, L_k), m)$ be a bipartite Ω-system. Suppose $L \subseteq L_1 \cup L_2 \cup L_3 \cup P(L_4) \cup \cdots \cup P(L_m)$ has a signature k-mate B. Then $B \cap \big(L_1 \cup L_2 \cup L_3 \cup P_4 \cup \cdots \cup P_m\big) = B \cap L$.*

Proof. As B is a signature, it intersects each of $C_4, \ldots, C_m, L_{m+1}, \ldots, L_k$. Hence,

$$k - 3 \geq |B - L| \geq \sum_{j=4}^{m} |B \cap C_j| + \sum_{j=m+1}^{k} |B \cap L_j| \geq k - 3,$$

so equality holds throughout, implying that $B - L \subseteq C_4 \cup \cdots \cup C_m \cup L_{m+1} \cup \cdots \cup L_k$, implying the result. □

8.2. **The mate proposition.**

Proposition 8.4. *Let $((G, \Sigma, \{s, t\}), \mathcal{L} = (L_1, \ldots, L_k), m)$ be a bipartite Ω-system, where $\Omega \in \delta(s)$. For each $i \in [m]$, let $\widetilde{P}_i \subseteq L_i$ be a connected st-join such that $\widetilde{P}_i \cap \Sigma \subseteq \{\Omega\}$, and if $\Omega \in \widetilde{P}_i$, then $\widetilde{P}_i \cap \delta(s) = \{\Omega\}$. Suppose, for each $i \in [m]$, there exists a k-mate B_i of $\widetilde{P}_i \cup \{\Omega\}$. Then one of $B_1, \ldots, B_m$ is not a signature.*

To prove this proposition, we will need a lemma, for which we introduce some notations. For $i \in [m]$, let $Q_i := \widetilde{P_i} \cup \{\Omega\}$. Given two signatures B_i, B_ℓ, we choose $U_{i\ell} \subseteq V(G) - \{s,t\}$ such that $\delta(U_{i\ell}) = B_i \triangle B_\ell$. For each $i \in [m]$, define $\widetilde{C_i}$ as follows: if $\widetilde{P_i}$ is odd then $\widetilde{C_i} := \emptyset$, and otherwise $\widetilde{C_i}$ is an odd circuit contained in the odd cycle $L_i \triangle \widetilde{P_i} = L_i - \widetilde{P_i}$.

Lemma 8.5. *Let $((G, \Sigma, \{s,t\}), \mathcal{L} = (L_1, \ldots, L_k), m)$ be a bipartite Ω-system, where G is connected and Ω has ends s, s'. Let $J \subseteq [m]$ be an index subset of size at least three. Suppose, for each $i \in J$, there exists a signature k-mate B_i for Q_i. Then, for each $i \in J$, the following hold:*

(1) B_i is a k-mate of L_i, and so B_i is a cap of L_i in $\mathcal{L}$,
(2) for $\ell \in [m]$ such that $\widetilde{C_\ell} \neq \emptyset$, $|B_i \cap \widetilde{C_\ell}| = 1$,
(3) for $\ell \in [m] - \{i\}$, $B_i \cap Q_\ell = \{\Omega\}$.

Now pick $j \in J$ and let $S := \bigcap(U_{ij} : i \in J, i < j)$. Then,

(4) $\Omega \notin \delta(S)$,
(5) $\delta(S) \subseteq \bigcup(B_i : i \in J, i \leq j)$,
(6) for distinct $i, \ell \in J - \{j\}$, $S \cap U_{i\ell} = \emptyset$,
(7) $Q_j \cap \delta(S) = (Q_j \cap B_j) - \{\Omega\}$,
(8) for $\ell \in [m] - \{j\}$, $Q_\ell \cap \delta(S) = \emptyset$.

Next take $L \in \{L_{m+1}, \ldots, L_k\}$ and $C \in \{\widetilde{C_1}, \ldots, \widetilde{C_m}\}$. Then,

(9) if $L \cap \delta(S) \neq \emptyset$, then $|L \cap \delta(S)| = 2$ and $|L \cap \delta(S) \cap B_j| = 1$,
(10) if $C \cap \delta(S) \neq \emptyset$, then $|C \cap \delta(S)| = 2$,
(11) if $C \cap \delta(S) \neq \emptyset$ and, for some $i, \ell \in J$ such that $i < \ell < j$, $C \cap \delta(S) \subseteq B_i \cup B_\ell$, then $V(C) \subseteq U_{ij} \cup U_{\ell j}$.

Proof. **(1)** If $i \in J \cap [3]$, then $Q_i \subseteq L_i$, and so B_i is clearly a k-mate of L_i. Otherwise, when $i \in J - [3]$, $B_i \cap \widetilde{C_i} \neq \emptyset$ as B_i is a signature, and so

$$|B_i - L_i| \leq |B_i - \widetilde{P_i}| - |B_i \cap \widetilde{C_i}| \leq (k-2) - 1 = k - 3,$$

implying that B_i is a k-mate of L_i. Hence, by proposition 3.1, B_i is a cap of L_i in $\mathcal{L}$. **(2)** Thus, if $\ell \neq i$ then $|B_i \cap \widetilde{C_\ell}| = 1$ (note B_i is a signature and $\widetilde{C_\ell}$ is an odd circuit). If $\ell = i$ and $i \notin [3]$, we have

$$k - 3 \leq |B_i \cap \widetilde{C_4}| + \cdots + |B_i \cap \widetilde{C_m}| + |B_i \cap L_{m+1}| + \cdots + |B_i \cap L_k| \leq |B_i - Q_i| \leq k - 3,$$

so equality holds throughout, in particular, $|B_i \cap \widetilde{C_i}| = 1$. Otherwise, when $\ell = i$ and $i \in [3]$, then

$$k - 3 \leq |B_i \cap L_4| + \cdots + |B_i \cap L_k| \leq |B_i - Q_i| \leq k - 3,$$

so equality holds throughout, in particular, the middle equality implies that $B_i \cap \widetilde{C_i} = \{\Omega\}$.

(3) Note $|B_i \cap L_\ell| = 1$. If $\ell \in [3]$, then $B_i \cap L_\ell = \{\Omega\}$ and so $B_i \cap Q_\ell = \{\Omega\}$. Otherwise, $\ell \in [m] - [3]$. By (2), $|B_i \cap \widetilde{C_\ell}| = 1$ and so $B_i \cap P_\ell = \emptyset$, implying that $B_i \cap Q_\ell = \{\Omega\}$.

(4) Note $\Omega \in B_i, i \in J$. In particular, for all $i \in J$ such that $i < j$, $\Omega \notin \delta(U_{ij})$ and so $s' \notin U_{ij}$. Thus $s' \notin S$, and since $s \notin S$, it follows that $\Omega \notin \delta(S)$.

(5) We have

$$\delta(S) \subseteq \bigcup(\delta(U_{ij}) : i \in J, i < j) \subseteq \bigcup(B_i : i \in J, i \leq j).$$

(6) Observe that

$$\delta(U_{i\ell} \triangle U_{\ell j} \triangle U_{ji}) = \delta(U_{i\ell}) \triangle \delta(U_{\ell j}) \triangle \delta(U_{ji}) = (B_i \triangle B_\ell) \triangle (B_\ell \triangle B_j) \triangle (B_j \triangle B_i) = \emptyset.$$

As G is connected, it follows that $U_{i\ell} \triangle U_{\ell j} \triangle U_{ji}$ is either $\emptyset$ or $V(G)$. However, as $s, t \notin U_{i\ell} \triangle U_{\ell j} \triangle U_{ji}$, it must be that $U_{i\ell} \triangle U_{\ell j} \triangle U_{ji} = \emptyset$. Hence, $U_{i\ell} \cap U_{\ell j} \cap U_{ji} = \emptyset$, and so in particular, $U_{i\ell} \cap S = \emptyset$.

(7) Since $\Omega \in Q_j \cap B_j$, we have

$$Q_j \cap \delta(U_{ij}) = Q_j \cap (B_j \triangle B_i) = (Q_j \cap B_j) \triangle (Q_j \cap B_i) = (Q_j \cap B_j) \triangle \{\Omega\} = (Q_j \cap B_j) - \{\Omega\}.$$

Thus,

$$Q_j \cap \delta(S) \subseteq \bigcup(Q_j \cap \delta(U_{ij}) : i \in J, i < j) = (Q_j \cap B_j) - \{\Omega\}.$$

Since $s, t \notin U_{ij}$ for all $i \in J$ with $i < j$ and since $Q_1, \ldots, Q_m$ are all connected, equality holds above.

(8) As $|J| \geq 3$, there exists $i \in J - \{j, \ell\}$. By (4) $B_i \cap Q_\ell = B_j \cap Q_\ell = \{\Omega\}$, and so as Q_ℓ is connected, $V(Q_\ell) \cap U_{ij} = \emptyset$. In particular, $V(Q_\ell) \cap S = \emptyset$, so $Q_\ell \cap \delta(S) = \emptyset$.

(9) Since L is connected, we can traverse its vertices in some order $s = v_0, v_1, v_2, \ldots, v_p = t$, where $L = \{e_x := \{v_{x-1}, v_x\} : 1 \leq x \leq p\}$. Choose $1 \leq x < y \leq p$ such that $e_x, e_y \in \delta(S)$ with $v_x, v_{y-1} \in S$. Either $B_j \cap L[s, v_x] = \emptyset$ or $B_j \cap L[v_{y-1}, t] = \emptyset$ (as $|B_j \cap L| = 1$). We assume $B_j \cap L[s, v_x] = \emptyset$, and the other case can be dealt with similarly. For $i \in J$ such that $i < j$, as $v_x \in U_{ij}$ and $s \notin U_{ij}$, it follows that $\delta(U_{ij}) \cap L[s, v_x] \neq \emptyset$, but $B_j \cap L[s, v_x] = \emptyset$, implying that $B_i \cap L[s, v_x] \neq \emptyset$. We claim that $e_y \in B_j$. As $v_y \notin S$, there exists $i \in J$ such that $i < j$ and $v_y \notin U_{ij}$ and so $e_y \in \delta(U_{ij})$. However, as $|B_i \cap L| = 1$ and $B_i \cap L[s, v_x] \neq \emptyset$, we get $B_i \cap L[v_{y-1}, t] = \emptyset$. In particular, $e_y \notin B_i$ and so $e_y \in B_j$. Since for all $i \in J$ such that $i \leq j$, $|B_i \cap L| = 1$, it follows that $L \cap \delta(S) = \{e_x, e_y\}$ and $L \cap \delta(S) \cap B_j = \{e_y\}$.

(10) As above, we traverse the vertices of C in some order $v_0, v_1, \ldots, v_{p-1}, v_p = v_0$, where $v_0 \in S$ and $C = \{e_x := \{v_{x-1}, v_x\} : 1 \leq x \leq p\}$. Assume there exist $1 \leq x < y \leq p$ such that $e_x, e_y \in \delta(S) - B_j$ with $v_x, v_{y-1} \notin S$. Then, for some $i \in J$ such that $i < j$, $v_x \notin U_{ij}$ and $e_x \in \delta(U_{ij})$. Since $e_x \notin B_j$, it follows that $e_x \in B_i$. Thus, as $|C \cap B_i| = 1$ and $e_y \notin B_j$, $e_y \notin \delta(U_{ij})$ and $v_{y-1} \in U_{ij}$. Let $C[v_x, v_{y-1}]$ be the $v_x v_{y-1}$-subpath of C not containing either of e_x, e_{y-1}. Then $C[v_x, v_{y-1}] \cap \delta(U_{ij}) \neq \emptyset$. Since

$C \cap B_i = \{e_x\}$, we get that $C[v_x, v_{y-1}] \cap B_j \neq \emptyset$. To summarize, if there exist $1 \leq x < y \leq p$ such that $e_x, e_y \in \delta(S) - B_j$ with $v_x, v_{y-1} \notin S$, then $C[v_x, v_{y-1}] \cap B_j \neq \emptyset$. Therefore, as $|C \cap B_j| = 1$, we get that $|C \cap \delta(S)| = 2$.

(11) By (10) $C \cap \delta(S) = \{e_x, e_y\}$ where $e_x \in B_i$ and $e_y \in B_\ell$. If $e_x \in B_j$ then $C \cap \delta(U_{ij}) = \emptyset$, but $V(C) \cap S \neq \emptyset$ and $S \subseteq U_{ij}$, implying that $V(C) \subseteq U_{ij} \subseteq U_{ij} \cup U_{\ell j}$, and we are done. Similarly, if $e_y \in B_j$ then $V(C) \subseteq U_{\ell j} \subseteq U_{ij} \cup U_{\ell j}$, and we are again done. Otherwise, $\{e_x, e_y\} \cap B_j = \emptyset$. As $e_x \in B_i - B_j$, it follows that $e_x \in \delta(U_{ij})$, and since $v_{x-1} \in S \subseteq U_{ij}$, we get $v_x \notin U_{ij}$. Also, as $|C \cap B_i| = 1$, we have $e_y \notin B_i$. This, together with the facts that $e_y \notin B_j$ and $v_y \in S \subseteq U_{ij}$, implies that $v_{y-1} \in U_{ij}$. Since $C \cap B_i = \{e_x\}$ and $|C \cap B_j| = 1$, there exists $z \in [y-1] - [x]$ such that

$$C \cap B_j = \{e_z\} \quad \text{and} \quad v_z, v_{z+1}, \ldots, v_{y-1} \in U_{ij}.$$

Similarly, we have

$$C \cap B_\ell = \{e_z\} \quad \text{and} \quad v_x, v_{x+1}, \ldots, v_{z-1} \in U_{\ell j}.$$

As a result, since $v_0, v_1, \ldots, v_{x-1}, v_y, v_{y+1}, \ldots, v_{p-1} \in S \subseteq U_{ij} \cap U_{\ell j}$, it follows that $V(C) \subseteq U_{ij} \cup U_{\ell j}$. □

We are now ready to prove the mate proposition 8.4.

Proof of proposition 8.4. We assume that Ω has ends s, s'. By identifying a vertex of each component with s, if necessary, we may assume that G is connected. Suppose, for a contradiction, that $B_1, \ldots, B_m$ are all signatures. We will be applying lemma 8.5 to the index set $[m]$. Notice first that as a corollary of parts (1)-(3), we have that $B_j \cap L_i \subseteq \widetilde{C_i} \cup \widetilde{P_i}$ for all $i, j \in [m]$. For distinct $i, j \in [m]$, choose $U_{ij} \subseteq V(G) - \{s, t\}$ such that $\delta(U_{ij}) = B_i \triangle B_j$. For each $j \in \{3, \ldots, m\}$, let

$$S_j := \bigcap (U_{ij} : 1 \leq i < j).$$

Let $C \in \{\widetilde{C_1}, \ldots, \widetilde{C_m}\}$ and $S_j \in \{S_3, \ldots, S_m\}$. We say C *is bad for* S_j if

$$|C \cap \delta(S_j)| = 2 \quad \text{and} \quad C \cap \delta(S_j) \cap B_j = \emptyset.$$

Claim 1. *One of $S_3, \ldots, S_m$ has no bad circuit.*

Proof. Let C be a bad circuit for some $S_j, 3 \leq j \leq m$. Then by lemma 8.5 parts (2) and (5),

$$C \cap \delta(S_j) \subseteq B_i \cup B_\ell, \qquad \text{for some } 1 \leq i < \ell < j.$$

Therefore, by lemma 8.5(11), $V(C) \subseteq U_{ij} \cup U_{\ell j}$. In particular, $s \notin V(C)$ and

$$V(C) \cap S_{j+1} = V(C) \cap S_{j+2} = \cdots = V(C) \cap S_m = \emptyset,$$

since by lemma 8.5(6), $(U_{ij} \cup U_{\ell j}) \cap S = \emptyset$, for all $S \in \{S_{j+1}, \ldots, S_m\}$. As a result, $C \notin \{\widetilde{C_1}, \widetilde{C_2}, \widetilde{C_3}\}$ and C is not bad for any of $S_{j+1}, \ldots, S_m$. Thus every circuit is bad for at most one of $S_3, \ldots, S_m$ and every bad circuit is one of $\widetilde{C_4}, \ldots, \widetilde{C_m}$. Thus, one of $S_3, \ldots, S_m$ has no bad circuit. ◇

Choose $j \in \{3, \ldots, m\}$ so that S_j has no bad circuit, and let $B := B_j \triangle \delta(S_j)$. Notice that for each $i \in [m]$, $B \cap L_i \subseteq \widetilde{C_i} \cup \widetilde{P_i}$.

Claim 2. *B is a cover of size $k-2$.*

Proof. It is clear that B is a cover. It remains to show that $|B| = k-2$. By lemma 8.5,

$$B \subseteq \bigcup (B_i : 1 \le i \le j) \subseteq \bigcup_{i=1}^{k} L_i.$$

The first inclusion follows from part (5) and the second inclusion follows from part (1). Therefore, as $\Omega \in B$, it suffices to show that, for all $i \in [k]$, $|B \cap L_i| = 1$. Observe that, for all $i \in [k] - \{j\}$, $|B_j \cap L_i| = 1$.

Take $i \in [k] - [m]$. If $L_i \cap \delta(S_j) = \emptyset$, then $|L_i \cap B| = |L_i \cap B_j| = 1$. Otherwise, when $L_i \cap \delta(S_j) \neq \emptyset$, lemma 8.5(9) implies $|L_i \cap \delta(S_j)| = 2$ and $|L_i \cap \delta(S_j) \cap B_j| = 1$, so $|L_i \cap B| = |L_i \cap (B_j \triangle \delta(S_j))| = 1$.

Next take $i \in [m]$. We will first consider $\widetilde{C_i} \cap B$, given that $\widetilde{C_i} \neq \emptyset$. If $\widetilde{C_i} \cap \delta(S_j) = \emptyset$, then $|\widetilde{C_i} \cap B| = |\widetilde{C_i} \cap B_j| = 1$. Otherwise, $\widetilde{C_i} \cap \delta(S_j) \neq \emptyset$. Then, by lemma 8.5(10), $|\widetilde{C_i} \cap \delta(S_j)| = 2$. By our choice of S_j, $\widetilde{C_i}$ is not bad for S_j, and so $|\widetilde{C_i} \cap \delta(S_j) \cap B_j| = 1$. Thus, $|\widetilde{C_i} \cap B| = |\widetilde{C_i} \cap (B_j \triangle \delta(S_j))| = 1$. We next consider $(\{\Omega\} \cup \widetilde{P_i}) \cap B$. If $i \neq j$, then by lemma 8.5,

$$\begin{aligned}(\{\Omega\} \cup \widetilde{P_i}) \cap B &= (\{\Omega\} \cup \widetilde{P_i}) \cap (B_j \triangle \delta(S_j)) \\ &= (\{\Omega\} \cup \widetilde{P_i}) \cap B_j \quad \text{by part (8)} \\ &= \{\Omega\} \quad \text{by part (3)}.\end{aligned}$$

On the other hand, if $i = j$, then by lemma 8.5,

$$\begin{aligned}(\{\Omega\} \cup \widetilde{P_j}) \cap B &= (\{\Omega\} \cup \widetilde{P_j}) \cap (B_j \triangle \delta(S_j)) \\ &= [(\{\Omega\} \cup \widetilde{P_j}) \cap B_j] \triangle [(\{\Omega\} \cup \widetilde{P_j}) \cap \delta(S_j)] \\ &= \{\Omega\} \quad \text{by part (7)}.\end{aligned}$$

Since whenever $\Omega \in \widetilde{P_i}$ then $\widetilde{C_i} = \emptyset$,

$$|L_i \cap B| = |\widetilde{C_i} \cap B| + |\widetilde{P_i} \cap B| = 1,$$

as $L_i \cap B \subseteq \widetilde{C_i} \cup \widetilde{P_i}$. ◇

By claim 2, $|B| = k - 2$. However, B is cover and so $|B| \geq \tau(G, \Sigma) \geq k$, a contradiction. □

8.3. **The odd-K_5 lemma.** The following lemma is essentially due to Schrijver [11], and the presentation follows Geelen and Guenin [3].

Lemma 8.6 ([11, 3]). *Let $G = (V, E)$ be a graph and let Ω be an edge of G with ends s, s'. Let U_0, U_1, U_2, U_3 be a partition of $V(G)$, and let P_1, P_2, P_3 be internally vertex-disjoint ss'-paths in $G \setminus \Omega$ such that*

(i) $s, s' \in U_0$, and for $i \in \{0, 1, 2, 3\}$, U_i is a stable set in $G \setminus \Omega$,
(ii) for $i \in [3]$, $V(P_i) \subseteq U_0 \cup U_i$, and
(iii) for distinct $i, j \in [3]$, there is a path between P_i and P_j in $G[U_i \cup U_j]$.

Then $(G, E(G))$ has a $\widetilde{K_5}$ minor.

8.4. **Mates and connectivity.**

Proposition 8.7. *Let $(G, \Sigma, \{s, t\})$ be a signed graft and $(L_1, \dots, L_k)$ be an (Ω, k)-packing, where $\Omega \in \delta(s)$. Suppose that for $i = 1, 2$ there exists a signature B_i that is a k-mate of L_i. Let $U \subseteq V(G) - \{s, t\}$ such that $B_1 \triangle B_2 = \delta(U)$. For $i = 1, 2$ let $W_i = V(L_i) \cap U$. Then there exists a path P in $G[U]$ between a vertex in W_1 and a vertex in W_2 such that $P \cap (B_1 \cup B_2) = \emptyset$.*

Proof. Suppose first that there is no path in $G[U]$ between W_1 and W_2. Then there exists $U' \subset U$ such that $W_1 \subseteq U'$, $W_2 \subseteq U - U'$ and there is no edge of G with one end in U' and one end in $U - U'$. Then $B = B_1 \triangle \delta(U')$ is a signature of $(G, \Sigma, \{s, t\})$ where $B \subseteq B_1 \cup B_2$ and $B \cap (L_1 \cup L_2) = \{\Omega\}$, contradicting proposition 3.4 part (4).

Thus there exists a path P in $G[U]$ between W_1 and W_2 with minimum number of edges in $B_1 \cup B_2$. Suppose P has an edge $e \in B_i$ for some $i \in [2]$. Then $e \in B_1 \cap B_2$ as $e \notin \delta(U)$. Since $s \notin U$, $e \neq \Omega$. Proposition 3.1 implies that for some $j \in [k] - [3]$, $e \in L_j$ and $B_1 \cap L_j = B_2 \cap L_j = \{e\}$. Hence, since $e \in E(G[U])$ and $s, t \notin U$, e must belong to an odd circuit C contained in $L_j \cap E(G[U])$. But then replacing P by $P \triangle C$ we obtain a new walk in $G[U]$ between W_1 and W_2 with fewer edges in $B_1 \cup B_2$, contradicting our choice of P. □

8.5. **Acyclicity and flows.**

Proposition 8.8. *Consider an acyclic digraph whose edges can be written as the union of dipaths $Q_1, \dots, Q_n$ rooted from some vertex x. Suppose that $Q_1, \dots, Q_n$ use distinct arcs incident with x. Consider the following partial ordering defined on the vertices: for vertices u, v, $u \leq v$ if there is a uv-dipath. For every $i \in [n]$, let v_i be the second smallest vertex of Q_i that also lies on a dipath in*

$\{Q_1, \ldots, Q_n\} - \{Q_i\}$ *(assuming v_i exists). Then there exists an index subset $I \subseteq [n]$ of size at least two such that, for each $i \in I$, the following hold:*

- *$v_i \leq v_1$, and there is no $j \in [n]$ such that $v_j < v_i$, and*
- *for each $j \in [n]$, $v_i = v_j$ if and only if $j \in I$.*

Proof. Suppose such an index subset does not exist. In particular, for any index $i \in [n]$ such that $v_i \leq v_1$, there exists $\pi(i) \in [n] - \{i\}$ such that $v_i \in V(Q_{\pi(i)})$ and $v_i > v_{\pi(i)}$. Then one can construct the infinite chain $v_1 > v_{\pi(1)} > v_{\pi(\pi(1))} > \ldots$, a contradiction as $>$ is a partial ordering on the vertices of the acyclic digraph. □

Remark 8.9. *Let $(\vec{H}, \{\Omega\}, \{s,t\})$ be a directed signed graft, where $\Omega \in \delta(s)$ and $\vec{H} \setminus \Omega$ is acyclic. Suppose $E(\vec{H})$ can be written as the union of pairwise Ω-disjoint edge sets L_1, L_2, L_3, $P_4, \ldots, P_m$ where $m \geq 3$, L_1, L_2, L_3 are directed minimal odd st-joins and $P_4, \ldots, P_m$ are even st-dipaths. Let L be a directed minimal odd st-join. Then the following hold:*

(1) there exist pairwise Ω-disjoint edge sets $L'_1, L'_2, L'_3, P'_4, \ldots, P'_m$ where $L'_1 = L$, L'_2, L'_3 are directed minimal odd st-joins, $P'_4, \ldots, P'_m$ are even st-dipaths, and the number of non-simple minimal odd st-joins among L'_1, L'_2, L'_3 is equal to that of L_1, L_2, L_3,

(2) if exactly one of L_1, L_2, L_3 is non-simple, then L is simple if and only if L is Ω-disjoint from a directed odd circuit,

(3) if at least two of L_1, L_2, L_3 are non-simple, then L is Ω-disjoint from a directed odd circuit.

9. Preliminaries for non-simple bipartite Ω-systems

In this section, we lay the groundwork to prove proposition 2.6, namely that a minimal non-simple bipartite Ω-system has an F_7 or $\widetilde{K_5}$ minor.

9.1. Signature mates.

Proposition 9.1. *Let $((G, \Sigma, \{s,t\}), \mathcal{L} = (L_1, \ldots, L_k), m, \vec{H})$ be a non-simple bipartite Ω-system. Let $L \subseteq E(\vec{H})$ be a directed minimal odd st-join that is Ω-disjoint from a directed circuit $C \subseteq E(\vec{H})$. Let B be a k-mate of L. Then B is not an st-cut and $B \cap E(\vec{H}) = B \cap L$.*

Proof. Since $\vec{H} \setminus \Omega$ is acyclic, we can write $E(\vec{H})$ as the union of $L'_1, L'_2, L'_3, P'_4, \ldots, P'_m$ such that, for

$$\mathcal{L}' = (L'_1, L'_2, L'_3, L'_4 := P'_4 \cup C_4, \ldots, L'_m := P'_m \cup C_m, L_{m+1}, \ldots, L_k),$$

$((G, \Sigma, \{s,t\}), \mathcal{L}', m, \vec{H})$ is a non-simple bipartite Ω-system, $L'_1 = L$ and $C(L'_2) = C$. By proposition 3.2, $B \subseteq L \cup L'_4 \cup \cdots \cup L'_m \cup L_{m+1} \cup \cdots \cup L_k$. Since $B \cap L'_2 \neq \emptyset$ and $B \cap L'_2 \subseteq \{\Omega\}$, it follows that $B \cap L_2' = \{\Omega\}$, so $B \cap C = \{\Omega\}$. Hence, B is not an st-cut, so it is a signature. Moreover, by proposition 8.3, $B \cap E(\vec{H}) = B \cap L$. □

Proposition 9.2. *Let $((G, \Sigma, \{s,t\}), \mathcal{L} = (L_1, \ldots, L_k), m, \vec{H})$ be a non-simple bipartite Ω-system. Choose an even st-dipath P of $\vec{H}$ such that $P \cup \{\Omega\}$ has a k-mate B. Then B is not an st-cut and $B \cap E(\vec{H}) = \{\Omega\} \cup (B \cap P)$.*

Proof. Since $\vec{H} \setminus \Omega$ is acyclic, we can write $E(\vec{H})$ as the union of $L_1', L_2', L_3', P_4', \ldots, P_m'$ such that, for

$$\mathcal{L}' = (L_1', L_2', L_3', L_4' := P_4' \cup C_4, \ldots, L_m' := P_m' \cup C_m, L_{m+1}, \ldots, L_k),$$

$((G, \Sigma, \{s,t\}), \mathcal{L}', m, \vec{H})$ is a non-simple bipartite Ω-system and $P(L_1') = P$. By proposition 3.2, $B \subseteq \{\Omega\} \cup P \cup L_4' \cup \cdots \cup L_m' \cup L_{m+1} \cup \cdots \cup L_k$, and $\Omega \in B$ as B intersects L_2'. Then $B \cap C(L_1') = \{\Omega\}$, implying that B is not an st-cut, so it is a signature. Moreover, by proposition 8.3 and the fact that $\Omega \in B$, it follows that $B \cap E(\vec{H}) = \{\Omega\} \cup (B \cap P)$. □

9.2. **Two disentangling lemmas.**

Lemma 9.3. *Let $\big((G, \Sigma, \{s,t\}), \mathcal{L} = (L_1, \ldots, L_k), m, \vec{H}\big)$ be a minimal non-simple bipartite Ω-system. Take disjoint subsets $I_d, I_c \subseteq E(\vec{H} \setminus \Omega)$ and $T' \subseteq \{s,t\}$ where*

(1) I_c is non-empty, if I_c contains an st-path then $T' = \emptyset$, and if not then $T' = \{s,t\}$,

(2) every signature or st-cut disjoint from I_c intersects I_d in an even number of edges,

(3) if $T' = \emptyset$, there is a directed subgraph $\vec{H}'$ of $\vec{H}/I_c \setminus I_d$ that is the union of directed odd circuits L_1', L_2', L_3' where

- *$\Omega \in L_1' \cap L_2' \cap L_3'$ and L_1', L_2', L_3' are pairwise Ω-disjoint,*
- *$\vec{H}' \setminus \Omega$ is acyclic.*

(4) if $T' = \{s,t\}$, there is a directed subgraph $\vec{H}'$ of $\vec{H}/I_c \setminus I_d$ that is the union of directed minimal odd st-joins L_1', L_2', L_3' and even st-dipaths $P_4', \ldots, P_m'$, where

- *$\Omega \in L_1' \cap L_2' \cap L_3'$, $\Omega \notin P_4' \cup \ldots \cup P_m'$ and $L_1', L_2', L_3', P_4', \ldots, P_m'$ are pairwise Ω-disjoint,*
- *one of L_1', L_2', L_3' is non-simple,*
- *$\vec{H}' \setminus \Omega$ is acyclic.*

Then one of the following does not hold:

(i) $I_d \cup \{\Omega\}$ does not have a k-mate,

(ii) for every directed odd T'-join L' of $\vec{H}'$ Ω-disjoint from a directed odd circuit, either $L' \cup I_d$ contains a directed odd st-join of $\vec{H}$ Ω-disjoint from a directed odd circuit or $L' \cup I_d$ has a k-mate in $(G, \Sigma, \{s,t\})$ disjoint from I_c.

Proof. Suppose otherwise. Let $(G', \Sigma', T') := (G, \Sigma, \{s,t\})/I_c \setminus I_d$ where $\Sigma' = \Sigma$; this signed graft is well-defined by (1). For $i \in [m]-[3]$, let $L'_i := L_i - P_i$ if $T' = \emptyset$, and let $L'_i := (L_i - P_i) \cup P'_i$ otherwise. Let $\mathcal{L}' := (L'_1, \ldots, L'_m, L_{m+1}, \ldots, L_k)$. If $T' = \emptyset$, let $m' := 3$, and if not, let $m' := m$. We will show that $((G', \Sigma', T'), \mathcal{L}', m', \vec{H}')$ is a non-simple bipartite Ω-system, and this will yield a contradiction with the minimality of the original non-simple bipartite Ω-system, thereby finishing the proof.

(NS1) We first show that $((G', \Sigma', T'), \mathcal{L}', m')$ is a bipartite Ω-system. **(B1)** By (2) every signature or T'-cut of (G', Σ', T') has the same parity as $\tau(G, \Sigma, \{s,t\})$, implying that (G', Σ', T') is an Eulerian signed graft. **(B2)** It also implies that $k, \tau(G, \Sigma, \{s,t\}), \tau(G', \Sigma', T')$ have the same parity, so every minimal cover of (G', Σ', T') has the same size parity as k. We claim that $\tau(G', \Sigma', T') \geq k$. Let B' be a minimal cover of (G', Σ', T'). If $\Omega \notin B'$, then

$$|B'| \geq \sum (|B' \cap L'| : L' \in \mathcal{L}') \geq k.$$

Otherwise, $\Omega \in B'$. In this case, $B' \cup I_d$ contains a cover B of $(G, \Sigma, \{s,t\})$. By (i), $I_d \cup \{\Omega\}$ does not have a k-mate, so

$$k - 2 \leq |B - (I_d \cup \{\Omega\})| \leq |B - I_d| - 1 \leq |B'| - 1,$$

and since $|B'|, k$ have the same parity, it follows that $|B'| \geq k$. Thus, $\mathcal{L}'$ is an (Ω, k)-packing. When $T' = \emptyset$, $m' = 3$. When $T' = \{s,t\}$, then $m' = m$ and for $j \in [m'] - [3]$, L'_j contains even st-path P'_j and some odd circuit in $L'_j - P'_j$, and for $j \in [k] - [m']$, L_j remains connected in G'. **(B3)** is clear from construction.

(NS2) and **(NS3)** follow from (3) and (4). **(NS4)** Let L' be a directed odd T'-join of $\vec{H}'$ that is Ω-disjoint from a directed odd circuit. We claim that $L' \cup I_d$ has a k-mate in $(G, \Sigma, \{s,t\})$ disjoint from I_c. By (ii), we may assume that $L' \cup I_d$ contains a directed odd st-join L of $\vec{H}$ that is Ω-disjoint from a directed odd circuit. Since $((G, \Sigma, \{s,t\}), \mathcal{L}, m, \vec{H})$ is a non-simple bipartite Ω-system, it follows that L has a k-mate B. By proposition 9.1, $B \cap E(\vec{H}) = B \cap L$, implying that $B \cap I_c = \emptyset$, as claimed. So B is a k-mate of $L' \cup I_d$ disjoint from I_c. $B - I_d$ contains a minimal cover B' of (G', Σ', T'), and since

$$|B' - L'| \leq |(B - I_d) - L'| \leq |B - L| \leq k - 3,$$

it follows that B' is a k-mate of L'. □

We will need an analogue of this lemma for the case $T = \emptyset$. As the proof is almost the same (and less intricate), we leave the proof as an exercise:

Lemma 9.4. *Let* $\big((G, \Sigma, \emptyset), \mathcal{L} = (L_1, \ldots, L_k), 3, \vec{H}\big)$ *be a minimal non-simple bipartite* Ω*-system, where* $\Omega \in \delta(s)$*. Take disjoint subsets* $I_d, I_c \subseteq E(\vec{H} \setminus \Omega)$ *where*

(1) I_c *is non-empty,*

(2) every signature disjoint from I_c intersects I_d in an even number of edges,

(3) there is a directed subgraph $\vec{H}'$ of $\vec{H}/I_c \setminus I_d$ that is the union of directed odd circuits L_1', L_2', L_3' where

$\Omega \in L_1' \cap L_2' \cap L_3'$ *and* L_1', L_2', L_3' *are pairwise* Ω*-disjoint,*

$\vec{H}' \setminus \Omega$ *is acyclic.*

Then one of the following does not hold:

(i) $I_d \cup \{\Omega\}$ *does not have a signature k-mate,*

(ii) for every directed odd cycle L' of $\vec{H}'$ Ω-disjoint from a directed odd circuit, either $L' \cup I_d$ contains a directed odd cycle of $\vec{H}$ Ω-disjoint from a directed odd circuit or $L' \cup I_d$ has a signature k-mate in $(G, \Sigma, \emptyset)$ disjoint from I_c.

9.3. **Setup for the proof of proposition 2.6.** Let $((G, \Sigma, T), \mathcal{L} = (L_1, \ldots, L_k), m, \vec{H})$ be a minimal non-simple bipartite Ω-system. We know that $\vec{H} \setminus \Omega$ is acyclic, and by (B3), every odd circuit in $\vec{H}$ contains Ω and no even st-path in $\vec{H}$ contains Ω. Hence,

Remark 9.5. *Let C be a directed odd circuit and let P be an even st-dipath in $\vec{H}$. Then C and P share exactly one vertex, namely s.*

There are three possibilities:

I: all three of L_1, L_2, L_3 are non-simple (see §10),

II: exactly two of L_1, L_2, L_3 are non-simple (see §11),

III: exactly one of L_1, L_2, L_3 is non-simple (see §12).

We will assume throughout this section that Ω has ends s, s'.

10. Non-simple bipartite Ω-system - part I

Here we prove proposition 2.6 when all of L_1, L_2, L_3 are non-simple. By remark 9.5, for $i \in [3]$ and $j \in [m]$, C_i and P_j share exactly one vertex, namely s.

Claim 1. *There exists $j \in [m]$ such that $P_j \cup \{\Omega\}$ has no k-mate.*

Proof. Suppose otherwise. Then $T = \{s, t\}$, as $\tau(G, \Sigma, T) \geq k$ (so $\{\Omega\}$ has no k-mate). Then by the mate proposition 8.4 there exists $i \in [m]$ such that the k-mate of $P_i \cup \{\Omega\}$ is an st-cut, contradicting proposition 9.2. ◇

By swapping the roles of P_1 and P_j in $\mathcal{L}$, if necessary, we may assume that $j = 1$.

Claim 2. $T = \emptyset$.

Proof. Suppose for a contradiction that $T = \{s, t\}$. Let $I_d := P_1$ and $I_c := P_2 \cup \ldots \cup P_m$. Let $T' := \emptyset$, and for $j \in [3]$ let $L'_j := C_j$, and let $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of L'_1, L'_2, L'_3. It is clear that (1)-(4) of the disentangling lemma 9.3 hold. By claim 1, $P_1 \cup \{\Omega\} = I_d \cup \{\Omega\}$ has no k-mate, so (i) holds. Let L' be a directed odd cycle of $\vec{H}'$. Then it is clear that $L' \cup I_d$ contains a directed minimal odd st-join L of $\vec{H}$ such that $L' \subseteq L$. By remark 8.9(3), L and so L' is Ω-disjoint from a directed odd circuit, and since I_d is Ω-disjoint from every directed odd circuit by remark 9.5, we get that $L' \cup I_d$ is Ω-disjoint from a directed odd circuit, so (ii) holds as well, a contradiction with the disentangling lemma 9.3. ◇

The rest of this part is dedicated to finding a $\widetilde{K_5}$ minor in $(G, \Sigma, T = \emptyset)$, and our arguments are very similar to the treatment of Geelen and Guenin [3], except for our use of Menger's theorem in claim 4.

We may assume that in $\vec{H}$, Ω is directed from s to s', and for $i \in [3]$, $L_i - \{\Omega\}$ is an $s's$-dipath. Consider the following partial ordering defined on the vertices of $\vec{H}$: for $u, v \in V(\vec{H})$, $u \leq v$ if there is a uv-dipath in $\vec{H} \setminus \Omega$; this partial ordering is well-defined as $\vec{H} \setminus \Omega$, by (NS3). For each $i \in [3]$, let v_i be the second smallest vertex of $L_i - \{\Omega\}$ that lies on a dipath in $\{L_1, L_2, L_3\} - \{L_i\}$ By proposition 8.8, there exists an index subset $I \subseteq [3]$ of size at least two such that, for each $i \in I$ and $j \in [3]$, $v_j = v_i$ if and only if $j \in I$. We may assume that $1 \in I$.

Claim 3. *For each $i \in I$, $L_i[s', v_i] \cup \{\Omega\}$ has a signature k-mate.*

Proof. Suppose otherwise. Let $I_d := L_i[s', v_i]$ and $I_c := \bigcup(L_j[s', v_j] : j \in I, j \neq i)$. For $i \in [3]$ let $L'_i := L_i - (I_c \cup I_d)$, and let $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of L'_1, L'_2, L'_3. It is easily seen that (1)-(3) of the disentangling lemma 9.4 hold. By our hypothesis, (i) holds. Let L' be a directed odd cycle of $\vec{H}'$. Then $L' \cup I_c$ contains a directed odd circuit of $\vec{H}$, implying that $L' \cup I_d$ also contains a directed odd circuit of $\vec{H}$, which by remark 8.9(3) is Ω-disjoint from a directed odd circuit. Hence, (ii) holds as well, a contradiction to the disentangling lemma 9.4. ◇

Claim 4. *There exist an $s's$-dipath P and a $v_1 s$-dipath Q in $\vec{H} \setminus \{\Omega\}$ that are internally vertex-disjoint.*

Proof. Suppose otherwise. Then $s \neq v_1$ and there exists a vertex $v \in V(\vec{H}) - \{s, s'\}$ such that there is no $s's$-dipath in $\vec{H} \setminus v$. One of the following holds:

(a) there exists an $s'v$-dipath R in $\vec{H}$ such that $R \cup \{\Omega\}$ has no k-mate:

Let $I_d := R$, $I_c := \bigcup(L_i[s', v] : i \in [3]) - R$, for $i \in [3]$ let $L'_i := L_i - (I_c \cup I_d)$, and let $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of L'_1, L'_2, L'_3.

(b) for every $s'v$-dipath R in $\vec{H}$, $R \cup \{\Omega\}$ has a (signature) k-mate:

Let $I_d := \emptyset$, $I_c := \bigcup(L_i[v,s] : i \in [3])$, for $i \in [3]$ let $L'_i := L_i[s',v] \cup \{\Omega\}$, and let $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of L'_1, L'_2, L'_3.

It is not difficult to check that in either of the cases above, (1)-(3) and (i)-(ii) of the disentangling lemma 9.4 hold, a contradiction. ◇

After redefining $\mathcal{L}$, if necessary, we may assume that $\{1,2\} \subseteq I$ and $P = L_3 - \{\Omega\}$.

Claim 5. *$(L_i - \{\Omega\} : i \in [3])$ are pairwise internally vertex-disjoint.*

Proof. It suffices to prove that $Q = \emptyset$. Suppose not. Let $I_c := Q$, $I_d := \emptyset$, for $i \in [2]$ let $L'_i := L_i[s', v_i] \cup \{\Omega\}$, and let $L'_3 := L_3$. Let $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of L'_1, L'_2, L'_3. Note that $(L'_i - \{\Omega\} : i \in [3])$ are pairwise internally vertex-disjoint. By our hypothesis, claim 3, (NS4), and proposition 8.3, (1)-(3) and (i)-(ii) of the disentangling lemma 9.4 hold, a contradiction. ◇

Claim 6. *$(G, \Sigma, T = \emptyset)$ has a $\widetilde{K_5}$ minor.*

Proof. By identifying a vertex of each component with s, if necessary, we may assume that G is connected. By (NS4), for each $i \in [3]$, there exists a signature k-mate B_i of L_i. For distinct $i, j \in [3]$, let $U_{ij} \subseteq V(G) - \{s,t\}$ such that $\delta(U_{ij}) = B_i \triangle B_j$; by proposition 8.7, there exists a shortest path P_{ij} between L_i and L_j in $G[U_{ij}] \setminus (B_i \cup B_j)$. To finish proving the claim, we will use the odd-K_5 lemma 8.6 to prove that $L_1 \cup L_2 \cup L_3 \cup P_{12} \cup P_{23} \cup P_{31}$ has a $\widetilde{K_5}$ minor.

Observe that

$$\emptyset = (B_1 \triangle B_2) \triangle (B_2 \triangle B_3) \triangle (B_3 \triangle B_1) = \delta(U_{12}) \triangle \delta(U_{23}) \triangle \delta(U_{31}) = \delta(U_{12} \triangle U_{23} \triangle U_{31}),$$

implying that $U_{12} \triangle U_{23} \triangle U_{31}$ is either $\emptyset$ or $V(G)$, as G is connected. However, $s, t \notin U_{12} \triangle U_{23} \triangle U_{31}$, implying that $U_{12} \triangle U_{23} \triangle U_{31} = \emptyset$. As a result, there exist pairwise disjoint subsets $U_1, U_2, U_3 \subseteq V(G)$ such that, for distinct $i, j \in [3]$, $U_{ij} = U_i \cup U_j$. Let $U_0 := V(G) - (U_1 \cup U_2 \cup U_3)$. Since $L_1 \cap (B_2 \cup B_3) = \{\Omega\}$, it follows that $L_1 \cap \delta(U_{23}) = \emptyset$, and since L_1 is connected, it must be that $V(L_1) \subseteq U_0 \cup U_1$. Similarly, $V(L_2) \subseteq U_0 \cup U_2$ and $V(L_3) \subseteq U_0 \cup U_3$. Let $B := B_1 \triangle B_2 \triangle B_3$, which is a signature for (G, Σ, T). Observe that the edges in B are precisely those with ends in different sets among U_0, U_1, U_2, U_3. Now contract all the edges of G not in B and apply the odd-K_5 lemma 8.6 to conclude that $L_1 \cup L_2 \cup L_3 \cup P_{12} \cup P_{23} \cup P_{31}$, and in turn (G, Σ, T), has a $\widetilde{K_5}$ minor. ◇

11. Non-simple bipartite Ω-system - part II

Here we prove proposition 2.6 when exactly two of L_1, L_2, L_3, say L_1 and L_2, are non-simple. Observe that $T \neq \emptyset$. Recall that $T = \{s,t\}$ and Ω has ends s, s'.

Claim 1. *There exists $j \in [m] - \{3\}$ such that $P_j \cup \{\Omega\}$ has no k-mate.*

Proof. Suppose otherwise. As P_3 is a directed odd st-join of $\vec{H}$ that is Ω-disjoint from directed odd circuit C_1, it has a k-mate. Thus by the mate proposition 8.4 there exists $i \in [m]$ such that the k-mate of $P_i \cup \{\Omega\}$ is an st-cut, contradicting propositions 9.1 and 9.2. ◇

By swapping the roles of P_1 and P_j in $\mathcal{L}$, if necessary, we may assume that $j = 1$. Observe that $P_1 \cup \cdots \cup P_m$ is acyclic, as $\vec{H} \backslash \Omega$ is so. Consider the following partial ordering: for $u, v \in V(P_1 \cup \cdots \cup P_m)$, $u \leq v$ if there is a uv-dipath in $P_1 \cup \cdots \cup P_m$. For $i \in [m]$ let v_i be the second largest vertex of P_i that lies on another st-dipath in $\{P_1, \ldots, P_m\} - \{P_i\}$.

Claim 2. $s < v_3$.

Proof. Suppose otherwise. In other words, P_3 is internally vertex-disjoint from each one of $P_1, P_2, P_4, \ldots, P_m$. Let $I_d := P_1$ and $I_c := P_2 \cup P_4 \cup P_5 \cup \ldots \cup P_m$. Let $T' := \emptyset$, for $j \in [2]$ let $L'_j := C_j$, let $L'_3 := P_3$, and let $\vec{H}' \subseteq \vec{H} \backslash I_d / I_c$ be the union of L'_1, L'_2, L'_3. It is clear that (1)-(4) of the disentangling lemma 9.3 hold. By claim 1, $P_1 \cup \{\Omega\} = I_d \cup \{\Omega\}$ has no k-mate, so (i) holds. Let L' be a directed odd cycle of $\vec{H}'$. Then it is clear that $L' \cup I_d$ contains a directed minimal odd st-join L of $\vec{H}$ such that $L' \subseteq L$. By remark 8.9(3), L is Ω-disjoint from a directed odd circuit, so by remark 9.5, $L' \cup I_d$ is Ω-disjoint from a directed odd circuit, implying in turn that (ii) holds, a contradiction with the disentangling lemma 9.3. ◇

By proposition 8.8 there exists an index subset $I \subseteq [m]$ of size at least two such that, for each $i \in I$,

- $v_i \geq v_3$, and there is no $j \in [m]$ such that $v_j > v_i$,
- for each $j \in [m]$, $v_i = v_j$ if and only if $j \in I$.

For $i \in I$, since $v_i \geq v_3 > s$ by claim 2, $P_i[v_i, t]$ is contained in an odd st-dipath of $\vec{H}$, and since $I \cap ([m] - \{3\}) \neq \emptyset$, $P_i[v_i, t]$ is also contained in an even st-dipath of $\vec{H}$

Claim 3. *For each $i \in I$ and $j \in [2]$, $P_i[v_i, t]$ and C_j have no vertex in common.*

Proof. Since $P_i[v_i, t]$ is contained in an even st-dipath of $\vec{H}$, the claim follows from remark 9.5 and the fact that $v_i > s$. ◇

As a result, for each $i \in I$, the internal vertices of $P_i[v_i, t]$ have degree two in $\vec{H}$.

Claim 4. *For each $i \in I$, $P_i[v_i, t] \cup \{\Omega\}$ has a k-mate. In particular, $1 \notin I$.*

Proof. Suppose otherwise. Let $I_d := P_i[v_i, t]$ and $I_c := \bigcup(P_j[v_j, t] : j \in I - \{i\})$. Let $T' := \{s, t\}$, for $j \in [3]$ let $L'_j := L_j - (I_c \cup I_d)$, and for $j \in [m] - [3]$ let $P'_j := P_j - (I_c \cup I_d)$. Let $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of $L'_1, L'_2, L'_3, P'_4, \ldots, P'_m$. It is clear that (1)-(4) of the disentangling lemma 9.3 hold. By assumption, $I_d \cup \{\Omega\}$ has no k-mate, so (i) holds. Let L' be a directed odd st-join of $\vec{H}'$. Then it is clear that $L' \cup I_d$ contains a directed minimal odd st-join L of $\vec{H}$ such that $L' \subseteq L$. By remark 8.9(3), L is Ω-disjoint from a directed odd circuit, so by remark 9.5, $L' \cup I_d$ is also Ω-disjoint from a directed odd circuit, so (ii) holds as well, a contradiction with the disentangling lemma 9.3. $\diamond$

Claim 5. *Fix $i \in I$. Then there exists an $s'v_i$-dipath in $\vec{H} \setminus (C_1 \cup C_2)$ that is vertex-disjoint from P_1.*

Proof. Let v be the smallest vertex on P_1 for which there exists a vv_i-dipath R in $\vec{H} \setminus \Omega$ such that $V(R) \cap V(P_1) = \{v\}$. Since R is contained in an even st-dipath, namely $P_1[s,v] \cup R \cup P_i[v_i,t]$, it follows from remark 9.5 that R and $C_1 \cup C_2$ have at most one vertex in common, namely s. Our choice of v and R implies the following:

($\star$) if $w \in V(R)$ and Q is an sw-dipath in $\vec{H} \setminus \Omega$, then Q and $P_1[v,t]$ have a vertex in common.

Suppose for a contradiction that there is no $s'v_i$-dipath in $\vec{H} \setminus (C_1 \cup C_2)$ that is vertex-disjoint from P_1. This fact, together with ($\star$) and remark 9.5, implies the following:

($\star\star$) if $w \in V(R)$ and Q is an $s'w$-dipath in $\vec{H}$, then Q and $P_1[v,t]$ have a vertex in common.

Let $I_d := P_1[v,t]$ and $I_c := R \cup [\bigcup(P_j[v_j,t] : j \in I)]$. For $i \in [3]$ let L'_i be $L_i - (I_c \cup I_d)$ minus any directed circuit that does not use Ω, and for $i \in [m] - [3]$ let P'_i be $P_i - (I_c \cup I_d)$ minus any directed circuit that does not use Ω. If $v = s$, let $T' := \emptyset$ and $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of L'_1, L'_2, L'_3. Otherwise, when $v \neq s$, let $T' := \{s,t\}$ and $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of $L'_1, L'_2, L'_3, P'_4, \ldots, P'_m$. It is not hard to see that (1)-(4) of the disentangling lemma 9.3 hold. By claim 1, $P_1[v,t] \cup \{\Omega\} = I_d \cup \{\Omega\}$ has no k-mate, so (i) holds. Let L' be a directed odd T'-join of $\vec{H}'$. Then $L' \cup I_c$ contains a directed odd st-join L of $\vec{H}$ such that $L' \subseteq L$. Choose $w \in V(R)$ (if any) such that L contains an $s'w$-dipath Q in $\vec{H}$ and $V(Q) \cap V(R) = \{w\}$. Then ($\star\star$) implies that $(L - I_c) \cup I_d$, and therefore $L' \cup I_d$, contains a directed minimal odd st-join of $\vec{H}$. By remark 8.9(3), L is Ω-disjoint from a directed odd circuit, so by remark 9.5, $L' \cup I_d$ is Ω-disjoint from a directed odd circuit, and so (ii) holds as well, a contradiction with the disentangling lemma 9.3. $\diamond$

After redefining $\mathcal{L}$, if necessary, we may assume that $3 \in I$ and that $P_3[s',v_3]$ is vertex-disjoint from P_1. (See remark 8.9(1).)

Claim 6. *$(G, \Sigma, \{s,t\})$ has an F_7 minor.*

Proof. For $i \in I$ let B_i be a k-mate of $P_i[v_i,t] \cup \{\Omega\}$, whose existence is guaranteed by claim 4. For each $i \in I$, since B_i is also a k-mate for odd st-dipath $P_3[s,v_3] \cup P_i[v_i,t]$, proposition 9.1 implies that B_i is a signature. Take $j \in I - \{3\}$. Choose $U \subseteq V(G) - \{s,t\}$ such that $B_3 \triangle B_j = \delta(U)$. Then by proposition 8.7 there exists a path P in $G[U]$ between $V(P_3[v_3,t])\cap U$ and $V(P_j[v_j,t])\cap U$ such that $P\cap(B_3\cup B_j)=\emptyset$, and P is minimal subject to this property. Observe that $L_1\cup P_3[s',v_3]$ has no vertex in common with U. It is easy (and is left as an exercise) to see that $C_1\cup P_1\cup P_3\cup P_j[v_j,t]\cup P$ has an F_7 minor. $\diamond$

12. Non-simple bipartite Ω-system - part III

Here we prove proposition 2.6 when exactly one of L_1,L_2,L_3, say L_1, is non-simple. This will complete the proof of proposition 2.6. Observe that $T\neq\emptyset$, so $T=\{s,t\}$, and recall that Ω has ends s,s'.

Observe that $P_1\cup\cdots\cup P_m$ is acyclic, as $\vec{H}\setminus\Omega$ is so. Consider the following partial ordering: for $u,v\in V(P_1\cup\cdots\cup P_m)$, $u\le v$ if there is a uv-dipath in $P_1\cup\cdots\cup P_m$. For $i\in[m]$ let v_i be the second largest vertex of P_i that lies on another st-dipath in $\{P_1,\dots,P_m\}-\{P_i\}$. By proposition 8.8 there exists an index subset $I\subseteq[m]$ of size at least two such that, for each $i\in I$,

- $v_i\ge v_3$, and there is no $j\in[m]$ such that $v_j>v_i$,
- for each $j\in[m]$, $v_i=v_j$ if and only if $j\in I$.

Claim 1. *For each $i\in I$, C_1 and $P_i[v_i,t]$ have no vertex of $V(G)-\{s'\}$ in common.*

Proof. Suppose otherwise. Then it follows from remark 9.5 that

$$(\diamond)\quad I=\{2,3\}\quad\text{and}\quad V(P_i)\cap V(P_j)=\{s,t\}\quad\forall\, i\in I,\forall\, j\in[m]-I.$$

Let $Q_1:=C_1-\{\Omega\},Q_2:=P_2-\{\Omega\}$ and $Q_3:=P_3-\{\Omega\}$. For each $i\in[3]$, let u_i be the second smallest (*not* largest) vertex of Q_i that also lies on one of $\{Q_1,Q_2,Q_3\}-\{Q_i\}$. Then by proposition 8.8, there exists an index subset J of $\{1,2,3\}$ of size at least two such that, for each $j\in J$ and $i\in[3]$, $u_i=u_j$ if and only if $i\in J$.

Subclaim 1. *For each $j\in J$, $Q_j[s',u_j]\cup\{\Omega\}$ has a k-mate.*

Proof of Subclaim. Suppose otherwise. Let $I_d:=Q_j[s',u_j]$ and $I_c:=\bigcup(Q_i[s',u_i]:i\in J-\{j\})$. Let $T':=\{s,t\}$, and for $i\in[3]$, let $L_i':=L_i-(I_c\cup I_d)$. Let $\vec{H}'\subseteq\vec{H}\setminus I_d/I_c$ be the union of $L_1',L_2',L_3',P_4,\dots,P_m$. It is clear that (1)-(4) of the disentangling lemma 9.3 hold. By assumption, $I_d\cup\{\Omega\}$ has no k-mate, so (i) holds. Let L' be a directed odd st-join of $\vec{H}'$ that is Ω-disjoint from a directed odd circuit, i.e. L' is an odd st-dipath of $\vec{H}'$ by remark 8.9(2). Then it is clear that $L'\cup I_d$ contains an odd st-dipath of $\vec{H}$, which by remark 8.9(2) is Ω-disjoint from a directed odd circuit, so (ii) holds as well, a contradiction with the disentangling lemma 9.3. $\triangledown$

Subclaim 2. *Fix $j \in J$. Then there exist an $s't$-dipath P and a $u_j t$-dipath Q in $\vec{H} \setminus s$ that are internally vertex-disjoint.*

Proof of Subclaim. Suppose otherwise. Then by Menger's theorem there exists a vertex $v \in V(\vec{H} \setminus s) - \{s', t\}$ such that there is no $s't$-dipath in $\vec{H} \setminus \{s, v\}$. Note that $v \in V(C_1)$, since C_1 and $P_2[v_2, t]$ have a vertex in common. One of the following holds:

(a) there exists an $s'v$-dipath R in $\vec{H} \setminus s$ such that $R \cup \{\Omega\}$ has no k-mate:
Let $I_d := R$, $I_c := \bigcup(Q_i[s', v] : i \in [3]) - R$, $T' := \{s, t\}$, for $i \in [3]$ let $L'_i := L_i - (I_c \cup I_d)$, and let $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of $L'_1, L'_2, L'_3, P_4, \dots, P_m$.

(b) for every $s'v$-dipath R in $\vec{H} \setminus s$, $R \cup \{\Omega\}$ has a k-mate:
Let $I_d := \emptyset$, $I_c := P_1 \cup P_2[v, t] \cup P_3[v, t] \cup P_4 \cup \dots \cup P_m$, $T' := \emptyset$, for $i \in [3]$ let $L'_i := Q_i[s', v] \cup \{\Omega\}$, and let $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of L'_1, L'_2, L'_3.

It is not difficult to check that in either of the cases above, (1)-(4) and (i), (ii) of the disentangling lemma 9.3 hold, which is the desired contradiction. ▽

Together with ($\diamond$), subclaim 2 implies that $J \neq \{1, 2, 3\}$, so because $|J| \geq 2$, we get that $|J| = 2$. We may assume that $J = \{1, 2\}$. Let $I_d := \emptyset$, $I_c := P_1 \cup Q \cup P_4 \cup \dots \cup P_m$, $T' := \emptyset$, $L'_1 := Q_1[s', u_1] \cup \{\Omega\}$, $L'_2 := Q_2[s', u_2] \cup \{\Omega\}$, $L'_3 := P \cup \{\Omega\}$, and let $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of L'_1, L'_2, L'_3. It is not difficult to check that (1)-(4) and (i), (ii) of the disentangling lemma 9.3 hold, which is a contradiction. ◇

Claim 2. *There exists $j \in [m] - \{2, 3\}$ such that $P_j \cup \{\Omega\}$ has no k-mate.*

Proof. Suppose otherwise. Observe that P_2, P_3, being odd st-dipaths in $\vec{H}$ Ω-disjoint from C_1, have k-mates. Thus by the mate proposition 8.4 there exists $i \in [m]$ such that the k-mate of $P_i \cup \{\Omega\}$ is an st-cut, contradicting propositions 9.1 and 9.2. ◇

By swapping the roles of P_1 and P_j in $\mathcal{L}$, if necessary, we may assume that $j = 1$.

Claim 3. *For each $i \in I$, $P_i[v_i, t] \cup \{\Omega\}$ has a k-mate. In particular, $1 \notin I$.*

Proof. Suppose otherwise. Let $I_d := P_i[v_i, t]$ and $I_c := \bigcup(P_j[v_j, t] : j \in I - \{i\})$. Let $T' := \{s, t\}$, for $j \in [3]$ let $L'_j := L_j - (I_c \cup I_d)$, and for $j \in [m] - [3]$ let $P'_j := P_j - (I_c \cup I_d)$. Let $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of $L'_1, L'_2, L'_3, P'_4, \dots, P'_m$. It is clear that (1)-(4) of the disentangling lemma 9.3 hold. By assumption, $I_d \cup \{\Omega\}$ has no k-mate, so (i) holds. Let L' be a directed odd st-join of $\vec{H}'$ that is Ω-disjoint from a directed odd circuit, i.e. L' is an odd st-dipath of $\vec{H}'$ by remark 8.9(2). Then it is clear that $L' \cup I_d$ contains an odd st-dipath of $\vec{H}$, which by remark 8.9(2) is Ω-disjoint from a directed odd circuit, so (ii) holds as well, a contradiction with the disentangling lemma 9.3. ◇

Claim 4. *Fix $i \in I$. Then there exists an $s'v_i$-dipath in $\vec{H} \setminus C_1$ that is vertex-disjoint from P_1.*

Proof. Let v be the second smallest vertex in P_1 for which there exists a vv_i-dipath R in $\vec{H}$ such that $V(R) \cap V(P_1) = \{v\}$. Since R is contained in an even st-dipath, namely $P_1[s,v] \cup R \cup P_i[v_i,t]$, it follows from remark 9.5 that R and C_1 have no vertex in common. Suppose for a contradiction that there is no $s'v_i$-dipath in $\vec{H} \setminus C_1$ that is vertex-disjoint from P_1. This fact, together with our choice of v and R, implies the following:

> ($\star$) if $w \in V(R)$ and Q is an $s'w$-dipath in $\vec{H} \setminus s$, then Q and $P_1[v,t]$ have a vertex in common.

Let $I_d := P_1[v,t]$ and $I_c := R \cup [\bigcup(P_j[v_j,t] : j \in I)]$. For $i \in [3]$ let L'_i be $L_i - (I_c \cup I_d)$ minus any directed circuit that does not use Ω, and for $i \in [m] - [3]$ let P'_i be $P_i - (I_c \cup I_d)$ minus any directed circuit that does not use Ω. Let $T' := \{s,t\}$ and $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of $L'_1, L'_2, L'_3, P'_4, \ldots, P'_m$. It is not hard to see that (1)-(4) of the disentangling lemma 9.3 hold. By claim 2, $I_d \cup \{\Omega\}$ has no k-mate, so (i) holds. Let L' be a directed odd T'-join of $\vec{H}'$ that is Ω-disjoint from a directed odd circuit, i.e. L' is an odd st-dipath of $\vec{H}'$ by remark 8.9(2). Then $L' \cup I_c$ contains an odd st-dipath L of $\vec{H}$. Choose $w \in V(R)$ (if any) such that L contains an $s'w$-dipath Q in $\vec{H}$ and $V(Q) \cap V(R) = \{w\}$. Then ($\star$) implies that $(L - I_c) \cup I_d$, and therefore $L' \cup I_d$, contains an odd st-dipath of $\vec{H}$, which by remark 8.9(2) is Ω-disjoint from a directed odd circuit, so (ii) holds as well, a contradiction with the disentangling lemma 9.3. ◇

After redefining $\mathcal{L}$, if necessary, we may assume that $3 \in I$ and that $P_3[s', v_3]$ is vertex-disjoint from P_1. (See remark 8.9(1).)

Claim 5. *$(G, \Sigma, \{s,t\})$ has an F_7 minor.*

Proof. For $i \in I$, let B_i be a k-mate of $P_i[v_i,t] \cup \{\Omega\}$, whose existence is guaranteed by claim 3. For each $i \in I$, since B_i is also a k-mate for odd st-dipath $P_3[s,v_3] \cup P_i[v_i,t]$, proposition 9.1 implies that B_i is a signature. Take $j \in I - \{3\}$. Choose $U \subseteq V(G) - \{s,t\}$ such that $B_3 \triangle B_j = \delta(U)$. Then by proposition 8.7 there exists a path P in $G[U]$ between $V(P_3[v_3,t]) \cap U$ and $V(P_j[v_j,t]) \cap U$ such that $P \cap (B_3 \cup B_j) = \emptyset$, and P is minimal subject to this property. Observe that $L_1 \cup P_3[s', v_3]$ has no vertex in common with U. It is easy (and is left as an exercise) to see that $C_1 \cup P_1 \cup P_3 \cup P_j[v_j,t] \cup P$ has an F_7 minor. ◇

13. A preliminary for simple bipartite and cut Ω-systems: the linkage lemma

The presentation of this section follows Thomassen [17]. Let H_0 be a plane graph such that the unbounded face is bounded by a circuit C_0 on four vertices s_1, s_2, t_1, t_2, in this cyclic order. Suppose further that every other face is bounded by a triangle, and every triangle is a facial circuit. For each triangle Δ of H_0 we add K^Δ, a possibly empty complete graph disjoint from H_0, and we join all its vertices to all the vertices of Δ. The resulting graph is called an (s_1, s_2, t_1, t_2)-*web* with *frame* C_0 and *rib* H_0.

Lemma 13.1 ([13, 17]). *Let H be a graph and take four distinct vertices s_1, s_2, t_1, t_2. Suppose there are no two vertex-disjoint paths P_1, P_2 such that, for $i = 1, 2$, P_i is an $s_i t_i$-path. Then H is a spanning subgraph of an (s_1, s_2, t_1, t_2)-web.*

14. Simple bipartite Ω-system of flavour (SF1)

In this section, we prove proposition 2.11.

14.1. A disentangling lemma.

Lemma 14.1. *Let $\big((G, \Sigma, \{s, t\}), \mathcal{L} = (L_1, \ldots, L_k), m, \vec{H}\big)$ be a minimal simple bipartite Ω-system of flavour (SF1), where $\Omega \in \delta(s)$, and assume there is no non-simple bipartite Ω-system whose associated signed graft is a minor of $(G, \Sigma, \{s, t\})$. Take disjoint subsets $I_d, I_c \subseteq E(\vec{H} \setminus \Omega)$ and $T' \subseteq \{s, t\}$ where*

(1) I_c is non-empty, if I_c contains an st-path then $T' = \emptyset$, and if not then $T' = \{s, t\}$,

(2) every signature or st-cut disjoint from I_c intersects I_d in an even number of edges,

(3) if $T' = \emptyset$, there is a directed subgraph $\vec{H}'$ of $\vec{H}/I_c \setminus I_d$ that is the union of directed odd circuits L'_1, L'_2, L'_3 where

$\Omega \in L'_1 \cap L'_2 \cap L'_3$ and L'_1, L'_2, L'_3 are pairwise Ω-disjoint,

$\vec{H}' \setminus \Omega$ is acyclic,

(4) if $T' = \{s, t\}$, there is a directed subgraph $\vec{H}'$ of $\vec{H}/I_c \setminus I_d$ that is the union of odd st-dipaths L'_1, L'_2, L'_3 and even st-dipaths $P'_4, \ldots, P'_m$, where

$\Omega \in L'_1 \cap L'_2 \cap L'_3$, $\Omega \notin P'_4 \cup \ldots \cup P'_m$ and $L'_1, L'_2, L'_3, P'_4, \ldots, P'_m$ are pairwise Ω-disjoint,

$\vec{H}'$ is acyclic.

Then one of the following does not hold:

(i) $I_d \cup \{\Omega\}$ does not have a k-mate,

(ii) for every directed odd T'-join L' of $\vec{H}'$, $L' \cup I_d$ contains an odd st-dipath of $\vec{H}$.

Proof. Suppose otherwise. Let $(G', \Sigma', T') := (G, \Sigma, \{s, t\})/I_c \setminus I_d$ where $\Sigma' = \Sigma$; this signed graft is well-defined by (1). For $i \in [m] - [3]$, let $L'_i := L_i - P_i$ if $T' = \emptyset$, and let $L'_i := (L_i - P_i) \cup P'_i$ otherwise. Let $\mathcal{L}' := (L'_1, \ldots, L'_m, L_{m+1}, \ldots, L_k)$. If $T' = \emptyset$, let $m' := 3$, and if not, let $m' := m$. We will show that $((G', \Sigma', T'), \mathcal{L}', m', \vec{H}')$ is either a non-simple bipartite Ω-system or a simple bipartite Ω-system, and this will yield a contradiction, thereby finishing the proof.

(B1) By (2), every signature or T'-cut of (G',Σ',T') has the same parity as $\tau(G,\Sigma,\{s,t\})$, implying that (G',Σ',T') is an Eulerian signed graft. **(B2)** It also implies that $k,\tau(G,\Sigma,\{s,t\})$ and $\tau(G',\Sigma',T')$ have the same parity, so every minimal cover of (G',Σ',T') has the same size parity as k. We claim that $\tau(G',\Sigma',T')\geq k$. Let B' be a minimal cover of (G',Σ',T'). If $\Omega\notin B'$, then

$$|B'|\geq \sum\left(|B'\cap L'|: L'\in\mathcal{L}'\right)\geq k.$$

Otherwise, $\Omega\in B'$. In this case, $B'\cup I_d$ contains a cover B of $(G,\Sigma,\{s,t\})$. By (i), $I_d\cup\{\Omega\}$ does not have a k-mate, so

$$k-2\leq |B-(I_d\cup\{\Omega\})|\leq |B-I_d|-1\leq |B'|-1,$$

and since $|B'|,k$ have the same parity, it follows that $|B'|\geq k$. Thus, $\mathcal{L}'$ is an (Ω,k)-packing. When $T'=\emptyset$, $m'=3$. When $T'=\{s,t\}$, then $m'=m$ and for $j\in[m']-[3]$, L'_j contains even st-path P'_j and some odd circuit in $L'_j-P'_j$, and for $j\in[k]-[m']$, L_j remains connected in G'. **(B3)** follows from construction.

Suppose first that $T'=\emptyset$. We will show that $((G',\Sigma',T'),\mathcal{L}',m',\vec{H}')$ is a non-simple bipartite Ω-system. **(NS1)** holds as (B1)-(B3) hold. **(NS2)** holds as $T'=\emptyset$. **(NS3)** follows from (3). **(NS4)** Let L' be a directed odd T'-join of $\vec{H}'$ that is Ω-disjoint from a directed odd circuit. By (ii), $L'\cup I_d$ contains an odd st-dipath L of $\vec{H}$. Since $((G,\Sigma,\{s,t\}),\mathcal{L},m,\vec{H})$ is of flavour (SF1), L has a signature k-mate B. By proposition 8.3, $B\cap E(\vec{H})=B\cap L$, implying that $B\cap I_c=\emptyset$. Thus, $B-I_d$ contains a minimal cover B' of (G',Σ',T'), and since

$$|B'-L'|\leq |(B-I_d)-L'|\leq |B-L|\leq k-3,$$

it follows that B' is a k-mate of L'.

Suppose now that $T'=\{s,t\}$. We will show that $((G',\Sigma',\{s,t\}),\mathcal{L}',m,\vec{H}')$ is a simple bipartite Ω-system. **(S1)** holds as (B1)-(B4) hold. **(S2)** follows from (4). **(S3)** Let L' be an odd st-dipath in $\vec{H}'$. By (ii), $L'\cup I_d$ contains an odd st-dipath L of $\vec{H}$. Since $((G,\Sigma,\{s,t\}),\mathcal{L},m,\vec{H})$ is a simple bipartite Ω-system of flavour (SF1), L has a signature k-mate B. By proposition 8.3, $B\cap E(\vec{H})=B\cap L$, implying that $B\cap I_c=\emptyset$. Then $B-I_d$ contains a minimal cover B' of $(G',\Sigma',\{s,t\})$, and since

$$|B'-L'|\leq |(B-I_d)-L'|\leq |B-L|\leq k-3,$$

it follows that B' is a k-mate of L'. □

14.2. **The proof of proposition 2.11.** Let $\big((G,\Sigma,\{s,t\}),(L_1,\ldots,L_k),m,\vec{H}\big)$ be a minimal simple bipartite Ω-system of flavour (SF1), where Ω has ends s,s', and assume there is no non-simple bipartite Ω-system whose associated signed graft is a minor of $(G,\Sigma,\{s,t\})$.

Claim 1. *$m \geq 4$.*

Proof. By (SF1), each one of P_1, P_2, P_3 has a signature k-mate, so the result follows from the mate proposition 8.4. $\diamond$

Claim 2. *There is an odd circuit C in $\vec{H} \setminus t$.*

Proof. Suppose otherwise. Let $I_c := P_4$ and $I_d := \emptyset$. Let $T' := \emptyset$, for $i \in [3]$ let $L'_i := P_i$, and let $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of L'_1, L'_2, L'_3. It is clear that (1)-(4) of the disentangling lemma 14.1 hold. As $\tau(G, \Sigma, \{s,t\}) \geq k$, it follows that $I_d \cup \{\Omega\} = \{\Omega\}$ does not have a k-mate, so (i) holds. Moreover, our assumption implies that P_4 is internally vertex-disjoint from each of P_1, P_2, P_3. This implies that every directed odd circuit of $\vec{H}'$ is an odd st-dipath in $\vec{H}$, so (ii) holds, a contradiction with the disentangling lemma 14.1. $\diamond$

Consider the following partial ordering on $V(\vec{H})$: $u \leq v$ if there exists a uv-dipath in $\vec{H}$. For $j \in [m]$ let v_j be the second largest vertex of P_j that lies on another st-dipath in $\{P_1, \ldots, P_m\} - \{P_j\}$. By proposition 8.8 there exists an index subset $I \subseteq [m]$ of size at least two such that, for each $i \in I$,

- $v_i \geq v_1$, and there is no $j \in [m]$ such that $v_j > v_i$,
- for each $j \in [m]$, $v_i = v_j$ if and only if $j \in I$.

After redefining $\mathcal{L}$, if necessary, we may assume that $1 \in I$.

Claim 3. *For each $i \in I$, $P_i[v_i, t] \cup \{\Omega\}$ has a k-mate.*

Proof. Let $I_d := P_i[v_i, t]$ and $I_c := \bigcup(P_j[v_j, t] : j \in I - \{i\})$. Let $T' := \{s,t\}$, for $j \in [3]$ let $L'_j := P_j - (I_c \cup I_d)$, and for $j \in [m] - [3]$ let $P'_i := P_i - (I_c \cup I_d)$. Let $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of $L'_1, L'_2, L'_3, P'_4, \ldots, P'_m$. Clearly, (1)-(4) and (ii) of the disentangling lemma 14.1 hold. Hence the lemma implies that (i) does not hold, proving the claim. $\diamond$

Claim 4. *There are two vertex-disjoint paths P, Q in H, where P is between s, t and Q is between s', v_1.*

Proof. Suppose otherwise.

Assume first that $s' = v_1$. Then, for each $j \in [m]$, $s' \in V(P_j)$. By (SF1), for each $j \in [m]$, $P_j[s', t] \cup \{\Omega\}$ has a signature k-mate B_j. However, for each $j \in [m]$, B_j is also a signature k-mate for $P_j \cup \{\Omega\}$. This is a contradiction with the mate proposition 8.4.

Thus, $s' \neq v_1$. By the linkage lemma 13.1, H is a spanning subgraph of an (s, v_1, t, s')-web with frame C_0 and rib H_0. Fix a plane drawing of H_0, where the unbounded face is bounded by C_0. After redefining $\mathcal{L}$, if necessary, we may assume the following:

($\star$) for every $s'v_1$-dipath P of $\vec{H}$, the number of rib vertices that are on the same side of P as s is at least as large as that of $P_1[s', v_1]$.

For $j \in [m] - [3]$, let u_j be the largest rib vertex on P_j that also lies on $P_1[s', v_1]$. Observe that if $j \in I \cap ([m] - [3])$, then $u_j = v_j$. For $j \in [m] - [3]$ let $R_j := P_j[u_j, t]$, for $j \in [3] \cap I$ let $R_j := P_j[v_j, t]$, and for $j \in [3] - I$ let $R_j := P_j[s', t]$. Observe that a k-mate for $R_j \cup \{\Omega\}, j \in [m]$ is also a k-mate for any odd st-dipath of $\vec{H}$ containing $R_j \cup \{\Omega\}$. Hence, by (SF1), every k-mate for $R_j \cup \{\Omega\}, j \in [m]$ must be a signature. However, every k-mate for $R_j \cup \{\Omega\}, j \in [m]$ is also a k-mate for $P_j \cup \{\Omega\}$. Hence, by the mate proposition 8.4, there exists $i \in [m]$ such that $R_i \cup \{\Omega\}$ has no k-mate. By (S3) and claim 3, $i \notin I \cup [3]$. Observe that ($\star$) implies the following:

($\star\star$) if $w \in V(P_1[u_i, t])$ and Q is an $s'w$-dipath, then Q and R_i have a vertex in common.

Let $I_d := R_i$ and $I_c := P_1[u_i, t]$. For $j \in [3]$ let L'_j be $P_j - (I_c \cup I_d)$ minus any directed circuit that does not use Ω, and for $j \in [m] - [3]$ let P'_j be $P_j - (I_c \cup I_d)$ minus any directed circuit that does not use Ω. Let $T' := \{s, t\}$ and $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of $L'_1, L'_2, L'_3, P'_4, \ldots, P'_m$. It is clear that (1)-(4) of the disentangling lemma 14.1 hold. By the choice of R_i, (i) holds as well. To show (ii) holds, let L' be an odd st-dipath of $\vec{H}'$. Then $L' \cup I_c$ contains an odd st-dipath of $\vec{H}$, and by ($\star\star$), $L' \cup I_d$ also contains an odd st-dipath of $\vec{H}$, so (ii) holds, a contradiction with lemma 14.1. $\diamond$

Claim 5. *$(G, \Sigma, \{s, t\})$ has an F_7 minor.*

Proof. For $i \in I$, let B_i be a k-mate of $P_i[v_i, t] \cup \{\Omega\}$, whose existence is guaranteed by claim 3. For each $i \in I$, since B_i is also a k-mate for odd st-dipath $P_1[s, v_1] \cup P_i[v_i, t]$, (SF1) implies that B_i is a signature. Take $j \in I - \{1\}$. Choose $U \subseteq V(G) - \{s, t\}$ such that $B_1 \triangle B_j = \delta(U)$. Then by proposition 8.7 there exists a path R in $G[U]$ between $V(P_1[v_1, t]) \cap U$ and $V(P_j[v_j, t]) \cap U$ such that $R \cap (B_1 \cup B_j) = \emptyset$, and R is minimal subject to this property. Observe that $P \cup Q \cup C$ has no vertex in common with U. It is easy (and is left as an exercise) to see that $C \cup P \cup Q \cup P_1[v_1, t] \cup P_j[v_j, t] \cup R$ has an F_7 minor. $\diamond$

15. A preliminary for cut Ω-systems: the shore proposition

The following proposition can be the thought of as the second half of the mate proposition 8.4:

Proposition 15.1. *Let $((G, \Sigma, \{s, t\}), \mathcal{L} = (L_1, \ldots, L_k), m, H)$ be a bipartite Ω-system, where Ω has ends s, s'. For each $i \in [m]$, let $\widetilde{P}_i \subseteq L_i$ be a connected st-join such that $\widetilde{P}_i \cap \Sigma \subseteq \{\Omega\}$, and if $i \in [3]$, $\Omega \in \widetilde{P}_i$ and $\widetilde{P}_i \cap \delta(s) = \{\Omega\}$. Suppose there exist $B_1, \ldots, B_m$ and $U \subseteq V(G) - \{t\}$ with $s \in U$ such that* *(i) for $i \in [m]$, B_i is a k-mate of $\widetilde{P_i} \cup \{\Omega\}$,*

(ii) exactly one of $B_1, \dots, B_m$, say B_ℓ, is not a signature, and $B_\ell = \delta(U)$,

(iii) there is no proper subset W of U with $s \in W$ such that $\delta(W)$ is a k-mate of $\widetilde{P_\ell} \cup \{\Omega\}$,

(iv) for $i \in [m]$, $B_i \cap P_i$ has no edge in $G[U]$.

Then, for every component of $\widetilde{P_\ell}$ in $G[U]$, there is a path P in $G[U]$ between s and a vertex of the component such that $P \cap (B_1 \cup \cdots \cup B_{\ell-1} \cup B_{\ell+1} \cup \cdots \cup B_m) = \emptyset$.

Proof. For each $i \in [3]$, let $\widetilde{C_i} := \emptyset$, and for each $i \in [m] - [3]$, let $\widetilde{C_i}$ be an odd circuit contained in the odd cycle $L_i \triangle \widetilde{P}_i = L_i - \widetilde{P}_i$. By identifying a vertex of each component with s, if necessary, we may assume that G is connected. For $n \geq 1$, let $[n]' := [n] - \{\ell\}$. We will be applying lemma 8.5 to the index set $[m]'$. For distinct $i, j \in [m]'$, choose $U_{ij} \subseteq V(G) - \{s, t\}$ such that $\delta(U_{ij}) = B_i \triangle B_j$. Observe that $[m]'$ contains $m-1$ ordered indices; for every index j other than the two smallest indices in $[m]'$, let

$$S_j := \bigcap (U_{ij} : i \in [m]', i < j).$$

By definition, each S_j is the intersection of at least two sets. Take $C \in \{\widetilde{C_4}, \dots, \widetilde{C_m}\}$ and an S_j. We say C *is bad for* S_j if

$$|C \cap \delta(S_j)| = 2 \quad \text{and} \quad C \cap \delta(S_j) \cap B_j = \emptyset.$$

We need a few preliminaries.

Claim 1. *Each circuit in $\{C_4, \dots, C_m\}$ is bad for at most one S_j.*

Proof. Suppose that $C \in \{C_4, \dots, C_m\}$ is bad for S_j and that it is not bad for any S_i with $i < j$. By lemma 8.5(5), there exist distinct $p, q \in [j-1]'$ such that $C \cap \delta(S_j) \subseteq B_p \cup B_q$. By lemma 8.5(11), $V(C) \subseteq U_{jp} \cup U_{jq}$, and subsequently by lemma 8.5(6), $V(C) \cap S_r = \emptyset$ for $r > j$. As a result, C cannot be bad for any S_r with $r > j$. $\diamond$

Claim 2. *Each S_j has a bad circuit.*

Proof. Suppose for a contradiction that some S_j has no bad circuit, and let $B := B_j \triangle \delta(S_j)$. We will prove that B is a cover of size $k-2$, which will yield a contradiction as $|B| \geq \tau(G, \Sigma) \geq k$. It is clear that B is a cover. By lemma 8.5,

$$B \subseteq \bigcup (B_i : i \in [m]', i \leq j) \subseteq \bigcup (L_i : i \in [k]').$$

The first inclusion follows from part (5), and the second inclusion follows from part (1) together with the fact that for each $i \in [m]'$, $B_i \cap \widetilde{P_\ell} \subseteq \{\Omega\}$. Therefore, as $\Omega \in B$ and $|L_\ell \cap B| = 1$, it suffices to show that, for all $i \in [k]'$, $|L_i \cap B| = 1$. Keep in mind that, for all $i \in [k] - \{j\}$, $|L_i \cap B_j| = 1$.

Take $i \in [k]-[m]$. If $L_i \cap \delta(S_j) = \emptyset$, then $|L_i \cap B| = |L_i \cap B_j| = 1$. Otherwise, when $L_i \cap \delta(S_j) \neq \emptyset$, lemma 8.5 part (9) implies $|L_i \cap \delta(S_j)| = 2$ and $|L_i \cap \delta(S_j) \cap B_j| = 1$, so $|L_i \cap B| = |L_i \cap (B_j \triangle \delta(S_j))| = 1$.

Next take $i \in [m]'$. We will first consider $\widetilde{C_i} \cap B$, given that $\widetilde{C_i} \neq \emptyset$. If $\widetilde{C_i} \cap \delta(S_j) = \emptyset$, then $|\widetilde{C_i} \cap B| = |\widetilde{C_i} \cap B_j| = 1$. Otherwise, $\widetilde{C_i} \cap \delta(S_j) \neq \emptyset$. Then, by lemma 8.5(10), $|\widetilde{C_i} \cap \delta(S_j)| = 2$. By our choice of S_j, $\widetilde{C_i}$ is not bad for S_j, so $|\widetilde{C_i} \cap \delta(S_j) \cap B_j| = 1$. Thus, $|\widetilde{C_i} \cap B| = |\widetilde{C_i} \cap (B_j \triangle \delta(S_j))| = 1$. We next consider $(\{\Omega\} \cup \widetilde{P_i}) \cap B$. If $i \neq j$, then by lemma 8.5,

$$\begin{aligned}(\{\Omega\} \cup \widetilde{P_i}) \cap B &= (\{\Omega\} \cup \widetilde{P_i}) \cap (B_j \triangle \delta(S_j)) \\ &= (\{\Omega\} \cup \widetilde{P_i}) \cap B_j \quad \text{by part (8)} \\ &= \{\Omega\} \quad \text{by part (3).}\end{aligned}$$

On the other hand, if $i = j$, then by lemma 8.5,

$$\begin{aligned}(\{\Omega\} \cup \widetilde{P_j}) \cap B &= (\{\Omega\} \cup \widetilde{P_j}) \cap (B_j \triangle \delta(S_j)) \\ &= [(\{\Omega\} \cup \widetilde{P_j}) \cap B_j] \triangle [(\{\Omega\} \cup \widetilde{P_j}) \cap \delta(S_j)] \\ &= \{\Omega\} \quad \text{by part (7).}\end{aligned}$$

Since whenever $\Omega \in \widetilde{P_i}$ then $\widetilde{C_i} = \emptyset$, $|L_i \cap B| = |\widetilde{C_i} \cap B| + |\widetilde{P_i} \cap B| = 1$. ◇

Let $\mathfrak{U} := \bigcup (U_{ij} : i, j \in [m]', i \neq j)$.

Claim 3. *For each $j \in [m]-[3]$, $V(\widetilde{C_j}) \subseteq \mathfrak{U}$*

Proof. Claims 1 and 2 imply that each circuit of $\widetilde{C_4}, \ldots, \widetilde{C_m}$ is bad for an S_j (of which there are $m-3$ many). The claim now follows from lemma 8.5(11). ◇

Claim 4. *Let $e \in E(G)$ be an edge with both ends in $V(G) - \mathfrak{U}$, and let $i \in [m]'$. If $e \in B_i$, then $e \in B_1 \cap \cdots \cap B_{\ell-1} \cap B_{\ell+1} \cap \cdots \cap B_m$.*

Proof. As e has both ends in $V(G) - \mathfrak{U}$, for each distinct $p, q \in [m]'$, we have $e \notin \delta(U_{pq}) = B_p \triangle B_q$, proving the claim. ◇

Claim 5. *Let $e \in E(G)$ be an edge with both ends in $U - \mathfrak{U}$ such that $e \in B_1 \cup \cdots \cup B_{\ell-1} \cup B_{\ell+1} \cup \cdots \cup B_m$. Then $e \in L_{m+1} \cup \cdots \cup L_k$.*

Proof. As $e \neq \Omega$, $e \notin L_1 \cup L_2 \cup L_3$. By (iv), $e \notin \widetilde{P_4} \cup \cdots \cup \widetilde{P_m}$. By claim 3, $e \notin \widetilde{C_4} \cup \cdots \cup \widetilde{C_m}$. The claim now follows from proposition 3.1. ◇

Claim 6. *For each $i \in [m]$, $\widetilde{P_i}$ has no vertex in common with $U \cap \mathfrak{U}$.*

Proof. Observe that $\widetilde{P_\ell}$ has no vertex in common with $\mathfrak{U}$, for $\widetilde{P_\ell} \cap \delta(\mathfrak{U}) = \emptyset$ and $\widetilde{P_\ell}$ is connected. We may therefore assume $i \in [m]'$, and for a contradiction, assume $\widetilde{P_i}$ has a vertex v in common with $U \cap \mathfrak{U}$. Since Since $|\widetilde{P_i} \cap \delta(U)| = 1$, the edges of $\widetilde{P_i}[s,v]$ belong to $G[U]$, so by (iv), $\widetilde{P_i}[s,v] \cap B_i = \emptyset$. Since $u \in \mathfrak{U}$, there exist distinct $p,q \in [m]'$ such that $u \in U_{pq}$. Since $\widetilde{P_i}[s,v] \cap B_i = \emptyset$, we may assume that $p \neq i$ and $\widetilde{P_i}[s,v] \cap B_p \neq \emptyset$. However, as B_p is a signature, $\widetilde{P_i} \cap B_p \subseteq \{\Omega\}$, a contradiction as $\Omega \in \delta(U)$. ◇

Claim 7. *For every component of $\widetilde{P_\ell}$ in $G[U]$, there is a path P in $G[U - \mathfrak{U}]$ between s and a vertex of the component such that $P \cap (B_1 \cup \cdots \cup B_{\ell-1} \cup B_{\ell+1} \cup \cdots \cup B_m) = \emptyset$.*

Proof. Suppose otherwise. By claim 4, there exists $W \subseteq (U - \mathfrak{U}) - \{s\}$ where $\widetilde{P_\ell} \cap \delta(W) \neq \emptyset$ such that, for every edge $e \in E(G)$ with one end in W and another in $(U - \mathfrak{U}) - W$, we have $e \in B_1 \cap \cdots \cap B_{\ell-1} \cap B_{\ell+1} \cap \cdots \cap B_m$. Let $U' := U - W$. We will show that $\delta(U')$ is a cap of L_ℓ in $\mathcal{L}$.

(T1) and **(T2)** clearly hold. **(T3)** We have

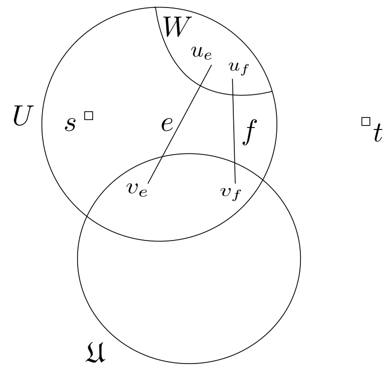
$$\delta(U') \subseteq \delta(U) \cup \delta(W) \subseteq (B_1 \cup \cdots \cup B_m) \cup \delta(\mathfrak{U}) \subseteq B_1 \cup \cdots \cup B_m \subseteq L_1 \cup \cdots \cup L_k.$$

In fact, the argument of the last inclusion can be replaced with

$$(\widetilde{P_1} \cup \cdots \cup \widetilde{P_m}) \cup (\widetilde{C_4} \cup \cdots \cup \widetilde{C_m}) \cup (L_{m+1} \cup \cdots \cup L_k).$$

(T4) Let $i \in [m]'$. When $i \in [3]$, we have $V(L_i) \cap U = \{s\}$, implying that $L_i \cap \delta(U') = \{\Omega\}$. Otherwise, when $i \in [m] - [3]$, claim 3 implies that $\widetilde{C_i} \cap \delta(U') = \emptyset$ and claims 5 and 6 imply that $|\widetilde{P_i} \cap \delta(U')| = |\widetilde{P_i} \cap \delta(U)| = 1$, so $|L_i \cap \delta(U')| = 1$.

Let $i \in [k] - [m]$. Recall that L_i is a connected odd st-join. If $L_i \cap \delta(W) = \emptyset$, then $|L_i \cap \delta(U')| = |L_i \cap \delta(U)| = 1$. We may therefore assume that $L_i \cap \delta(W) \neq \emptyset$. We claim that $|L_i \cap \delta(W)| = 2$ and that one of the edges in $L_i \cap \delta(W)$ belongs to $\delta(U)$. Note that this will prove that $|L_i \cap \delta(U')| = 1$. If $L_i \cap \delta(W)$ contains an edge e with one end in W and another in $U' - \mathfrak{U}$, then $e \in B_1 \cap \cdots \cap B_{\ell-1} \cap B_{\ell+1} \cap \cdots \cap B_m$. However, $|L_i \cap B_1| = \cdots = |L_i \cap B_{\ell-1}| = |L_i \cap B_{\ell+1}| = \cdots = |L_i \cap B_m| = 1$, so $|L_i \cap \delta(W)| = 2$ and the edge in $(L_i \cap \delta(W)) - \{e\}$ belongs to $\delta(U)$, and we are done. Otherwise, it suffices to show that L_i does not contain two edges e, f, each with one end in $U \cap \mathfrak{U}$ and another in W. Suppose otherwise. Let v_e, v_f be the ends of e, f in $U \cap \mathfrak{U}$, respectively, and let u_e, u_f be the ends of e, f in W, respectively.



Since $e,f \in \delta(\mathfrak{U})$, each of e,f belongs to $\cup_{j\in[m]'} B_j$. Since L_i intersects each one of $B_j, j \in [m]'$ exactly once, there are distinct $p,q \in [m]'$ such that $e \in B_p, f \in B_q$ and $\{e,f\} \subseteq B_p \triangle B_q = \delta(U_{pq})$. Since $|L_i \cap \delta(U)| = 1$ and L_i is connected, we get that L_i contains a path Q in $G[U]$ containing the vertex s and edges e,f. Since $L_i \cap \delta(W)$ does not contain an edge with one end in W and another in $U' - \mathfrak{U}$, it follows that $Q \cap \delta(W)$ does not contain an edge with one end in W and another in $U' - \mathfrak{U}$, implying in turn that $|Q \cap \delta(U_{pq})| \geq 3$, so $|L_i \cap \delta(U_{pq})| \geq 3$, a contradiction. Hence, $|L_i \cap \delta(U')| = 1$.

Moreover, $L_\ell \cap \delta(U') \subsetneq L_\ell \cap \delta(U)$, and since $\tau(G,\Sigma) \geq k$, it follows that $|L_\ell \cap \delta(U')| \geq 3$. As a result, (T4) holds, so $\delta(U')$ is a cap of L_ℓ in $\mathcal{L}$. Proposition 3.1 therefore implies that $\delta(U')$ is a k-mate of L_ℓ, but $\delta(U') \cap L_\ell = \delta(U') \cap \widetilde{P_\ell}$, so $\delta(U')$ is a k-mate for $\widetilde{P_\ell}$, a contradiction with (iii). $\diamond$

Note that claim 7 finishes the proof of the shore proposition. $\square$

16. Primary cut Ω-system

16.1. **Signature mates and the brace proposition.**

Proposition 16.1. *Let $((G,\Sigma,\{s,t\}),\mathcal{L} = (L_1,\dots,L_k),m,(U_1,\dots,U_n),\vec{H})$ be a primary cut Ω-system. Let P be an odd st-dipath with $V(P) \cap U_n = \{s\}$, and let B be a k-mate of it. Then B is not an st-cut.*

Proof. After redefining $\mathcal{L}$, if necessary, we may assume that $P = P_2 = L_2$. (Note the acyclicity condition in (C3).) Suppose, for a contradiction, that B is an st-cut. Choose $W \subseteq V(G) - \{t\}$ with $s \in W$ such that $B = \delta(W)$. Since L_2 is simple, it follows that $\delta(U_n \cap W) \cap L_2 = \{\Omega\}$. As the brace and the base of L_1 intersect $\delta(W)$ at only Ω, it follows that $q,d \in U_n - W$, and since the residue of L_1 is a connected qd-join, it follows that $\delta(U_n \cap W) \cap L_1 = \{\Omega\}$, contradicting proposition 3.4 part (4). $\square$

Proposition 16.2. *Let $((G,\Sigma,\{s,t\}),\mathcal{L} = (L_1,\dots,L_k),m,\mathcal{U} = (U_1,\dots,U_n),\vec{H})$ be a minimal cut Ω-system that is primary. Let P^+ be an st-dipath in $\vec{H}^+ \setminus \Omega$. Then P^+ and the brace share no vertex outside U_n.*

Proof. After redefining $\mathcal{L}$, if necessary, we may assume that $P := P^+ \cap E(\vec{H})$ is the base for one of $P_4, \dots, P_m$. (Note the acyclicity condition in (C3).) Suppose for a contradiction that P^+ and the brace share a vertex outside U_n.

In the first case, assume that P is the base for one of $L_{n+3}, \dots, L_m$, say $P = Q_{n+3}$. Let x be the closest vertex to t on Q_{n+3} that belongs to the both of D and $V(G) - U_n$. Let $L_1' := D[s,x] \cup Q_{n+3}[x,t]$ and $L_{n+3}' := (Q_{n+3}[s,x] \cup D[x,d] \cup R \cup Q) \cup C_{n+3}$. Let

$$\mathcal{L}' := (L_1', L_2, L_3, \dots, L_{n+2}, L_{n+3}', L_{n+4}, \dots, L_k).$$

Note that $\mathcal{U}$ is a secondary cut structure for $((G, \Sigma, \{s,t\}), \mathcal{L}', m)$, where the base for L_{n+3}' is Q. Let $\vec{H}' := \vec{H} \setminus (Q_{n+3}[s,x] \cup D[x,d])$. Then it is easily seen that $((G, \Sigma, \{s,t\}), \mathcal{L}', m, \mathcal{U}, \vec{H}')$ is a secondary cut structure, contradicting the minimality of the original Ω-system.

In the remaining case, assume that $P = Q_j$ for some $j \in [n+2] - [3]$. Let x be the closest vertex to t on Q_j that belongs to the both of D and $V(G) - U_n$. Let $L_1' := D[s,x] \cup Q_j[x,t]$ and $L_j' := (R_j \cup P[q_j, x] \cup D[x,d] \cup R \cup Q) \cup C_j$. Let

$$\mathcal{L}' := (L_1', L_2, \dots, L_{j-1}, L_{j+1}, \dots, L_{n+2}, L_j', L_{n+3}, \dots, L_k)$$
$$\mathcal{U}' := (U_1, \dots, U_{j-4}, U_{j-2}, \dots, U_n).$$

Then $\mathcal{U}'$ is a secondary cut structure for $((G, \Sigma, \{s,t\}), \mathcal{L}', m)$, where the base for L_j' is Q, and $\delta(U_n)$ is a k-mate for $L_j' - C_j$. Let $\vec{H}' := \vec{H} \setminus (Q_j[q_j, x] \cup D[x,d])$. Then it is easily seen that $((G, \Sigma, \{s,t\}), \mathcal{L}', m, \mathcal{U}', \vec{H}')$ is a secondary cut structure, contradicting the minimality of the original Ω-system. □

16.2. **A disentangling lemma.**

Lemma 16.3. *Let $((G, \Sigma, \{s,t\}), \mathcal{L} = (L_1, \dots, L_k), m, \mathcal{U} = (U_1, \dots, U_{n-1}, U), \vec{H})$ be a minimal cut Ω-system that is primary, and assume there is no non-simple bipartite Ω-system whose associated signed graft is a minor of $(G, \Sigma, \{s,t\})$. Take disjoint subsets $I_d, I_c \subseteq E(\vec{H} \setminus \Omega)$ and $T' \subseteq \{s,t\}$ where*

(1) I_c is non-empty, if I_c contains an st-path then $T' = \emptyset$, and if not then $T' = \{s,t\}$,

(2) every signature or st-cut disjoint from I_c intersects I_d in an even number of edges,

(3) if $T' = \emptyset$, there is a directed subgraph $\vec{H}'$ of $\vec{H}/I_c \setminus I_d$ that is the union of directed odd circuits L_1', L_2', L_3' where

$\Omega \in L_1' \cap L_2' \cap L_3'$ and L_1', L_2', L_3' are pairwise Ω-disjoint,

$\vec{H}' \setminus \Omega$ is acyclic,

(4) if $T' = \{s,t\}$, then $I_d, I_c \subseteq E(\vec{H} \setminus U)$ and there is a directed subgraph $\vec{H}'$ of $\vec{H}/I_c \setminus I_d$ that is the union of D', Q', odd st-dipaths L_2', L_3', and dipaths $Q_4', \dots, Q_m'$, where

- *D' is an sd-dipath containing Ω with $V(D')\cap U=\{s,d\}$, Q' is a qt-dipath with $V(Q')\cap U=\{q\}$, and D',Q' have no vertex outside U in common,*
- *for $i=4,\dots,n+2$, Q'_i is a $q_{i-3}t$-dipath with $V(Q'_i)\cap U_{i-3}=\{q_{i-3}\}$, and for $i=n+3,\dots,m$, Q'_i is an even st-dipath,*
- *$D',Q',L'_2,L'_3,Q'_4,\dots,Q'_m$ are pairwise Ω-disjoint,*
- *$D',Q',Q'_4,\dots,Q'_m$ coincide with $D,Q,Q_4,\dots,Q_m$ on $E(G[U])\cup\delta(U)$, respectively,*
- *the following digraph is acyclic: start from $\vec{H}'$, for each q_i add arc (s,q_i), and if $d\neq q$, add arc (d,q).*

Then one of the following does not hold:

(i) $I_d\cup\{\Omega\}$ does not have a k-mate,

(ii) if $T'=\emptyset$, then for every directed odd circuit L' of $\vec{H}'$, either $L'\cup I_d$ contains an odd st-dipath P of $\vec{H}$ with $V(P)\cap U=\{s\}$, or $L'\cup I_d$ has a k-mate in $(G,\Sigma,\{s,t\})$ disjoint from I_c,

(iii) if $T'=\{s,t\}$, then for every odd st-dipath P' of $\vec{H}'$ with $V(P')\cap U=\{s\}$, either $P'\cup I_d$ contains an odd st-dipath of $\vec{H}$, or $P'\cup I_d$ has a k-mate in $(G,\Sigma,\{s,t\})$ disjoint from I_c.

Proof. Suppose otherwise. Let $(G',\Sigma',T'):=(G,\Sigma,\{s,t\})/I_c\setminus I_d$ where $\Sigma'=\Sigma$; this signed graft is well-defined by (1). Let $\mathcal{L}':=(L'_1,\dots,L'_m,L_{m+1},\dots,L_k)$, where $L'_1,\dots,L'_m$ are defined as follows. If $T'=\emptyset$, let $m':=3$, and for $i\in[m]-[3]$, let $L'_i:=L_i-P_i$. Otherwise, when $T'=\{s,t\}$, let $m':=m$, $L'_1:=D'\cup Q'\cup R$, and for $i\in[m]-[3]$, let $L'_i:=(L_i-Q_i)\cup Q'_i$.

We will first show that $((G',\Sigma',T'),\mathcal{L}',m')$ is a bipartite Ω-system. **(B1)** By (2), every signature of (G',Σ',T') has the same parity as $\tau(G,\Sigma,\{s,t\})$, implying that (G',Σ',T') is an Eulerian signed graft. **(B2)** It also implies that $k,\tau(G,\Sigma,\{s,t\})$ and $\tau(G',\Sigma',T')$ have the same parity, so every minimal cover of (G',Σ',T') has the same size parity as k. We claim that $\tau(G',\Sigma',T')\geq k$. Let B' be a minimal cover of (G',Σ',T'). If $\Omega\notin B'$, then

$$|B'|\geq\sum\left(|B'\cap L'|:L'\in\mathcal{L}'\right)\geq k.$$

Otherwise, $\Omega\in B'$. In this case, $B'\cup I_d$ contains a cover B of $(G,\Sigma,\{s,t\})$. By (i), $I_d\cup\{\Omega\}$ does not have a k-mate, so

$$k-2\leq|B-(I_d\cup\{\Omega\})|\leq|B-I_d|-1\leq|B'|-1,$$

and since $|B'|,k$ have the same parity, it follows that $|B'|\geq k$. Thus, $\mathcal{L}'$ is an (Ω,k)-packing. When $T'=\emptyset$ then $m'=3$. When $T=\{s,t\}$, then $m'=m$ and for $j\in[m']-[3]$, L'_j contains an even st-path in the bipartite st-join L'_j-C_j and some odd circuit in C_j, and for $j\in[k]-[m']$, L_j remains connected in G'. **(B3)** follows from construction.

Suppose first that $T' = \emptyset$. We will show that $((G', \Sigma', \emptyset), \mathcal{L}', 3, \vec{H}')$ is a non-simple bipartite Ω-system, yielding a contradiction. **(NS1)** holds as (B1)-(B3) hold. **(NS2)** holds as $T' = \emptyset$. **(NS3)** follows from (3). **(NS4)** Let L' be a directed odd circuit of $\vec{H}'$. If $L' \cup I_d$ has a k-mate B in $(G, \Sigma, \{s, t\})$ disjoint from I_c, then $B - I_d$ contains a minimal cover B' of $(G', \Sigma', \emptyset)$, and since

$$|B' - L'| \leq |(B - I_d) - L'| = |B - (L' \cup I_d)| \leq k - 3,$$

it follows that B' is a k-mate of L'. Otherwise by (ii) $L' \cup I_d$ contains an odd st-dipath P of $\vec{H}$ with $V(P) \cap U = \{s\}$. Since $((G, \Sigma, \{s, t\}), \mathcal{L}, m, \mathcal{U}, \vec{H})$ is a primary cut Ω-system, P has a k-mate B which by proposition 16.1 is a signature. By proposition 8.3, $B \cap E(\vec{H}) = B \cap P$, implying that $B \cap I_c = \emptyset$. Thus, $B - I_d$ contains a minimal cover B' of $(G', \Sigma', \emptyset)$, and since

$$|B' - L'| \leq |(B - I_d) - L'| \leq |B - P| \leq k - 3,$$

it follows that B' is a k-mate of L'.

Suppose otherwise that $T' = \{s, t\}$. To obtain a contradiction, we will show that $((G', \Sigma', \{s, t\}), \mathcal{L}', m, \mathcal{U}, \vec{H}')$ is a primary cut Ω-system. **(C1)** holds because (B1)-(B3) are true. **(C2)**-**(C3)** follow from (4). **(C4)** Let P' be an odd st-dipath in $\vec{H}'$ with $V(P') \cap U = \{s\}$. If $P' \cup I_d$ has a k-mate B in $(G, \Sigma, \{s, t\})$ disjoint from I_c, then $B - I_d$ contains a minimal cover B' of $(G', \Sigma', \{s, t\})$, and since

$$|B' - P'| \leq |(B - I_d) - P'| = |B - (P' \cup I_d)| \leq k - 3,$$

it follows that B' is a k-mate of P'. Otherwise by (iii) $P' \cup I_d$ contains an odd st-dipath P of $\vec{H}$. As $I_d \subseteq E(\vec{H} \setminus U)$, it follows that $V(P) \cap U = \{s\}$. Since $((G, \Sigma, \{s, t\}), \mathcal{L}, m, \mathcal{U}, \vec{H})$ is a primary cut Ω-system, P has a k-mate B. By proposition 16.1, B is a signature, so by proposition 8.3, $B \cap E(\vec{H}) = B \cap P$, implying that $B \cap I_c = \emptyset$. Thus $B - I_d$ contains a minimal cover B' of $(G', \Sigma', \{s, t\})$, and since

$$|B' - P'| \leq |(B - I_d) - P'| \leq |B - P| \leq k - 3,$$

it follows that B' is a k-mate of P'. □

16.3. **The proof of proposition 2.14.** In this section, we prove proposition 2.14. We assume Ω has ends s, s'. Reset $C_1 := D$ and $Q_1 := Q$. Let Q_1^+ be the st-dipath obtained from Q_1 after adding arc (s, q). For $i = 4, \ldots, n+2$, let Q_i^+ be the st-dipath obtained from Q_i after adding (s, q_{i-3}) to it. Let $\vec{H}^+$ be the union of C_1, arc (d, q) if $d \neq q$, and st-dipaths $Q_1^+, Q_2, Q_3, Q_4^+, \ldots, Q_{n+2}^+, Q_{n+3}, \ldots, Q_m$. For $u, v \in V(Q_1^+ \cup Q_2 \cup Q_3 \cup Q_4^+ \cup \ldots \cup Q_{n+2}^+ \cup Q_{n+3} \cup \ldots \cup Q_m)$, $u \leq v$ if there is a uv-dipath in $Q_1^+ \cup Q_2 \cup Q_3 \cup Q_4^+ \cup \ldots \cup Q_{n+2}^+ \cup Q_{n+3} \cup \ldots \cup Q_m$; this partial ordering is well-defined as $\vec{H}^+$ is acyclic, by (C3). For $i \in [m]$, let v_i be the second largest vertex of the i^{th} st-dipath that lies on one of the other st-dipaths. By proposition 8.8 there exists an index subset $I \subseteq [m]$ of size at least two such that, for each $i \in I$,

- $v_i \geq v_3$, and there is no $j \in [m]$ such that $v_j > v_i$,
- for each $j \in [m]$, $v_i = v_j$ if and only if $j \in I$.

Claim 1. *For each $i \in I$, U and $Q_i[v_i, t]$ have no vertex in common.*

Proof. Suppose otherwise. Among the arcs of $\vec{H}$ in $\delta(U)$, there is only one arc, say e, entering U, and e is the arc in $(C_1 \cap \delta(U)) - \{\Omega\}$. However, $(Q_1 \cup \cdots \cup Q_m) \cap C_1 = \{\Omega\}$, implying that $e \notin \bigcup(Q_j : j \in [m])$. In particular, $Q_i[v_i, t]$ does not enter U, so $v_i \in U$. As $v_i \geq v_3$, there is a $v_3 v_i$-dipath $P \subset \bigcup(Q_j : j \in [m])$. However, $v_3 \in V(Q_3[s', t])$, so $v_3 \notin U$, implying that $e \in P \subset \bigcup(Q_j : j \in [m])$, a contradiction. ◇

Claim 2. *For each $i \in I$, C_1 and $Q_i[v_i, t]$ have no vertex of $V(G) - \{s'\}$ in common.*

Proof. Suppose otherwise. Then it follows from the brace proposition 16.2 and the acyclicity of $\vec{H}^+$ that

$$(\diamond) \quad I = \{2, 3\} \quad \text{and} \quad V(Q_i) \cap V(Q_j) \subseteq \{s, t\} \quad \forall\, i \in I, \forall\, j \in [m] - I.$$

Let $X_1 := C_1 - \{\Omega\}, X_2 := Q_2 - \{\Omega\}$ and $X_3 := Q_3 - \{\Omega\}$. For each $i \in [3]$, let u_i be the second smallest vertex of X_i that also lies on one of $\{X_1, X_2, X_3\} - \{X_i\}$. Then by proposition 8.8, there exists an index subset $J \subseteq [3]$ of size at least two such that, for each $j \in J$ and $i \in [3]$, $u_i = u_j$ if and only if $i \in J$. Observe that, for each $j \in J$, $X_j[s', u_j] \subseteq E(\vec{H} \setminus U)$, and as $(\diamond)$ holds, each internal vertex of $X_i[s', u_i]$ has degree 2.

Subclaim 1. *For each $j \in J$, $X_j[s', u_j] \cup \{\Omega\}$ has a k-mate.*

Proof of Subclaim. Suppose otherwise. Let $I_d := X_j[s', u_j]$ and $I_c := \bigcup(X_i[s', u_i] : i \in J - \{j\})$. Let $T' := \{s, t\}$, $D' := C_1 - (I_c \cup I_d)$, and for $i = 2, 3$, let $L'_i := L_i - (I_c \cup I_d)$. Let $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of $D', Q, L'_2, L'_3, Q_4, \ldots, Q_m$. It is clear that (1)-(4) of the disentangling lemma 16.3 hold. By assumption, $I_d \cup \{\Omega\}$ has no k-mate, so (i) holds. However, since each internal vertex of $X_i[s', u_i]$ has degree 2, so (ii) and (iii) hold as well, a contradiction with the disentangling lemma 16.3. ▽

Subclaim 2. *Fix $j \in J$. Then there exist an $s't$-dipath X and a $u_j t$-dipath Y in $\vec{H}$ that are internally vertex-disjoint.*

Proof of Subclaim. Suppose otherwise. Then there exists a vertex $v \in V(\vec{H}) - \{s', t\}$ such that there is no $s't$-dipath in $\vec{H} \setminus v$. Note that $v \in V(C_1)$. By proposition 16.1, one of the following holds:

(a) there exists an $s'v$-dipath Z in $\vec{H}$ such that $Z \cup \{\Omega\}$ has no k-mate:

Let $I_d := Z$, $I_c := \bigcup(X_i[s', v] : i \in [3]) - Z$, $T' := \{s, t\}$, $D' := C_1 - (I_c \cup I_d)$, for $i = 2, 3$ let $L'_i := L_i - (I_c \cup I_d)$, and let $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of $D', Q, L'_2, L'_3, Q_4, \dots, Q_m$.

(b) for every $s'v$-dipath Z in $\vec{H}$, $Z \cup \{\Omega\}$ has a signature k-mate, and $m > 3$:

Let $I_d := \emptyset$, $I_c := Q_2[v, t] \cup Q_3[v, t] \cup Q_4 \cup R_4$, $T' := \emptyset$, for $i \in [3]$ let $L'_i := Q_i[s', v] \cup \{\Omega\}$, and let $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of L'_1, L'_2, L'_3.

(c) for every $s'v$-dipath Z in $\vec{H}$, $Z \cup \{\Omega\}$ has a signature k-mate, and $m = 3$.

It is not difficult to check that in either of the cases (a), (b) above, (1)-(4) and (i) of the disentangling lemma 16.3 hold, and as ($\diamond$) holds, (ii) and (iii) hold as well, which cannot be the case. (For (b), note that $V(R_4) \subseteq U$.) Hence, (c) holds. For each $j \in [3]$, let B_j be a signature k-mate for $Q_j[s', v] \cup \{\Omega\}$, which is also a signature k-mate for L_j. However, this is in contradiction with the mate proposition 8.4. (Observe that L_1 is a connected odd st-join with $L_1 \cap \delta(s) = \{\Omega\}$.) $\triangledown$

Hence, in particular, $|J| = 2$ and after redefining $\mathcal{L}$, if necessary, we may assume $J = \{1, 2\}$ and $X = X_3$.

Subclaim 3. *$m > 3$.*

Proof of Subclaim. By subclaim 1, for $j = 1, 2$, there exists a k-mate B_j of $Q_j[s', u_j] \cup \{\Omega\}$, and by (C4), Q_3 has a k-mate B_3. By proposition 16.1, B_1, B_2, B_3 are signatures, and for $j \in [3]$, B_j is also a k-mate for L_j. The result now follows from the mate proposition 8.4. $\triangledown$

Now let $I_d := \emptyset$, $I_c := Y \cup Q_4 \cup R_4$, $T' := \emptyset$, $L'_1 := Q_1[s', u_1] \cup \{\Omega\}$, $L'_2 := Q_2[s', u_2] \cup \{\Omega\}$, $L'_3 := P_3$, and let $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of L'_1, L'_2, L'_3. (Note that $V(R_4) \subseteq U$.) It is easy to check that (1)-(4) and (i)-(iii) of the disentangling lemma 16.3 hold, which is a contradiction. $\diamond$

Claim 3. *For each $i \in I$, $Q_i[v_i, t] \cup \{\Omega\}$ has a signature k-mate.*

Proof. Suppose otherwise. Since $v_i \geq v_3$, $Q_i[v_i, t] \cup \{\Omega\}$ is contained in an odd st-dipath P such that $V(P) \cap U = \{s\}$. Hence, by proposition 16.1, $Q_i[v_i, t] \cup \{\Omega\}$ has no k-mate at all. Let $I_d := Q_i[v_i, t]$ and $I_c := \bigcup(Q_j[v_j, t] : j \in I - \{i\})$. Let $T' := \{s, t\}$, $Q' := Q_1 - (I_c \cup I_d)$, for $j = 2, 3$ let $L'_j := L_j - (I_c \cup I_d)$, and for $j \in [m] - [3]$ let $Q'_j := Q_j - (I_c \cup I_d)$. Let $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of $D, Q', L'_2, L'_3, Q'_4, \dots, Q'_m$. It is clear that (1)-(4) and (ii), (iii) of the disentangling lemma 16.3 hold. However, $I_d \cup \{\Omega\}$ has no k-mate, so (i) holds, contradicting the disentangling lemma 16.3. $\diamond$

After redefining $\mathcal{L}$, if necessary, we may assume that $3 \in I$.

Claim 4. *There exist vertex-disjoint paths X and Y in $\vec{H}$ such that X is an $s'v_3$-path in $\vec{H} \setminus U$ and Y connects a vertex of U to t.*

Proof. Suppose otherwise.

Assume first that $s' = v_3$. Then, for each $j \in [m]$, $s' \in V(Q_j)$ and by claim 1, $Q_j[s', t]$ has no vertex in common with U. Hence, for each $j \in [m]$, by (C4) and proposition 16.1, $Q_j[s', t] \cup \{\Omega\}$ has a signature k-mate B_j. However, B_1 is also a signature k-mate for L_1, and for each $j \in [m] - [1]$, B_j is also a signature k-mate for $P_j \cup \{\Omega\}$. (Note $P_j - E(G[U])$ contains all the edges of $Q_j - E(G[U])$.) This is a contradiction with the mate proposition 8.4.

Thus, $s' \neq v_3$. Let $\vec{H}^\star$ be the digraph obtained from $\vec{H}$ after shrinking U to a single vertex $u^\star$ and removing all loops. Notice that every odd st-dipath in $\vec{H}$ whose intersection with U is $\{s\}$, is a $u^\star t$-dipath in $\vec{H}^\star$ that uses Ω, and vice-versa. Also, note that the acyclicity condition in (C3) implies that $\vec{H}^\star \setminus u^\star$ is acyclic. By the linkage lemma 13.1, $H^\star$ is a spanning subgraph of a $(u^\star, v_3, t, s')$-web with frame C_0 and rib $H_0^\star$. Fix a plane drawing of $H_0^\star$, where the unbounded face is bounded by C_0. After redefining $\mathcal{L}$, if necessary, we may assume the following:

> ($\star$) for every $s'v_3$-dipath P of $\vec{H}^\star \setminus u^\star$, the number of rib vertices that are on the same side of P as u^* is at least as large as that of $Q_3[s', v_3]$.

For $j \in [m] - \{2, 3\}$, let u_j be the largest rib vertex on Q_j that also lies on $Q_3[s', v_3]$. Observe that if $j \in I \cap ([m] - \{2, 3\})$, then $u_j = v_j$. For $j \in [m] - \{2, 3\}$ let $X_j := Q_j[u_j, t]$, for $j \in \{2, 3\} \cap I$ let $X_j := Q_j[v_j, t]$, and for $j \in \{2, 3\} - I$ let $X_j := Q_j[s', t]$. For each $j \in [m]$, since $X_j \cup \{\Omega\}$ is contained in a $u^\star t$-dipath of $\vec{H}^\star$, proposition 16.1 implies that every k-mate for $X_j \cup \{\Omega\}$ (if any) must be a signature. However, every k-mate for $X_j \cup \{\Omega\}, j \in [m]$ is also a k-mate for $P_j \cup \{\Omega\}$. Hence, by the mate proposition 8.4, there exists $i \in [m]$ such that $X_i \cup \{\Omega\}$ has no k-mate. By (C4) and claim 3, $i \notin I \cup \{2, 3\}$. Observe that ($\star$) implies the following:

> ($\star\star$) if $w \in V(Q_3[u_i, t])$ and P is an $s'w$-dipath in $\vec{H}^\star \setminus u^\star$, then P and X_i have a vertex in common.

Observe that ($\star\star$), together with the brace proposition 16.2, implies that $D = C_1$ is vertex-disjoint from $Q_3[u_i, t]$.

Let $I_d := X_i$ and $I_c := Q_3[u_i, t]$. Let $T' := \{s, t\}$, let Q' be $Q_1 - (I_c \cup I_d)$ minus any directed circuit (of $\vec{H}$) it contains, for $j \in \{2, 3\}$ let L_j' be $Q_j - (I_c \cup I_d)$ minus any directed circuit it contains, and for $j \in [m] - [3]$ let Q_j' be $Q_j - (I_c \cup I_d)$ minus any directed circuit it contains. Let $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of $D, Q', L_2', L_3', Q_4', \ldots, Q_m'$. It is clear that (1)-(4) and (ii) of the disentangling lemma 16.3 hold. By the choice of X_i, (i) holds as well. To show (iii) holds, let P' be an odd st-dipath of $\vec{H}'$ with $V(P') \cap U = \{s\}$. Then $P' \cup I_c$ contains an odd st-dipath of $\vec{H}$, so $P' \cup I_c$ contains a $u^\star t$-dipath of $\vec{H}^\star$ containing Ω and by ($\star\star$), $P' \cup I_d$ also contains a $u^\star t$-dipath of $\vec{H}^\star$ containing Ω, implying in turn that $P' \cup I_d$ contains an odd st-dipath of $\vec{H}$. Hence, (iii) holds, a contradiction with the disentangling lemma 16.3. $\diamond$

For each $i \in I$, let B_i be an extremal k-mate of $Q_i[v_i, t] \cup \{\Omega\}$. Note that $B_i \cap Q_i[v_i, t] \neq \emptyset$. As $v_i \geq v_3$, $Q_i[v_i, t] \cup \{\Omega\}$ is contained in an odd st-dipath P such that $V(P) \cap U = \{s\}$. Note that B_i is also a k-mate for P, so by proposition 16.1, B_i is a signature. Fix $z \in I - \{3\}$. Choose $W \subseteq V(G) - \{s, t\}$ such that $\delta(W) = B_3 \triangle B_z$. By proposition 8.7, there is a path in $G[W] \setminus B_3$ between Q_3 and Q_z. Moreover, by proposition 5.4, there is a path between s and each of d, q in $G[U] \setminus B_3$. We say that *property (S)* holds if there exist paths S_d, S_q, S in G such that

S_d is an sd-path and S_q is an sq-path, and they are contained in $G[U] \setminus B_3$,

S connects a vertex of Q_3 to a vertex of Q_z in $G[W] \setminus B_3$, and

each of S_d, S_q is vertex-disjoint from S.

Claim 5. *If property (S) holds, then* $(G, \Sigma, \{s, t\})$ *has an* F_7 *minor.*

Proof. Take X and Y from claim 4. Notice that each edge in $Y \cap \delta(U)$ belongs to either of $D, Q, P_4, \ldots, P_m$, so we may assume that, for some $u \in \{s, d, q\}$, Y is a ut-path. It is now easy (and is left as an exercise) to see that $C_1 \cup X \cup Y \cup S_d \cup S_q \cup Q_3[v_3, t] \cup Q_z[v_z, t] \cup S$ has an F_7 minor. $\diamond$

Claim 6. *Suppose property (S) does not hold. Then* $m \geq 4$.

Proof. Suppose for a contradiction that $m = 3$. By proposition 16.1, L_2 and L_3 have signature k-mates. As $m = 3$, the mate proposition 8.4 therefore implies that L_1 does not have a signature k-mate. Hence, by claim 3, $1 \notin I$ and so $I = \{2, 3\}$. Since property (S) does not hold, there is $u \in \{d, q\}$ for which there is no su-path contained in $G[U] \setminus (B_2 \cup B_3)$. Let $B_1 := \delta(U)$. Clearly, (i) and (ii) of the shore proposition 15.1 hold. By (PC5), (iii) also holds. Moreover, for $i \in \{2, 3\}$, $B_i \cap P_i = B_i \cap (Q_i[v_i, t] \cup \{\Omega\})$, so by claim 1, $B_i \cap P_i$ has no edge in $G[U]$, so (iv) holds. Thus, by the shore proposition 15.1, there is an su-path contained in $G[U] \setminus (B_2 \cup B_3)$, a contradiction. $\diamond$

Claim 7. *Suppose property (S) does not hold. Then there exist vertex-disjoint paths* X *and* Y *in* $\vec{H}$ *where* X *is an* $s'v_3$*-path and* Y *is an* st*-path.*

Proof. Suppose otherwise. By claim 6, $m \geq 4$ and by the brace proposition 16.2, none of $Q_4, \ldots, Q_m$ contains vertex s'. Thus, $s' \neq v_3$. By the linkage lemma 13.1, H is a spanning subgraph of an (s, v_3, t, s')-web with frame C_0 and rib H_0. Fix a plane drawing of H_0, where the unbounded face is bounded by C_0. After redefining $\mathcal{L}$, if necessary, we may assume the following:

($\star$) for every $s'v_3$-dipath P of $\vec{H}$ with $V(P) \cap U = \emptyset$, the number of rib vertices that are on the same side of P as s is at least as large as that of $Q_3[s', v_3]$.

For $j \in [m]-[3]$, let u_j be the largest rib vertex on Q_j that also lies on $Q_3[s', v_3]$. Observe that if $j \in I \cap ([m]-[3])$, then $u_j = v_j$. For $j \in [m]-[3]$ let $X_j := Q_j[u_j, t]$, for $j \in \{2,3\} \cap I$ let $X_j := Q_j[v_j, t]$, and for $j \in \{2,3\} - I$ let $X_j := Q_j[s', t]$. Observe that each $X_j, j \in [m]-\{1\}$ is contained in an odd st-dipath whose intersection with U is $\{s\}$. As a result, by proposition 16.1, every k-mate for $X_j \cup \{\Omega\}, j \in [m]-\{1\}$ (if any) must be a signature. However, every k-mate for $X_j \cup \{\Omega\}, j \in [m]-\{1\}$ is also a k-mate for $P_j \cup \{\Omega\}$. Hence, since property (S) does not hold, the (contrapositive equivalent of the) shore proposition 15.1 implies that, for some $i \in [m]-\{1\}$, $X_i \cup \{\Omega\}$ has no k-mate. By (C4) and claim 3, $i \notin I \cup \{2,3\}$. Observe that ($\star$) implies the following:

> ($\star\star$) if $w \in V(Q_3[u_i, t])$ and P is an $s'w$-dipath in $\vec{H} \setminus U$, then P and X_i have a vertex in common.

Note that ($\star\star$), together with the brace proposition 16.2, implies that $C_1 = D$ is vertex-disjoint from $Q_3[u_i, t]$.

Let $I_d := X_i$ and $I_c := Q_3[u_i, t]$. Let $T' := \{s, t\}$, let Q' be $Q_1 - (I_c \cup I_d)$ minus any directed circuit it contains, for $j \in \{2,3\}$ let L'_j be $Q_j - (I_c \cup I_d)$ minus any directed circuit it contains, and for $j \in [m]-[3]$ let Q'_j be $Q_j - (I_c \cup I_d)$ minus any directed circuit it contains. Let $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of $D, Q', L'_2, L'_3, Q'_4, \ldots, Q'_m$. It is clear that (1)-(4) and (ii) of the disentangling lemma 16.3 hold. By the choice of X_i, (i) holds as well. To show (iii) holds, let P' be an odd st-dipath of $\vec{H}'$ with $V(P') \cap U = \{s\}$. Then $P' \cup I_c$ contains an odd st-dipath of $\vec{H}$ whose intersection with U is $\{s\}$, so by ($\star\star$), $P' \cup I_d$ also contains an st-dipath of $\vec{H}$. Hence, (iii) holds, a contradiction with the disentangling lemma 16.3. ◇

Claim 8. *Suppose property (S) does not hold. If $\vec{H} \setminus t$ is non-bipartite, then $(G, \Sigma, \{s,t\})$ has an F_7 minor.*

Proof. Take X and Y from claim 7, and let C be an odd circuit in $\vec{H} \setminus t$. Note that $\Omega \in C$. By proposition 8.7, there is a shortest path S in $G[W] \setminus B_3$ between P_3 and P_z. Note that $S \cap E(H) = \emptyset$. It is easy (and is left as an exercise) to see that $C \cup X \cup Y \cup P_3[v_3, t] \cup P_z[v_z, t] \cup S$ has an F_7 minor. ◇

Notice that if $\vec{H} \setminus t$ is bipartite, then for all $i \in \{2,3\}$ and $j \in [m]-[3]$, Q_i and $Q_j \cup R_j$ are internally vertex-disjoint.

We say that *property (S')* holds if there exist vertex-disjoint paths S_d, S in G such that

S_d is an sd-path in $G[U] \setminus B_3$,

S connects a vertex of P_3 to a vertex of P_z in $G[W] \setminus B_3$.

Notice that if property (S') does not hold, then neither does property (S).

Claim 9. *Suppose property (S) does not hold, $\vec{H} \setminus t$ is bipartite, and property (S') holds. Then $(G, \Sigma, \{s,t\})$ has an F_7 minor.*

Proof. By claim 6, $m \geq 4$. Note that $Q_4 \cup R_4$ is internally vertex-disjoint from Q_3. It is easy to see that $C_1 \cup S_d \cup Q_3 \cup Q_z[v_z, t] \cup S \cup Q_4 \cup R_4$ has an F_7 minor. $\diamond$

Claim 10. *Suppose property (S') does not hold and that $\vec{H} \setminus t$ is bipartite. Then $(G, \Sigma, \{s,t\})$ has an F_7 minor.*

Proof. We will find an F_7 minor in a different way than we have done so far, by using edges from $L_{m+1}, \ldots, L_k$.

Since property (S') does not hold, there does not exist a path connecting a vertex of Q_3 to a vertex of Q_z in $G[W-U] \setminus B_3$. So there is a partition of $W-U$ into two parts W_3, W_z such that W_3 shares no vertex with Q_z, W_z shares no vertex with Q_3, and every edge with one end in W_3 and another in W_z belongs to B_3. Observe that $\delta(W_3) \cup \delta(W_z) \subseteq B_3 \cup B_z \cup \delta(U)$.

Subclaim 1. *There is no edge with one end in W_3 and another in W_z.*

Proof of Subclaim. Suppose otherwise, and let e be such an edge. Then $e \in B_3$, and since $e \notin \delta(W)$, it follows that $e \in B_z$. Note $e \in C_4 \cup \cdots \cup C_m \cup L_{m+1} \cup \cdots \cup L_k$, and since each of $L_{m+1}, \ldots, L_k$ is a connected odd st-join intersecting each of B_3, B_z exactly once, it follows that $e \in C_4 \cup \cdots \cup C_m$. We may assume $e \in C_4$. However, $C_4 \cap \delta(U) = \emptyset$, implying that there is another edge f of C_4 with one end in W_3 and another in W_z. But then $\{e, f\} \subseteq C_4 \cap B_3$, a contradiction as $|C_4 \cap B_3| = 1$. $\triangledown$

Given $L \in \{L_{m+1}, \ldots, L_k\}$ and $Q_j \in \{Q_3, Q_z\}$, we say that L *is bad for* Q_j if $|L \cap \delta(W_j)| = 2$, $L \cap \delta(W_j) \cap B_j = \emptyset$, and there exists a path in $G[W_j] \setminus B_3$ between Q_j and L.

Subclaim 2. *Each $L \in \{L_{m+1}, \ldots, L_k\}$ is bad for at most one of Q_3, Q_z.*

Proof of Subclaim. Suppose otherwise. Then $|L \cap \delta(W_3)| = |L \cap \delta(W_z)| = 2$, and by subclaim 1, L shares exactly four edges with $\delta(W_3) \cup \delta(W_z)$. However, $\delta(W_3) \cup \delta(W_z) \subseteq B_3 \cup B_z \cup \delta(U)$, implying that L shares at least two edges with one of $B_3, B_z, \delta(U)$, a contradiction. $\triangledown$

Subclaim 3. *Each of Q_3, Q_z has a bad odd st-join.*

Proof of Subclaim. We prove that Q_3 has a bad odd st-join, and proving Q_z has a bad odd st-join can be done similarly. Suppose for a contradiction that Q_3 has no bad odd st-join. Let W_3' be the set of all vertices in W_3 that are reachable from a vertex of Q_3 in $G[W_3] \setminus B_3$. A similar argument as in subclaim 1 shows that there is no edge with one end in W_3' and another in $W_3 - W_3'$. Moreover, our contrary assumption implies that, for every $L \in \{L_{m+1}, \ldots, L_k\}$ such that $L \cap \delta(W_3') \neq \emptyset$, we have

$$|L \cap \delta(W_3')| = 2 \quad \text{and} \quad |L \cap \delta(W_3') \cap B_3| = 1.$$

This implies that $B_3 \triangle \delta(W_3')$ is also a k-mate of $Q_3[v_3, t] \cup \{\Omega\}$. However, $(B_3 \triangle \delta(W_3')) \cap Q_3[v_3, t] = \emptyset$, contradicting the extremality of B_3. $\triangledown$

Subclaim 4. *$(G, \Sigma, \{s, t\})$ has an F_7 minor.*

Proof of Subclaim. Since property (S') does not hold, there is no path in $G[U - W] \setminus B_3$ between s and d. So there is a partition U_s, U_d of $U - W$ such that U_s contains s, U_d contains d, and every edge with one end in U_s and another in U_d belongs to B_3.

By proposition 5.4, there is a path S_d between s and d in $G[U] \setminus B_3$. By proposition 8.7, there is a shortest path S in $G[W] \setminus B_3$ between Q_3 and Q_z. Suppose S has ends $r_3 \in V(Q_3)$ and $r_z \in V(Q_z)$. Since property (S') does not hold, S and S_d have a vertex in common in $U \cap W$. After contracting edges in $G[U_s] \setminus B_3$, if necessary, we may assume that S_d and P_4 share only the vertex s. (We may assume $P_4 \subseteq Q_4 \cup R_4$.)

By subclaims 2 and 3, we may assume that L_{m+1} is bad for Q_3 and that L_{m+2} is bad for Q_z. After contracting the path between L_{m+1}, Q_3 in $G[W_3] \setminus B_3$ and the path between L_{m+2}, Q_z in $G[W_z] \setminus B_3$, we may assume that $r_3 \in V(L_{m+1})$ and $r_z \in V(L_{m+2})$. After contracting edges in $G[U_s] \setminus B_3$, if necessary, we may assume that L_{m+1} and each one of P_4, S_d share only the vertex s in U_s. Similarly, we may assume that L_{m+2} and S share only the vertex r_z in W_z.

To construct the desired F_7 minor, we will need three odd circuits and an even st-path, described as follows.

Even st-path: Our even st-path will be P_4. Recall that P_4 is internally vertex-disjoint from each one of $Q_2, Q_3, Q_z[v_z, t]$. Moreover, by the brace proposition 16.2, $V(P_4) \cap V(C_1) \subseteq \{s, d\}$. In fact, since property (S') does not hold, $V(P_4) \cap V(C_1) = \{s\}$. In fact, notice that P_4 has no vertex in common with $U_d \cup W$.

Middle odd circuit: Along S_d, let x be the closest vertex to d that also lies on S. Note that $x \in U \cap W$. Our middle circuit will be

$$C_{\text{middle}} := S_d[d, x] \cup S[x, r_3] \cup Q_3[r_3, s'] \cup C_1[s', d].$$

Observe that the even st-path P_4 is vertex-disjoint from C_{middle}. Moreover, $C_{\text{middle}} \cap B_3 = Q_3[r_3, s'] \cap B_3$, so C_{middle} is an odd circuit.

First odd circuit: Our first odd circuit C_{first} will be one contained in the odd cycle

$$S_d[s,x] \cup S[x,r_3] \cup L_{m+1}[r_3,s].$$

(The intersection of this cycle with B_3 is $L_{m+1}[r_3,s] \cap B_3$, so the cycle is indeed odd.) Note that C_{first} is contained in $G[U \cup W]$.

Last odd circuit: Our last odd circuit C_{last} will be one contained in the set

$$L_{m+2}[r_z,t] \cup Q_3[s',v_3] \cup Q_z[v_3,t]$$

whose intersection with B_3 is $B_3 \cap L_{m+2}[r_z,t]$, which has odd cardinality. Note that $V(C_{\text{last}})$ is contained in $(V(G)-(U \cup W)) \cup W_z$. However, as can be easily seen, C_{first} and C_{last} share no vertex in W_z. Hence, C_{first} and C_{last} have no vertex in common.

It is now quite easy to see that $(G, \Sigma, \{s,t\})$ has an F_7 minor, finishing the proof. $\triangledown$

$\diamond$

Observe that claims 5, 8, 9 and 10 finish the proof of proposition 2.14.

17. Secondary cut Ω-system

17.1. **Signature mates.**

Proposition 17.1. *Let $((G,\Sigma,\{s,t\}), \mathcal{L} = (L_1,\ldots,L_k), m, (U_1,\ldots,U_n), \vec{H})$ be a minimal cut Ω-system that is secondary. Let P be an odd st-dipath with $V(P) \cap U_n = \{s\}$, and let B be a k-mate of it. Then B is not an st-cut.*

Proof. After redefining $\mathcal{L}$, if necessary, we may assume that $P = Q_1$. Suppose for a contradiction that B is an st-cut. Choose $W \subseteq V(G) - \{t\}$ with $s \in W$ such that $B = \delta(W)$, and assume that there is no proper subset W' of W with $s \in W'$ such that $\delta(W')$ is a k-mate for $Q_1 = L_1$. Observe that $Q_1 \cap \delta(U_n) = \{\Omega\}$, and since Q_1 is an odd st-dipath, it follows that $Q_1 \cap \delta(U_n \cap W) = \{\Omega\}$. It now follows that $\delta(U_n \cap W)$ is also a k-mate for $L_{n+3} - C_{n+3}$. Hence, by the minimality condition of (SC3), it follows that $U_n \subset W$. Let $\mathcal{U} := (U_1,\ldots,U_n,W)$. Let d (resp. q) be the closest (resp. furthest) vertex to (resp. from) s on Q_1 that also belongs to $W - U_n$. It is easily seen that $\mathcal{U}$ is a primary cut structure for $((G,\Sigma,\{s,t\}), \mathcal{L}, m)$, where L_1 has brace $Q_1[s,d]$, residue $Q_1[d,q]$ and base $Q_1[q,t]$. Let $\vec{H}' := \vec{H} \setminus Q_1[d,q]$. Then it is easily seen that $((G,\Sigma,\{s,t\}), \mathcal{L}, m, \mathcal{U}, \vec{H}')$ is a primary cut structure, contradicting the minimality of the original Ω-system. $\square$

17.2. **A disentangling lemma.**

Lemma 17.2. *Let $((G,\Sigma,\{s,t\}),\mathcal{L}=(L_1,\ldots,L_k),m,\mathcal{U}=(U_1,\ldots,U_n),\vec{H})$ be a minimal cut Ω-system that is secondary, and assume there is no non-simple bipartite Ω-system whose associated signed graft is a minor of $(G,\Sigma,\{s,t\})$. Take disjoint subsets $I_d,I_c\subseteq E(\vec{H}\setminus\Omega)$ and $T'\subseteq\{s,t\}$ where*

(1) I_c is non-empty, if I_c contains an st-path then $T'=\emptyset$, and if not then $T'=\{s,t\}$,

(2) every signature or st-cut disjoint from I_c intersects I_d in an even number of edges,

(3) if $T'=\emptyset$, there is a directed subgraph $\vec{H}'$ of $\vec{H}/I_c\setminus I_d$ that is the union of directed odd circuits L_1',L_2',L_3' where

$\Omega\in L_1'\cap L_2'\cap L_3'$ and L_1',L_2',L_3' are pairwise Ω-disjoint,

$\vec{H}'\setminus\Omega$ is acyclic,

(4) if $T'=\{s,t\}$, then $I_d,I_c\subseteq E(\vec{H}\setminus U_n)$ and there is a directed subgraph $\vec{H}'$ of $\vec{H}/I_c\setminus I_d$ that is the union of odd st-dipaths L_1',L_2',L_3' and dipaths $Q_4',\ldots,Q_m'$, where

- for $i=4,\ldots,n+3$, Q_i' is a $q_{i-3}t$-dipath with $V(Q_i')\cap U_{i-3}=\{q_{i-3}\}$, and for $i=n+4,\ldots,m$, Q_i' is an even st-dipath,

- $L_1',L_2',L_3',Q_4',\ldots,Q_m'$ are pairwise Ω-disjoint,

- $L_1',L_2',L_3',Q_4',\ldots,Q_m'$ coincide with $L_1,L_2,L_3,Q_4,\ldots,Q_m$ on $E(G[U_n])\cup\delta(U_n)$, respectively,

- the following digraph is acyclic: start from $\vec{H}'$, and for each q_i add arc (s,q_i).

Then one of the following does not hold:

(i) $I_d\cup\{\Omega\}$ does not have a k-mate,

(ii) if $T'=\emptyset$, then for every directed odd circuit L' of $\vec{H}'$, either $L'\cup I_d$ contains an odd st-dipath P of $\vec{H}$ with $V(P)\cap U_n=\{s\}$, or $L'\cup I_d$ has a k-mate in $(G,\Sigma,\{s,t\})$ disjoint from I_c,

(iii) if $T'=\{s,t\}$, then for every odd st-dipath P' of $\vec{H}'$ with $V(P')\cap U_n=\{s\}$, either $P'\cup I_d$ contains an odd st-dipath of $\vec{H}$, or $P'\cup I_d$ has a k-mate in $(G,\Sigma,\{s,t\})$ disjoint from I_c.

Proof. Suppose otherwise. Let $(G',\Sigma',T'):=(G,\Sigma,\{s,t\})/I_c\setminus I_d$ where $\Sigma'=\Sigma$; this signed graft is well-defined by (1). Let $\mathcal{L}':=(L_1',\ldots,L_m',L_{m+1},\ldots,L_k)$, where $L_1',\ldots,L_m'$ are defined as follows. If $T'=\emptyset$, let $m':=3$, and for $i\in[m]-[3]$, let $L_i':=L_i-P_i$. Otherwise, when $T'=\{s,t\}$, let $m':=m$, and for $i\in[m]-[3]$, let $L_i':=(L_i-Q_i)\cup Q_i'$.

We will first show that $((G',\Sigma',T'),\mathcal{L}',m')$ is a bipartite Ω-system. **(B1)** By (2), every signature of (G',Σ',T') has the same parity as $\tau(G,\Sigma,\{s,t\})$, implying that (G',Σ',T') is an Eulerian signed graft. **(B2)** It also implies that $k,\tau(G,\Sigma,\{s,t\})$ and $\tau(G',\Sigma',T')$ have the same parity, so every minimal cover of (G',Σ',T') has the same size parity as k. We claim that $\tau(G',\Sigma',T')\geq k$. Let B' be a minimal cover of (G',Σ',T'). If $\Omega\notin B'$, then

$$|B'|\geq\sum\left(|B'\cap L'|:L'\in\mathcal{L}'\right)\geq k.$$

Otherwise, $\Omega \in B'$. In this case, $B' \cup I_d$ contains a cover B of $(G, \Sigma, \{s,t\})$. By (i), $I_d \cup \{\Omega\}$ does not have a k-mate, so

$$k - 2 \leq |B - (I_d \cup \{\Omega\})| \leq |B - I_d| - 1 \leq |B'| - 1,$$

and since $|B'|, k$ have the same parity, it follows that $|B'| \geq k$. Thus, $\mathcal{L}'$ is an (Ω, k)-packing. When $T' = \emptyset$ then $m' = 3$. When $T = \{s,t\}$, then $m' = m$ and for $j \in [m'] - [3]$, L'_j contains an even st-path in the bipartite st-join $L'_j - C_j$ and some odd circuit in C_j, and for $j \in [k] - [m']$, L_j remains connected in G'. **(B3)** follows from construction.

Suppose first that $T' = \emptyset$. We will show that $((G', \Sigma', \emptyset), \mathcal{L}', 3, \vec{H}')$ is a non-simple bipartite Ω-system, yielding a contradiction. **(NS1)** holds as (B1)-(B3) hold. **(NS2)** holds as $T' = \emptyset$. **(NS3)** follows from (3). **(NS4)** Let L' be a directed odd circuit of $\vec{H}'$. If $L' \cup I_d$ has a k-mate B in $(G, \Sigma, \{s,t\})$ disjoint from I_c, then $B - I_d$ contains a minimal cover B' of $(G', \Sigma', \emptyset)$, and since

$$|B' - L'| \leq |(B - I_d) - L'| = |B - (L' \cup I_d)| \leq k - 3,$$

it follows that B' is a k-mate of L'. Otherwise by (ii) $L' \cup I_d$ contains an odd st-dipath P of $\vec{H}$ with $V(P) \cap U_n = \{s\}$. Since $((G, \Sigma, \{s,t\}), \mathcal{L}, m, \mathcal{U}, \vec{H})$ is a minimal secondary cut Ω-system, P has a k-mate B which by proposition 17.1 is a signature. By proposition 8.3, $B \cap E(\vec{H}) = B \cap P$, implying that $B \cap I_c = \emptyset$. Thus, $B - I_d$ contains a minimal cover B' of $(G', \Sigma', \emptyset)$, and since

$$|B' - L'| \leq |(B - I_d) - L'| \leq |B - P| \leq k - 3,$$

it follows that B' is a k-mate of L'.

Suppose otherwise that $T' = \{s,t\}$. To obtain a contradiction, we will show that $((G', \Sigma', \{s,t\}), \mathcal{L}', m, \mathcal{U}, \vec{H}')$ is a secondary cut Ω-system. **(C1)** holds because (B1)-(B3) are true. **(C2)-(C3)** follow from (4). **(C4)** Let P' be an odd st-dipath in $\vec{H}'$ with $V(P') \cap U = \{s\}$. If $P' \cup I_d$ has a k-mate B in $(G, \Sigma, \{s,t\})$ disjoint from I_c, then $B - I_d$ contains a minimal cover B' of $(G', \Sigma', \{s,t\})$, and since

$$|B' - P'| \leq |(B - I_d) - P'| = |B - (P' \cup I_d)| \leq k - 3,$$

it follows that B' is a k-mate of P'. Otherwise by (iii) $P' \cup I_d$ contains an odd st-dipath P of $\vec{H}$. As $I_d \subseteq E(\vec{H} \setminus U)$, it follows that $V(P) \cap U = \{s\}$. Since $((G, \Sigma, \{s,t\}), \mathcal{L}, m, \mathcal{U}, \vec{H})$ is a minimal secondary cut Ω-system, P has a k-mate B. By proposition 17.1, B is a signature, so by proposition 8.3, $B \cap E(\vec{H}) = B \cap P$, implying that $B \cap I_c = \emptyset$. Thus $B - I_d$ contains a minimal cover B' of $(G', \Sigma', \{s,t\})$, and since

$$|B' - P'| \leq |(B - I_d) - P'| \leq |B - P| \leq k - 3,$$

it follows that B' is a k-mate of P'. □

17.3. **The proof of proposition 2.15.** In this section, we prove proposition 2.15. We assume Ω has ends s, s'. For $i = 4, \dots, n+3$, let Q_i^+ be the st-dipath obtained from Q_i after adding arc (s, q_{i-3}) to it. Let $\vec{H}^+$ be the union of $Q_1, Q_2, Q_3, Q_4^+, \dots, Q_{n+3}^+, Q_{n+4}, \dots, Q_m$. For $u, v \in V(\vec{H}^+)$, $u \le v$ if there is a uv-dipath in $\vec{H}^+$. This partial ordering is well-defined as $\vec{H}^+$ is acyclic, by (C3). For $i \in [m]$, let v_i be the second largest vertex of the i^{th} dipath that lies on one of the other st-dipaths. By proposition 8.8, there exists an index subset $I \subseteq [m]$ of size at least two such that, for each $i \in I$,

- $v_i \ge v_1$, and there is no $j \in [m]$ such that $v_j > v_i$,
- for each $j \in [m]$, $v_i = v_j$ if and only if $j \in I$.

Claim 1. *For each $i \in I$, $Q_i[v_i, t]$ and U_n have no vertex in common.*

Proof. Suppose otherwise. Since $\vec{H}$ has no arc entering U_n, it follows that $v_i \in U_n$. As $v_i \ge v_1$, there is a $v_1 v_i$-dipath $P \subset E(\vec{H})$. However, as $v_1 \in V(Q_1[s', t])$, so $v_1 \notin U$, implying that P has an arc that enters U_n, a contradiction. $\diamond$

Claim 2. *For each $i \in I$, $Q_i[v_i, t] \cup \{\Omega\}$ has a signature k-mate.*

Proof. Suppose otherwise. Since $v_i \ge v_1$, $Q_i[v_i, t] \cup \{\Omega\}$ is contained in an odd st-dipath P such that $V(P) \cap U_n = \{s\}$. Hence, by proposition 17.1, $Q_i[v_i, t] \cup \{\Omega\}$ has no k-mate at all. Let $I_d := Q_i[v_i, t]$ and $I_c := \bigcup(Q_j[v_j, t] : j \in I - \{i\})$. Let $T' := \{s, t\}$, for $j \in [3]$ let $L'_j := Q_j - (I_c \cup I_d)$, and for $j \in [m] - [3]$ let $Q'_j := Q_j - (I_c \cup I_d)$. Let $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of $L'_1, L'_2, L'_3, Q'_4, \dots, Q'_m$. It is clear that (1)-(4) and (ii), (iii) of the disentangling lemma 17.2 hold. However, $I_d \cup \{\Omega\}$ has no k-mate, so (i) holds, contradicting the disentangling lemma 17.2. $\diamond$

After redefining $\mathcal{L}$, if necessary, we may assume that $1 \in I$.

Claim 3. *If $m = 4$, then $I \subseteq [3]$.*

Proof. Suppose otherwise. By claim 2, there exists a signature k-mate B_4 for $Q_4[v_4, t] \cup \{\Omega\}$. By (C4) and proposition 17.1, for each $i \in [3]$, there exists a signature k-mate B_i for Q_i. However, B_1, B_2, B_3, B_4 contradict the mate proposition 8.4. $\diamond$

Claim 4. *Suppose $m = 4$. Then there exists $i \in [3]$ such that Q_i and Q_4 are not internally vertex-disjoint.*

Proof. Suppose for a contradiction that Q_4 is internally vertex-disjoint from $Q_1 \cup Q_2 \cup Q_3$. Notice that $I \subseteq [3]$, by claim 3.

Subclaim 1. *There exist an $s'v_1$-dipath X and an $s't$-dipath Y in $\vec{H}$ that are internally vertex-disjoint.*

Proof of Subclaim. Suppose otherwise. Then $s' \neq v_1$ and there exists a vertex $v \in V(\vec{H}) - \{s', t\}$ such that there is no $s't$-dipath in $\vec{H} \setminus v$. By proposition 17.1, one of the following holds:

(a) there exists an $s'v$-dipath Z in $\vec{H}$ such that $Z \cup \{\Omega\}$ has no k-mate:

Let $I_d := Z$, $I_c := \bigcup(Q_i[s', v] : i \in [3]) - Z$, $T' := \{s, t\}$, for $i \in [3]$ let $L'_i := Q_i - (I_c \cup I_d)$, and let $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of L'_1, L'_2, L'_3, Q_4.

(b) for every $s'v$-dipath Z in $\vec{H}$, $Z \cup \{\Omega\}$ has a signature k-mate:

Let $I_d := \emptyset$, $I_c := \bigcup(Q_i[v, t] : i \in [3])$, $T' := \{s, t\}$, for $i \in [3]$ let $L'_i := Q_i[s', v] \cup \{\Omega\}$, and let $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of L'_1, L'_2, L'_3, Q_4.

It is not difficult to check that in either of the cases above, (1)-(4) and (i)-(iii) of the disentangling lemma 17.2 hold, a contradiction. ▽

After redefining $\mathcal{L}$, if necessary, we may assume that $\{1, 2\} \subseteq I$ and $Y = Q_3[s', t]$. For $i = 1, 2$, let B_i be a signature k-mate for $Q_i[v_i, t] \cup \{\Omega\}$, whose existence is guaranteed by claim 2. Moreover, by (C4) and proposition 17.1, Q_3 has a signature k-mate B_3. Observe that by proposition 8.3, for each $i \in [3]$, $B_i \cap (Q_4 \cup X) = \emptyset$.

Subclaim 2. *There exists a path R between s and Q_4 in $G[U_n] \setminus (B_1 \cup B_2 \cup B_3)$.*

Proof of Subclaim. This is an immediate consequence of the shore proposition 15.1 and the fact that $m = 4$. ▽

Let $I_c := R \cup Q_4 \cup X$ and $I_d := \emptyset$. Let $T' := \emptyset$, for $i = 1, 2$ let $L'_i := Q_i[v_i, t] \cup \{\Omega\}$, and let $L'_3 := Q_3$. Let $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of L'_1, L'_2, L'_3. Note that L'_1, L'_2, L'_3 are internally vertex-disjoint in $\vec{H}'$ and have signature k-mates B_1, B_2, B_3, respectively. It is now clear that (1)-(4) and (i)-(iii) of the disentangling lemma 17.2 hold, a contradiction. ◇

Claim 5. *Suppose $m = 4$. Then there exists an $s'v_1$-dipath P in $\vec{H}$ that is vertex-disjoint from Q_4.*

Proof. By claim 3, $I \subseteq [3]$. Suppose for a contradiction that there is no $s'v_1$-dipath in $\vec{H}$ that is vertex-disjoint from Q_4. Let v be the smallest vertex of Q_4 outside U_n for which there exists a vv_1-dipath R in $\vec{H}$ such that $V(R) \cap V(Q_4) = \{v\}$. Our contrary assumption together with the choice of v and R, implies the following:

($\star$) if $w \in V(R)$ and Q is an $s'w$-dipath in $\vec{H}$, then Q and $Q_4[v, t]$ have a vertex in common.

Let $I_d := Q_4[v, t]$ and $I_c := R \cup [\bigcup(Q_j[v_j, t] : j \in I)]$. For $i \in [3]$ let L'_i be $Q_i - (I_c \cup I_d)$ minus any directed circuit, and let $Q'_4 := Q_4[q_n, t]$. Let $T' := \{s, t\}$ and $\vec{H}' \subseteq \vec{H} \setminus I_d / I_c$ be the union of L'_1, L'_2, L'_3, Q'_4. It is not hard to see that (1)-(4) and (ii) of the disentangling lemma 17.2 hold. By proposition 17.1 and the mate proposition 8.4, $I_d \cup \{\Omega\}$ has no k-mate, so (i) holds. Let P' be an odd st-dipath of $\vec{H}'$ for which $V(P') \cap U_n = \{s\}$. Then $P' \cup I_c$ contains an odd st-dipath P of $\vec{H}$. Choose $w \in V(R)$ (if any) such that P contains an $s'w$-dipath Q in $\vec{H}$ and $V(Q) \cap V(R) = \{w\}$. Then $(\star)$ implies that $(P - I_c) \cup I_d$, and therefore $P' \cup I_d$, contains an odd st-dipath of $\vec{H}$, so (iii) holds as well, a contradiction with the disentangling lemma 17.2 $\diamond$

Claim 6. *Suppose $m = 4$. Then $(G, \Sigma, \{s, t\})$ has an F_7 minor.*

Proof. Take P from claim 5. By claim 3, $I \subseteq [3]$. After redefining $\mathcal{L}$, if necessary, we may assume that $\{1, 2\} \subseteq I$ and that $P = Q_1[s', v_1]$. For each $i \in \{1, 2\}$, by claim 2, there exists a signature k-mate B_i for $Q_i[v_i, t] \cup \{\Omega\}$. Choose $W \subseteq V(G) - \{s, t\}$ such that $\delta(W) = B_1 \triangle B_2$. By proposition 8.7, there exists a shortest path R in $G[W] \setminus B_1$ between Q_1 and Q_2. By the shore proposition 15.1, there exists a path R_q in $G[U_n] \setminus (B_1 \cup B_2)$ between s and Q_4. By claim 4, there exists $i \in \{2, 3\}$ and vertex $v \in V(Q_i) \cap V(Q_4)$ such that $Q_i[s', v]$ is vertex-disjoint from $Q_1 \cup Q_2[v_2, t] \cup Q_4$. It is now easy (and is left as an exercise) to see that $R_q \cup Q_4 \cup Q_i[s', v] \cup Q_1 \cup Q_2[v_2, t] \cup R$ has an F_7 minor. $\diamond$

Claim 7. *There exist vertex-disjoint paths X and Y in $\vec{H}$ such that X is an $s'v_1$-path in $\vec{H} \setminus U_n$ and Y connects a vertex of U_n to t.*

Proof. Suppose otherwise.

Assume first that $s' = v_1$. Then, for each $j \in [m]$, $s' \in V(Q_j)$ and by claim 1, $Q_j[s', t]$ has no vertex in common with U_n. Hence, for each $j \in [m]$, by (C4) and proposition 17.1, $Q_j[s', t] \cup \{\Omega\}$ has a signature k-mate B_j. However, B_1 is also a signature k-mate for L_1, and for each $j \in [m] - [1]$, B_j is also a signature k-mate for $Q_j \cup \{\Omega\}$. This is a contradiction with the mate proposition 8.4.

Thus, $s' \neq v_1$. Let $\vec{H}^\star$ be the digraph obtained from $\vec{H}$ after shrinking U_n to a single vertex $u^\star$ and removing all loops. Notice that every odd st-dipath in $\vec{H}$ whose intersection with U_n is $\{s\}$, is a $u^\star t$-dipath in $\vec{H}^\star$ that uses Ω, and vice-versa. Also, note that the acyclicity condition in (C3) implies that $\vec{H}^\star \setminus u^\star$ is acyclic. By the linkage lemma 13.1, $H^\star$ is a spanning subgraph of a $(u^\star, v_1, t, s')$-web with frame C_0 and rib $H_0^\star$. Fix a plane drawing of $H_0^\star$, where the unbounded face is bounded by C_0. After redefining $\mathcal{L}$, if necessary, we may assume the following:

> $(\star)$ for every $s'v_3$-dipath P of $\vec{H}^\star \setminus u^\star$, the number of rib vertices that are on the same side of P as u^* is at least as large as that of $Q_1[s', v_1]$.

For $j \in [m] - [3]$, let u_j be the largest rib vertex on Q_j that also lies on $Q_1[s', v_1]$. Observe that if $j \in I \cap ([m] - [3])$, then $u_j = v_j$. For $j \in [m] - [3]$ let $X_j := Q_j[u_j, t]$, for $j \in [3] \cap I$ let $X_j := Q_j[v_j, t]$, and for $j \in [3] - I$ let $X_j := Q_j[s', t]$. For each $j \in [m]$, since $X_j \cup \{\Omega\}$ is contained in a $u^\star t$-dipath of $\vec{H}^\star$, proposition 17.1 implies that every k-mate for $X_j \cup \{\Omega\}$ (if any) must be a signature. However, every k-mate for $X_j \cup \{\Omega\}$, $j \in [m]$ is also a k-mate for $Q_j \cup \{\Omega\}$. Hence, by the mate proposition 8.4, there exists $i \in [m]$ such that $X_i \cup \{\Omega\}$ has no k-mate. By (C4) and claim 2, $i \notin I \cup [3]$. Observe that ($\star$) implies the following:

($\star\star$) if $w \in V(Q_1[u_i, t])$ and P is an $s'w$-dipath in $\vec{H}^\star \setminus u^\star$, then P and X_i have a vertex in common.

Let $I_d := X_i$ and $I_c := Q_1[u_i, t]$. Let $T' := \{s, t\}$, for $j \in [3]$ let L'_j be $Q_j - (I_c \cup I_d)$ minus any directed circuit it contains, and for $j \in [m] - [3]$ let Q'_j be $Q_j - (I_c \cup I_d)$ minus any directed circuit it contains. Let $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of $D, Q', L'_2, L'_3, Q'_4, \ldots, Q'_m$. It is clear that (1)-(4) and (ii) of the disentangling lemma 17.2 hold. By the choice of X_i, (i) holds as well. To show (iii) holds, let P' be an odd st-dipath of $\vec{H}'$ with $V(P') \cap U_n = \{s\}$. Then $P' \cup I_c$ contains an odd st-dipath of $\vec{H}$, so $P' \cup I_c$ contains a $u^\star t$-dipath of $\vec{H}^\star$ containing Ω and by ($\star\star$), $P' \cup I_d$ also contains a $u^\star t$-dipath of $\vec{H}^\star$ containing Ω, implying in turn that $P' \cup I_d$ contains an odd st-dipath of $\vec{H}$. Hence, (iii) holds, a contradiction with the disentangling lemma 17.2. $\diamond$

Claim 8. *Suppose $m \geq 5$. Then there exists $i \in [3]$ and $j \in [m] - \{1, 2, 3, n+3\}$ such that Q_i and Q_j are not internally vertex-disjoint.*

Proof. Suppose otherwise. Choose $j \in [m] - \{1, 2, 3, n+3\}$. Observe that $R_j \cup Q_j$ is internally vertex-disjoint from each of Q_1, Q_2, Q_3, and that by (C4) and propositions 17.1 and 8.3, every odd st-dipath contained in $Q_1 \cup Q_2 \cup Q_3$ has a signature k-mate disjoint from $R_j \cup Q_j$. With this in mind, let $I_c := R_j \cup Q_j$ and $I_d := \emptyset$. Let $T' := \emptyset$ and let $\vec{H}' \subseteq \vec{H} \setminus I_d/I_c$ be the union of L_1, L_2, L_3. It can be readily checked that (1)-(4) and (i)-(iii) of the disentangling lemma 17.2 hold, a contradiction. $\diamond$

For each $i \in I$, let B_i be an extremal k-mate of $Q_i[v_i, t] \cup \{\Omega\}$. Note that $B_i \cap Q_i[v_i, t] \neq \emptyset$. As $v_i \geq v_1$, $Q_i[v_i, t] \cup \{\Omega\}$ is contained in an odd st-dipath P such that $V(P) \cap U_n = \{s\}$. Note that B_i is also a k-mate for P, so by proposition 17.1, B_i is a signature. Fix $z \in I - \{1\}$. Choose $W \subseteq V(G) - \{s, t\}$ such that $\delta(W) = B_1 \triangle B_z$. By proposition 8.7, there is a path in $G[W] \setminus B_1$ between Q_1 and Q_z. Moreover, by proposition 5.4, there is a path between s and q_n in $G[U_n] \setminus B_1$. We say that *property (S)* holds if there exist paths S_n, S in G such that

S_n is an sq_n-path contained in $G[U_n] \setminus B_1$,

S connects a vertex of Q_1 to a vertex of Q_z in $G[W] \setminus B_1$, and

S_n and S are vertex-disjoint.

Claim 9. *Suppose $m \geq 5$ and property (S) holds. Then $(G, \Sigma, \{s, t\})$ has an F_7 minor.*

Proof. Take X and Y from claim 7. Notice that each edge in $Y \cap \delta(U_n)$ belongs to either of $Q_4, \ldots, Q_m$, so we may assume that, for some $u \in \{s, q_1, \ldots, q_n\}$, Y is a ut-path. By claim 8, there is an odd circuit

C in $(\vec{H}\cup R_1\cup\cdots\cup R_m)\setminus R_{n+3}$ that shares no vertex with $Q_1[v_1,t]\cup Q_z[v_z,t]$ in $V(G)-\{v_1\}$. It is now easy (and is left as an exercise) to see that $(C\cup S_n\cup X\cup Y\cup Q_1[v_1,t]\cup Q_z[v_z,t]\cup S\cup R_1\cup\ldots\cup R_m)-R_{n+3}$ has an F_7 minor. $\diamond$

Claim 10. *Suppose $m\geq 5$ and property (S) does not hold. Then there exist vertex-disjoint paths X and Y in $(H\cup R_1\cup\cdots\cup R_m)\setminus R_{n+3}$ where X is an $s'v_1$-path and Y is an st-path.*

Proof. Suppose otherwise. Since property (S) does not hold, the (contrapositive equivalent of the) shore proposition 15.1 implies that $s'\neq v_1$. Hence, by the linkage lemma 13.1, $(H\cup R_1\cup\cdots\cup R_m)\setminus R_{n+3}$ is a spanning subgraph of an (s,v_1,t,s')-web with frame C_0 and rib H_0. Fix a plane drawing of H_0, where the unbounded face is bounded by C_0. After redefining $\mathcal{L}$, if necessary, we may assume the following:

> ($\star$) for every $s'v_1$-dipath P of $\vec{H}$ with $V(P)\cap U_n=\emptyset$, the number of rib vertices that are on the same side of P as s is at least as large as that of $Q_1[s',v_1]$.

For $j\in[m]-\{1,2,3,n+3\}$, let u_j be the largest rib vertex on Q_j that also lies on $Q_1[s',v_1]$; such u_j exists as $R_j\cup Q_j$ intersects $Q_1[s',v_1]$, but R_j cannot have any vertex in common with $Q_1[s',v_1]$. Observe that if $j\in I\cap([m]-[3])$, then $u_j=v_j$. For $j\in[m]-\{1,2,3,n+3\}$ let $X_j:=Q_j[u_j,t]$, for $j\in[3]\cap I$ let $X_j:=Q_j[v_j,t]$, and for $j\in[3]-I$ let $X_j:=Q_j[s',t]$. Observe that each $X_j, j\in[m]-\{n+3\}$ is contained in an odd st-dipath whose intersection with U_n is $\{s\}$. As a result, by proposition 17.1, every k-mate for $X_j\cup\{\Omega\}, j\in[m]-\{n+3\}$ (if any) must be a signature. However, every k-mate for $X_j\cup\{\Omega\}, j\in[m]-\{n+3\}$ is also a k-mate for $P_j\cup\{\Omega\}$. Hence, since property (S) does not hold, the (contrapositive equivalent of the) shore proposition 15.1 implies that, for some $i\in[m]-\{n+3\}$, $X_i\cup\{\Omega\}$ has no k-mate. By (C4) and claim 2, $i\notin I\cup[3]$. Observe that ($\star$) implies the following:

> ($\star\star$) if $w\in V(Q_1[u_i,t])$ and P is an $s'w$-dipath in $\vec{H}\setminus U_n$, then P and X_i have a vertex in common.

Let $I_d:=X_i$ and $I_c:=Q_1[u_i,t]$. Let $T':=\{s,t\}$, for $j\in[3]$ let L'_j be $Q_j-(I_c\cup I_d)$ minus any directed circuit it contains, and for $j\in[m]-[3]$ let Q'_j be $Q_j-(I_c\cup I_d)$ minus any directed circuit it contains. Let $\vec{H}'\subseteq\vec{H}\setminus I_d/I_c$ be the union of $D,Q',L'_2,L'_3,Q'_4,\ldots,Q'_m$. It is clear that (1)-(4) and (ii) of the disentangling lemma 17.2 hold. By the choice of X_i, (i) holds as well. To show (iii) holds, let P' be an odd st-dipath of $\vec{H}'$ with $V(P')\cap U_n=\{s\}$. Then $P'\cup I_c$ contains an odd st-dipath of $\vec{H}$ whose intersection with U_n is $\{s\}$, so by ($\star\star$), $P'\cup I_d$ also contains an st-dipath of $\vec{H}$. Hence, (iii) holds, a contradiction with the disentangling lemma 17.2. $\diamond$

Claim 11. *Suppose $m\geq 5$ and property (S) does not hold. Then $(G,\Sigma,\{s,t\})$ has an F_7 minor.*

Proof. Take X and Y from claim 10. By proposition 8.7, there is a path S in $G[W] \setminus B_1$ between Q_1 and Q_z. By claim 8, there is an odd circuit C in $(\vec{H} \cup R_1 \cup \cdots \cup R_m) \setminus R_{n+3}$ that shares no vertex with $Q_1[v_1, t] \cup Q_z[v_z, t]$ in $V(G) - \{v_1\}$. It is now easy (and is left as an exercise) to see that $C \cup X \cup Y \cup Q_1[v_1, t] \cup Q_z[v_z, t] \cup S$ has an F_7 minor. ◇

Observe that claims 6, 9 and 11 finish the proof of proposition 2.15.

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