The Cycling Property for the Clutter of Odd st-Walks

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Abstract. A binary clutter is cycling if its packing and covering linear program have integral optimal solutions for all eulerian edge capacities. We prove that the clutter of odd st-walks of a signed graph is cycling if and only if it does not contain as a minor the clutter of odd circuits of K_5 nor the clutter of lines of the Fano matroid. Corollaries of this result include, of many, the characterization for weakly bipartite signed graphs [5], packing two-commodity paths [7, 10], packing T-joins with small $|T|$, a new result on covering odd circuits of a signed graph, as well as a new result on covering odd circuits and odd T-joins of a signed graft.

1 Introduction

A *clutter* $\mathcal{C}$ is a finite collection of sets, over some finite ground set $E(\mathcal{C})$, with the property that no set in $\mathcal{C}$ is contained in, or is equal to, another set of $\mathcal{C}$. This terminology was first coined by Edmonds and Fulkerson [2]. A *cover* B is a subset of $E(\mathcal{C})$ such that $B \cap C \neq \emptyset$, for all $C \in \mathcal{C}$. The *blocker* $b(\mathcal{C})$ is the clutter of the minimal covers. It is well known that $b(b(\mathcal{C})) = \mathcal{C}$ ([8, 2]). A clutter is *binary* if, for any $C_1, C_2, C_3 \in \mathcal{C}$, their symmetric difference $C_1 \triangle C_2 \triangle C_3$ contains, or is equal to, a set of $\mathcal{C}$. Equivalently, a clutter is binary if, for every $C \in \mathcal{C}$ and $B \in b(\mathcal{C})$, $|C \cap B|$ is odd ([8]). It is therefore immediate that a clutter is binary if and only if its blocker is.

Let $\mathcal{C}$ be a clutter and $e \in E(\mathcal{C})$. The *contraction* $\mathcal{C}/e$ and *deletion* $\mathcal{C} \setminus e$ are clutters on the ground set $E(\mathcal{C}) - \{e\}$ where $\mathcal{C}/e$ is the collection of minimal sets in $\{C - \{e\} : C \in \mathcal{C}\}$ and $\mathcal{C} \setminus e := \{C : e \notin C \in \mathcal{C}\}$. Observe that $b(\mathcal{C}/e) = b(\mathcal{C}) \setminus e$ and $b(\mathcal{C} \setminus e) = b(\mathcal{C})/e$. Contractions and deletions can be performed sequentially and the result does not depend on the order. A clutter obtained from $\mathcal{C}$ by a sequence of deletions E_d and a sequence of contractions E_c ($E_d \cap E_c = \emptyset$) is called a *minor* of $\mathcal{C}$ and is denoted $\mathcal{C} \setminus E_d/E_c$.

Given edge-capacities $w \in \mathbb{Z}_+^{E(\mathcal{C})}$ consider the linear program

$$(P) \quad \begin{cases} \min & \sum(w_e x_e : e \in E(\mathcal{C})) \\ \text{s.t.} & x(C) \geq 1, \quad C \in \mathcal{C} \\ & x_e \geq 0, \quad e \in E(\mathcal{C}), \end{cases}$$

and its dual

$$(D) \quad \begin{cases} \max & \sum(y_C : C \in \mathcal{C}) \\ \text{s.t.} & \sum(y_C : e \in C \in \mathcal{C}) \le w_e, \quad e \in E(\mathcal{C}) \\ & y_C \ge 0, \quad C \in \mathcal{C}. \end{cases}$$

A clutter is said to be *ideal* if, for every edge-capacities $w \in \mathbb{Z}_+^{E(\mathcal{C})}$, (P) has an optimal solution that is integral. A beautiful result of Lehman [9] states that a clutter is ideal if and only if its blocker is. Edge-capacities $w \in \mathbb{Z}_+^{E(\mathcal{C})}$ are said to be *eulerian* if, for every B and B' in $b(\mathcal{C})$, $w(B)$ and $w(B')$ have the same parity. Seymour [13] calls a binary clutter *cycling* if, for every eulerian edge-capacities $w \in \mathbb{Z}_+^{E(\mathcal{C})}$, (P) and (D) both have optimal solutions that are integral. It can be readily checked that if a clutter is cycling (or ideal) then so are all its minors ([13, 14]). Therefore, one can characterize the class of cycling clutters by excluding minor-minimal clutters that are not in this class. In this paper, we will only focus on binary clutters.

$\mathcal{O}_5$ is the clutter of the odd circuits of K_5. Let $\mathcal{L}_7$ be the clutter of the lines of the Fano matroid, i.e. $E(\mathcal{L}_7) = \{1,2,3,4,5,6,7\}$ and

$$\mathcal{L}_7 := \{\{1,2,7\},\{3,4,7\},\{5,6,7\},\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\}\}.$$

Let $\mathcal{P}_{10}$ be the collection of the postman sets of the Petersen graph, i.e. sets of edges which induce a subgraph whose odd degree vertices are the (odd degree) vertices of the Petersen graph. Observe that the four clutters $\mathcal{O}_5, b(\mathcal{O}_5), \mathcal{L}_7, \mathcal{P}_{10}$ are binary, and moreover, it can be readily checked that none of these clutters is cycling. Hence, if a binary clutter is cycling then it cannot have any of these clutters as a minor. The following excluded minor characterization is predicted.

Conjecture 1 (Cycling Conjecture) *A binary clutter is cycling if, and only if, it has none of the following minors:* $\mathcal{O}_5, b(\mathcal{O}_5), \mathcal{L}_7, \mathcal{P}_{10}$.

The Cycling Conjecture, as stated, can be found in Schrijver [12]. However, this conjecture was first proposed by Seymour [13] and then modified by A.M.H. Gerards and B. Guenin. It is worth mentioning that this conjecture contains the *four color theorem* [15]. None of our results in this paper have any apparent bearings on this theorem.

Consider a finite graph G, where parallel edges and loops are allowed. A *cycle* of G is the edge set of a subgraph of G where every vertex has even degree. A *circuit* of G is a minimal cycle, and a *path* is a circuit minus an edge. We define an *st-path* as follows: if $s \neq t$ then it is a path where s and t are the degree one vertices of the path; otherwise, when $s = t$ then it is just the singleton vertex s. Let Σ be a subset of its edges. The pair (G, Σ) is called a *signed graph.* We say a subset S of the edges is *odd* (resp. *even*) in (G, Σ) if $|S \cap \Sigma|$ is odd (resp. even). Let s, t be vertices of G. We call a subset of the edges of (G, Σ) an *odd st-walk* if it is either an odd st-path, or it is the union of an even st-path P and an odd circuit C where P and C share at most one vertex. Observe that when $s = t$ then an odd st-walk is simply an odd circuit. It is easy to see that clutters of odd st-walks are closed under taking minors. As is shown in [6] the clutter of odd st-walks is binary, and it does not have a minor isomorphic to $b(\mathcal{O}_5)$ or $\mathcal{P}_{10}$. In this paper, we verify the Cycling Conjecture for this class of binary clutters:

Theorem 2 *A clutter of odd st-walks is cycling if, and only if, it has no $\mathcal{O}_5$ and no $\mathcal{L}_7$ minor.*

2 Restating Theorem 2

One can view Theorem 2 as a packing and covering result. We need the following definition: two edges of a signed graph are *parallel* if they have the same end-vertices as well as the same sign. Now let $(G = (V, E), \Sigma)$ be a signed graph without any parallel edges, and choose $s, t \in V$. Let $\mathcal{C}$ be the clutter of the odd st-walks, over the ground set E, and choose edge-capacities $w \in \mathbb{Z}_+^E$. An *odd st-walk cover of* (G, Σ) is simply a cover for $\mathcal{C}$. When there is no ambiguity, we refer to an odd st-walk cover as just a cover.

Proposition 3 (Guenin [6]) *If a subset of the edges is a minimal cover then it is either an st-bond (a minimal st-cut) or it is of the form $\Sigma \triangle C$, where C is a cut with s and t on the same shore.*

The minimal covers of the latter form above are called *signatures*. Notice that if Σ' is a signature, then (G, Σ) and (G, Σ') have the same clutter of odd st-walks.

Reset (G, Σ) as follows: replace each edge e of (G, Σ) with w_e parallel edges. The *packing number* $\nu(G, \Sigma)$ of (G, Σ) is the maximum number of pairwise (edge-)disjoint odd st-walks. A dual parameter to the packing number is the *covering number* $\tau(G, \Sigma)$, which records the minimum size of a cover of (G, Σ). Consider a packing of $\nu(G, \Sigma)$ pairwise disjoint odd st-walk and a cover of size $\tau(G, \Sigma)$. As the cover intersects every odd st-walk in the packing, it follows that $\tau(G, \Sigma) \geq \nu(G, \Sigma)$. A natural question arises: when does equality hold? Theorem 2 gives sufficient conditions for a signed graph to satisfy $\tau(G, \Sigma) = \nu(G, \Sigma)$. To elaborate, observe that $\tau(G, \Sigma)$ is the value of (P) and $\nu(G, \Sigma)$ is the value of (D). For w to be eulerian is to say that every two minimal covers of (G, Σ) have the same parity. Therefore, Proposition 3 implies the following.

Remark 4 *Edge-capacities $w = \mathbf{1}$ are eulerian if, and only if,*

- *(i) $s = t$ and the degree of every vertex is even, or*
- *(ii) $s \neq t$, $\deg(s) - |\Sigma|$ and the degree of every vertex in $V - \{s, t\}$ are even.*

We call such signed graphs *st-eulerian*.

Just like how we defined minor operations for clutters, we now define minor operations for signed graphs. Let $e \in E$. Then the minor operations for $\mathcal{C}$ correspond to the following minor operations for (G, Σ): (1) *delete* e: replace (G, Σ) by $(G \setminus e, \Sigma - \{e\})$, (2) *contract* e: replace (G, Σ) by $(G/e, \Sigma')$, where Σ' is a signature of (G, Σ) that does not use the edge e. Observe that vertices s and

t move to wherever the edge contractions take them, and if s and t are ever identified then we say $s = t$. A signed graph (H, Γ) *is a minor of* (G, Σ) if it is isomorphic to a signed graph obtained from (G, Σ) by a sequence of edge deletions, edge contractions, and possibly deletion of isolated vertices and switching s and t. Note that if (H, Γ) is a minor of (G, Σ), then the clutter of odd st-walks of (H, Γ) is a minor of the clutter of odd st-walks of (G, Σ).

The two special clutters $\mathcal{O}_5$ and $\mathcal{L}_7$ that appear in Theorem 2 have the following representations: $\mathcal{O}_5$ is the clutter of odd st-walks of $\widetilde{K_5} := (K_5, E(K_5))$ where $s = t$ is one of the five vertices, and $\mathcal{L}_7$ is the clutter of odd st-walks of the signed graph F_7 with $s \neq t$, as shown in Figure 1. Observe that $\tau(\widetilde{K_5}) = 4 > 2 = \nu(\widetilde{K_5})$ and $\tau(F_7) = 3 > 1 = \nu(F_7)$. We can now restate Theorem 2 as

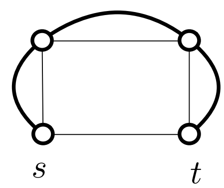

Fig. 1. Signed graph F_7: a representation of $\mathcal{L}_7$. Bold edges are odd.

follows, and in fact, we will prove this restatement instead of the original one:

Theorem 5 *Let (G, Σ) be a signed graph with $s, t \in V(G)$. If (G, Σ) is an st-eulerian signed graph that does not contain $\widetilde{K_5}$ or F_7 as a minor then $\tau(G, \Sigma) = \nu(G, \Sigma)$.*

3 Extensions of Theorem 2

Let $(G = (V, E), \Sigma)$ be a signed graph with $s, t \in V$. Suppose (G, Σ) is an st-eulerian signed graph that does not contain $\widetilde{K_5}$ or F_7 as a minor. If $s \neq t$ let τ_{st} be the size of a minimum st-bond, otherwise let $\tau_{st} := \tau(G, \Sigma)$. Observe that $\tau_{st} \geq \tau(G, \Sigma)$ as every st-bond is also a cover. Add $\tau_{st} - \tau(G, \Sigma)$ odd loops to (G, Σ) to obtain another st-eulerian signed graph (G', Σ'). Since neither $\widetilde{K_5}$ nor F_7 contain an odd loop, it follows that (G', Σ') also does not contain $\widetilde{K_5}$ or F_7 as a minor. Observe that $\tau(G', \Sigma') = \tau(G, \Sigma) + (\tau_{st} - \tau(G, \Sigma)) = \tau_{st}$ and so by Theorem 2, one can find a packing of τ_{st} pairwise disjoint odd st-walks in (G', Σ'). In (G, Σ) this packing corresponds to a collection of τ_{st} pairwise disjoint elements, $\tau(G, \Sigma)$ of which are odd st-walks and the remaining elements are even st-paths. Therefore, we get the following equivalent, and sharper, formulation of Theorem 5.

Theorem 6 *Let (G, Σ) be a signed graph with $s, t \in V(G)$. Suppose that (G, Σ) is an st-eulerian signed graph that does not contain $\widetilde{K_5}$ or F_7 as a minor. Then* *there exists a collection of $\tau_{st}(G,\Sigma)$ pairwise (edge-)disjoint elements, $\tau(G,\Sigma)$ of which are odd st-walks and the remaining elements are even st-paths.*

We can obtain a counterpart to Theorem 6 as follows: let τ_Σ be the size of a minimum signature. Observe that $\tau_\Sigma \geq \tau(G,\Sigma)$ and that $\tau(G,\Sigma) = \min\{\tau_{st}, \tau_\Sigma\}$. In contrast to above, this time we add $\tau_\Sigma - \tau(G,\Sigma)$ even edges between s and t to (G,Σ) to obtain another st-eulerian signed graph (G',Σ'). Notice, however, that we can no longer guarantee that (G',Σ') contains no $\widetilde{K_5}$ or F_7 minor. Observe that this is true if, and only if, (G,Σ) does not contain $\widetilde{K_5}, \widetilde{K_5}^0, \widetilde{K_5}^1, \widetilde{K_5}^2, \widetilde{K_5}^3$ or F_7^- as a minor, where

(*i*) for $i \in \{0,1,2,3\}$, $\widetilde{K_5}^i$ is the signed graph obtained from splitting a vertex, and its incident edges, of $\widetilde{K_5}$ into two vertices s,t, where s has degree i and t has degree $4-i$, and
(*ii*) F_7^- is the signed graph obtained from F_7 by deleting the edge between s and t.

Note that if we add an even edge to any of these signed graphs, then a $\widetilde{K_5}$ or an F_7 appears as a minor. It can be readily checked that if (G,Σ) does not contain any of these five signed graphs as a minor, then (G',Σ') contains no $\widetilde{K_5}$ or F_7 minor. Observe now that $\tau(G',\Sigma') = \tau(G,\Sigma) + (\tau_\Sigma - \tau(G,\Sigma)) = \tau_\Sigma$ and so by Theorem 2, one can find a packing of τ_Σ pairwise disjoint odd st-walks in (G',Σ'). In (G,Σ) this packing corresponds to a collection of τ_Σ pairwise disjoint elements, $\tau(G,\Sigma)$ of which are odd st-walks and the remaining elements are odd circuits. Thus, the following counterpart to Theorem 6 is obtained.

Theorem 7 *Let (G,Σ) be a signed graph with $s,t \in V(G)$. Suppose that (G,Σ) is an st-eulerian signed graph that does not contain $\widetilde{K_5}, \widetilde{K_5}^0, \widetilde{K_5}^1, \widetilde{K_5}^2, \widetilde{K_5}^3$ or F_7^- as a minor. Then in (G,Σ) there exists a collection of $\tau_\Sigma(G,\Sigma)$ pairwise (edge-)disjoint elements, $\tau(G,\Sigma)$ of which are odd st-walks and the remaining elements are odd circuits.*

4 Applications of Theorem 2

In this section, we discuss some applications of Theorem 2. Observe that a cycling clutter is also ideal. As a corollary, we get the following theorem:

Corollary 8 (Guenin [6]) *A clutter of odd st-walks is ideal if, and only if, it has no $\mathcal{O}_5$ and no $\mathcal{L}_7$ minor.*

When $s = t$ an odd st-walk is just an odd circuit. A signed graph is said to be *weakly bipartite* if the clutter of its odd circuits is ideal. The clutter of odd circuits does not contain an $\mathcal{L}_7$ minor [6]. Hence, we get the following two results as corollaries of Theorem 2:

Corollary 9 (Guenin [5]) *A signed graph is weakly bipartite if, and only if, it has no $\widetilde{K_5}$ minor.*

Corollary 10 (Geelen and Guenin [3]) *A clutter of odd circuits is cycling if, and only if, it has no $\mathcal{O}_5$ minor.*

Observe that $2w$ is eulerian for any $w \in \mathbb{Z}_+^{E(G)}$. As a result, the following result follows as a corollary of Theorem 2:

Theorem 11 *Suppose that $\mathcal{C}$ is a clutter of odd st-walks without an $\mathcal{O}_5$ or an $\mathcal{L}_7$ minor. Then, for any edge-capacities $w \in \mathbb{Z}_+^{E(G)}$, the linear program (P) has an optimal solution that is integral and its dual (D) has an optimal solution that is half-integral.*

To obtain more applications of Theorem 2, we will turn to its restatement Theorem 5, and naturally try to find nice classes of signed graphs without a $\widetilde{K_5}$ or an F_7 minor.

4.1 Signed graphs without $\widetilde{K_5}$ and F_7 minor

Let (G, Σ) be a signed graph with $s, t \in V$. Observe that if $s = t$ then (G, Σ) has no F_7 minor, and there are many classes of such signed graphs without a $\widetilde{K_5}$ minor. For instance, whenever G is planar or $|\Sigma| = 2$, (G, Σ) does not contain a $\widetilde{K_5}$ minor. Other classes of such signed graphs can be found in [4, 3]. In this section, we focus only on signed graphs (G, Σ) with distinct $s, t \in V$.

A *blocking vertex* is a vertex v whose deletion removes all the odd cycles, and a *blocking pair* is a pair of vertices $\{u, v\}$ whose deletion removes all the odd cycles.

Remark 12 *The following classes of signed graphs with $s \neq t$ do not contain $\widetilde{K_5}$ or F_7 as a minor:*

(1) *signed graphs with a blocking vertex,*
(2) *signed graphs where $\{s, t\}$ is a blocking pair,*
(3) *plane signed graphs with at most two odd faces,*
(4) *signed graphs that have an even face embedding on the projective plane, and s and t are connected with an odd edge,*
(5) *signed graphs where every odd st-walk is connected, and*
(6) *plane signed graphs with a blocking pair $\{u, v\}$ where s, u, t, v appear on a facial cycle in this cyclic order.*

Observe that class (5) contains (2) and (4). We will apply Theorem 5 to the first three classes, and in the first two cases, we obtain two well-known results. However, the third class will yield a new and interesting result on packing odd circuit covers. Notice that one can even apply Theorem 6 to these classes.

Observe further that the signed graphs in (1) and (2) do not contain $\widetilde{K_5}^0, \widetilde{K_5}^1, \widetilde{K_5}^2, \widetilde{K_5}^3$ or F_7^- as a minor either, so one may even consider applying Theorem 7 to these classes. We leave it to the reader to find out what Theorems 6 and 7 applied to these classes imply.

4.2 Class (1): packing T-joins with $|T| = 4$

Let H be a graph with vertex set W, and choose an even vertex subset T. A *T-join* of H is an edge subset whose odd degree vertices are (all) the vertices in T. A *T-cut* of H is an edge subset of the form $\delta(U)$ where $U \subseteq W$ and $|U \cap T|$ is odd. Observe that the blocker of the clutter of minimal T-joins is the clutter of minimal T-cuts.

We are now ready to prove the following result as a corollary of Theorem 2. However, it should be noted that this result (for T of size at most 8, in fact) is relatively easy to prove from first principles, as is shown in [1].

Corollary 13 (Cohen and Lucchesi [1]) *Let H be a graph and choose a vertex subset T of size 4. Suppose that every vertex of H not in T has even degree and that all the vertices in T have degrees of the same parity. Then the maximum number of pairwise (edge-) disjoint T-joins is equal to the minimum size of a T-cut.*

Proof. Suppose that $T = \{s, t, s', t'\}$. Identify s' and t' to obtain G, and let $\Sigma = \delta_H(s')$. Then the signed graph (G, Σ) contains a blocking vertex $s't'$, and so it belongs to class (i). By Remark 4, (G, Σ) is st-eulerian. Theorem 2 then implies that $\tau(G, \Sigma) = \nu(G, \Sigma)$. However, observe that an odd st-walk of (G, Σ) is a T-join of H, and a T-join in H contains an odd st-walk of (G, Σ). Hence, $\tau(G, \Sigma) = \nu(G, \Sigma)$ implies that the maximum number of pairwise disjoint T-joins is equal to the minimum size of a T-cut. □

4.3 Class (2): packing two-commodity paths

Corollary 14 (Hu [7], Rothschild and Whinston [10]) *Let H be a graph and choose two pairs (s_1, t_1) and (s_2, t_2) of vertices, where $s_1 \neq t_1$, $s_2 \neq t_2$, all of s_1, t_1, s_2, t_2 have the same parity, and all the other vertices have even degree. Then the maximum number of pairwise (edge-)disjoint paths, that are between s_i and t_i for some $i = 1, 2$, is equal to the minimum size of an edge subset whose deletion removes all s_1t_1- and s_2t_2-paths.*

Proof. Identify s_1 and s_2, as well as t_1 and t_2 to obtain G, and let $\Sigma = \delta_H(s_1) \triangle \delta_H(t_2)$. Let $s := s_1s_2 \in V(G)$ and $t := t_1t_2 \in V(G)$. Then the signed graph (G, Σ) has $\{s, t\}$ as a blocking pair, and so it belongs to class (2). Again by Remark 4 (G, Σ) is st-eulerian. Therefore, by Theorem 2 we get that $\tau(G, \Sigma) = \nu(G, \Sigma)$. However, observe that an odd st-walk of (G, Σ) is an s_it_i-path of H, for some $i = 1, 2$, and such a path in H contains an odd st-walk of (G, Σ). Thus, $\tau(G, \Sigma) = \nu(G, \Sigma)$ proves the corollary. □

4.4 Class (3): packing odd circuit covers

Theorem 15 *Let (H, Σ) be a plane signed graph with exactly two odd faces and choose distinct $g, h \in V(H)$. Let (G, Σ) be the signed graph obtained from* *identifying g and h in H, and suppose that every two odd circuits of (G, Σ) have the same size parity. Then in (G, Σ) the maximum number of pairwise disjoint odd circuit covers is equal to the size of a minimum odd circuit.*

(Here an odd circuit cover is simply a cover for the clutter of odd circuits.) As the reader may be wondering, what is the rationale behind the rather strange construction of (G, Σ) above? Interestingly, the clutter of minimal odd circuit covers is binary, and so the Cycling Conjecture predicts an excluded minor characterization for when this clutter is cycling. As we did with the clutter of odd st-walks, one can restate the Cycling Conjecture for the clutter of odd circuit covers as follows:

> *(?) for signed graphs (G, Σ) without a $\widetilde{K_5}$ minor such that every two odd circuits have the same parity, the maximum number of pairwise disjoint odd circuit covers is equal to the minimum size of an odd circuit. (?)*

The construction in the statement of Theorem 15 yields a signed graph (G, Σ) that has no $\widetilde{K_5}$ minor, and Theorem 15 verifies the restatement above for these classes of signed graphs.

Proof. Let H^* be the plane dual of H, and let P be an odd gh-path in (H, Σ). Let s and t be the two odd faces of (H, Σ). Consider the plane signed graph (H^*, P); note that this signed graph has precisely two odd faces, namely g and h, and so it belongs to (3). In particular, (H^*, P) contains no $\widetilde{K_5}$ and F_7 minor. Since every two odd circuits of (G, Σ) have the same parity, it follows from Remark 4 that (H^*, P) is st-eulerian. So Theorem 2 applies and we have $\tau(H^*, P) = \nu(H^*, P)$.

We claim that an odd cycle of (G, Σ) is an odd st-walk cover of (H^*, P), and vice-versa. Let L be an odd cycle of (G, Σ). If L is an odd cycle of (H, Σ) then L separates the two odd faces s and t, and so it is an st-cut in (H^*, P). Otherwise, L is an odd gh-path and so $L \triangle P$ is an even cycle of (H, Σ). However, an even cycle in (H, Σ) is a cut in (H^*, P) having s and t on the same shore. Hence, L is of the form $P \triangle \delta(U)$ where $s, t \in U \subseteq V(H^*)$. Therefore, in either cases, L is an odd st-walk cover of (H^*, P). Similarly, one can show that an odd st-walk cover of (H^*, P) is an odd cycle of (G, Σ). Therefore, since $b(b(\mathcal{C})) = \mathcal{C}$ for any clutter $\mathcal{C}$, it follows that an odd circuit cover of (G, Σ) is an odd st-walk of (H^*, P), and vice-versa.

Hence, $\tau(H^*, P)$ is the minimum size of an odd circuit of (G, Σ), and $\nu(H^*, P)$ is the maximum number of pairwise disjoint odd circuit covers of (G, Σ). Since $\tau(H^*, P) = \nu(H^*, P)$, the result follows. □

4.5 Clutter of odd circuits and odd T-joins

Here, we provide yet another application of Theorem 2. This result generalizes Theorem 15. Let $(G = (V, E), \Sigma)$ be a signed graph, and let $T \subseteq V$ be a subset of even size. We call the triple (G, Σ, T) a *signed graft*. Let $\mathcal{C}$ be the clutter over the ground set E that consists of odd circuits and minimal odd T-joins of (G, Σ, T). This minor-closed class of clutters is fairly large. For instance, if $T = \emptyset$ then $\mathcal{C}$ is the clutter of odd circuits, and if Σ is a T-cut then $\mathcal{C}$ is the clutter of T-joins.

Remark 16 *$\mathcal{C}$ is a binary clutter.*

Proof. Take any three elements C_1, C_2, C_3 of $\mathcal{C}$. If an even number of C_1, C_2, C_3 are odd circuits, then $C_1 \triangle C_2 \triangle C_3$ is an odd T-join and so it contains an element of $\mathcal{C}$. Otherwise, an odd number of C_1, C_2, C_3 are odd circuits, and so $C_1 \triangle C_2 \triangle C_3$ is an odd cycle and so it contains an element of $\mathcal{C}$. Since this is true for all C_1, C_2, C_3 in $\mathcal{C}$, it follows from definition that $\mathcal{C}$ is binary. □

Remark 17 *Minimal covers of $\mathcal{C}$ are of the form $\Sigma \triangle \delta(U)$, where $U \subseteq V$ and $|U \cap T|$ is even.*

Proof. Let B be a minimal cover of $\mathcal{C}$. Then B intersects every odd circuit of (G, Σ), and so $B \triangle \Sigma = \delta(U)$ for some $U \subseteq V$. The preceding remark showed $\mathcal{C}$ is binary, and so B intersects every odd T-join in an odd number of edges, so $|U \cap T|$ must be even. □

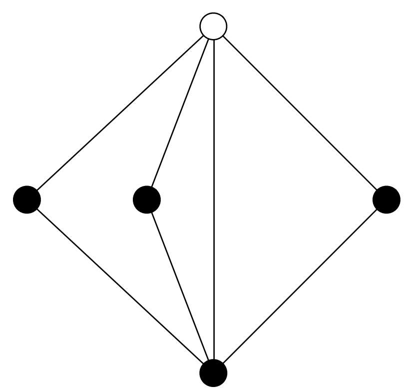

Fig. 2. Signed graft $\widetilde{F}_7$, where all edges are odd and filled-in vertices are in T. For this signed graft, the clutter of odd circuits and minimal odd T-joins isomorphic to $\mathcal{L}_7$.

Theorem 18 *Let (G, Σ, T) be a plane signed graft with exactly two odd faces that has no minor isomorphic to $\widetilde{F}_7$, depicted in Figure 2. Let $\mathcal{C}$ be the clutter of odd circuits and minimal odd T-joins, and suppose that every two elements of $\mathcal{C}$ have the same size parity. Then the maximum of pairwise disjoint minimal covers of $\mathcal{C}$ is equal to the minimum size of an element of $\mathcal{C}$.*

Proof. The proof is similar to the proof of Theorem 15. Let G^* be the plane dual of G, and let P be an odd T-join in (G, Σ, T). Let s and t be the two odd faces of (G, Σ, T). Since (G, Σ, T) has no minor isomorphic to $\widetilde{F}_7$, it follows that the signed graph (G^*, P) contains no F_7 minor, and since it is planar, it has no $\widetilde{K_5}$

minor either. Since every two elements of $\mathcal{C}$ have the same parity, it follows that (G^*, P) is st-eulerian. Hence, by Theorem 5, $\tau(G^*, P) = \nu(G^*, P)$.

We claim that $\mathcal{C}$ is the clutter of odd st-walk covers of (G^*, P), and vice-versa. Let $C \in \mathcal{C}$. If C is an odd circuit of (G, Σ, T), then C is an st-cut of G^*. Otherwise, C is an odd T-join and so $C \triangle P$ is an even cycle of (G, Σ). Thus, $C = P \triangle \delta(U)$ for some $U \subseteq V(G^*) - \{s, t\}$, i.e. C is a signature of (G^*, P).

Hence, $\tau(G^*, P)$ is the minimum size of an element of $\mathcal{C}$, and $\nu(G^*, P)$ is the maximum number of pairwise disjoint covers of $\mathcal{C}$. Since $\tau(G^*, P) = \nu(G^*, P)$, the result follows. □

Let us explain how this result implies Theorem 15. In the context of Theorem 15, let $T = \{g, h\}$. Observe that (H, Σ, T) is a plane signed graft with exactly two odd faces, and it has no minor isomorphic to $\widetilde{F_7}$ (for $|T| = 2$). However, the clutter of odd circuits and minimal odd T-joins of (H, Σ, T) is isomorphic to the clutter of odd circuits of (G, Σ). It is now easily seen that Theorem 18 implies Theorem 15.

5 Overview of the Proof of Theorem 2

A complete proof will appear in the full version. In this section, however, we provide an overview of our proof of Theorem 5, which is equivalent to Theorem 2. The proof follows a routine strategy. We start with an st-eulerian signed graph (G, Σ) that does not *pack*, i.e. $\tau(G, \Sigma) > \nu(G, \Sigma)$, and we will look for either of the *obstructions* $\widetilde{K_5}$, F_7 as a minor.

We say that a signed graph (H, Γ) is a *weighted minor* of (G, Σ) if (H, Γ) minus some parallel edges is a minor of (G, Σ). (Two edges are parallel if they have the same end vertices as well as the same parity.) Observe that if $\widetilde{K_5}$ or F_7 appears as a weighted minor of (G, Σ), then it is also present as a minor since neither of $\widetilde{K_5}, F_7$ contain parallel edges.

Among all st-eulerian non-packing weighted minors of (G, Σ), we pick one (G', Σ') with smallest $\tau(G', \Sigma')$, smallest $|V(G')|$ and largest $|E(G')|$, in this order of priority. Such a non-packing weighted minor exists. Indeed, if an edge has sufficiently many parallel edges, then it may be contracted while keeping (G', Σ') non-packing and $\tau(G', \Sigma')$ unchanged. Reset $(G, \Sigma) := (G', \Sigma')$ and let $\tau := \tau(G, \Sigma)$, $\nu := \nu(G, \Sigma)$. By identifying a vertex of each (connected) component with s, if necessary, we may assume that G is connected. (Notice that neither of the obstructions $\widetilde{K_5}$, F_7 has a cut-vertex.)

Remark 19 *There do not exist $\tau - 1$ pairwise disjoint odd st-walks in (G, Σ).*

Proof. Suppose otherwise. Remove some $\tau - 1$ pairwise disjoint odd st-walks in (G, Σ). Observe that what is left is an odd $\{s, t\}$-join because $|\Sigma|$, $\deg(s), \deg(t)$ and τ all have the same parity and all vertices other than s, t have even degree. Hence, since every odd $\{s, t\}$-join contains an odd st-walk, one can actually find τ pairwise disjoint odd st-walks in (G, Σ), contradicting the fact that (G, Σ) is non-packing. □

Let B be a cover of (G, Σ) of size τ. Choose an edge Ω as follows. If $s = t$ then let $\Omega \in E - B$, and since label s is irrelevant to our problem in this case, we may as well assume $\Omega \in \delta(s)$. Otherwise, when $s \neq t$, let $\Omega \in (\delta(s) \cup \delta(t)) - B$. Indeed, if such an edge does not exist, then $\delta(s) \cup \delta(t)$ is contained in the minimum cover B, implying that $\delta(s) \cup \delta(t) = \delta(s) = \delta(t)$, but this cannot be the case as G is connected and non-packing. Again, we may assume that Ω is incident to s. Let s' be the other end-vertex of Ω. Add two parallel edges Ω_1, Ω_2 to Ω to obtain (K, Γ); this st-eulerian signed graph must pack since $\tau(K, \Gamma) = \tau$ as B is also a minimum cover for (K, Γ), $V(K) = V(G)$ but $|E(K)| > |E(G)|$. Hence, (K, Γ) contains a collection $\{L_1, L_2, \ldots, L_\tau\}$ of pairwise disjoint odd st-walks. Observe that all of Ω, Ω_1 and Ω_2 must be used by the odd st-walks in $\{L_1, L_2, \ldots, L_\tau\}$, say by L_1, L_2, L_3, since otherwise one finds at least $\tau - 1$ disjoint odd st-walks in (G, Σ), which is not the case by the preceding remark. As a result, the sequence $(L_1, L_2, L_3, \ldots, L_\tau)$ corresponds to an *Ω-packing of odd st-walks* in (G, Σ), described as follows:

(*i*) $L_1, \ldots, L_\tau$ are odd st-walks in (G, Σ),
(*ii*) $\Omega \in L_1 \cap L_2 \cap L_3$ and $\Omega \notin L_4 \cup \cdots \cup L_\tau$, and
(*iii*) $(L_j - \{\Omega\} : 1 \leq j \leq \tau)$ are pairwise disjoint subsets of edges.

We fix an Ω-packing $(L_1, L_2, L_3, \ldots, L_\tau)$ having a minimum number of edges in their union.

We call T a *transversal* of a collection of sets if T picks exactly one element from each of the sets. For an odd st-walk L, we say that a minimal cover B is a *mate* of L if $|B - L| = \tau - 3$.

Lemma 20 *Let L be an odd st-walk such that $(G, \Sigma) \setminus L$ contains at least $\tau - 3$ pairwise disjoint odd st-walks collected in $\mathcal{L}$. Then L has a mate B, and $B - L$ is a transversal of $\mathcal{L}$.*

Proof. The signed graph $(G, \Sigma) \setminus L$ packs as it is st-eulerian and $\tau((G, \Sigma) \setminus L) < \tau$. Let B' be one of its minimum covers. By our assumption, $\tau((G, \Sigma) \setminus L) \geq \tau - 3$. Since both (G, Σ) and $(G, \Sigma) \setminus L$ are st-eulerian, it follows that $\tau((G, \Sigma) \setminus L)$ and τ have different parities, and so $\tau((G, \Sigma) \setminus L)$ is either $\tau - 3$ or $\tau - 1$. However, observe that the latter is not possible due to Remark 19 and the fact that $(G, \Sigma) \setminus L$ packs. As a result $|B'| = \tau((G, \Sigma) \setminus L) = \tau - 3$. It is now clear that $B' \cup L$ contains a mate for L, and that B' is a transversal of $\mathcal{L}$. □

Observe that if $L \subseteq L_1 \cup L_2 \cup L_3$ or $L \in \{L_4, \ldots, L_\tau\}$, then $(G, \Sigma) \setminus L$ does contain at least $\tau - 3$ pairwise disjoint odd st-walks. Thus, the preceding lemma guarantees the existence of a mate for any such odd st-walk. Vaguely speaking, mates are used as means to build connectivity, with appropriate signing, between the odd st-walks.

Let us call an odd st-walk L *simple* if it is an odd st-path P; otherwise when L is the union of an odd circuit C and an even st-path P, we call L a *non-simple* odd st-walk. By our definition then, when $s = t$ all the odd st-walks are non-simple. For each $1 \leq i \leq \tau$, either L_i is a simple odd st-walk P_i, or it is a non-simple odd st-walk $C_i \cup P_i$, where C_i is an odd circuit and P_i is an even st-path.

Lemma 21 *One of the following holds:*

(i) L_1, L_2 *and* L_3 *are simple,*
(ii) at least one of L_1, L_2, L_3 *is non-simple, and whenever* L_k *is non-simple for some* $1 \leq k \leq 3$*, then* $\Omega \in C_k$*,*
(iii) at least two of L_1, L_2, L_3 *are non-simple, and* $\Omega \in P_1 \cap P_2 \cap P_3$*.*

We analyze each of the three cases separately, and the techniques used to tackle each case are different. A major difference between our proof and the ones for Corollaries 8, 9 and 10 (see [6, 5, 11, 3]) is in where an obsruction is looked for. In any of the aforementioned proofs, only the first three sets of the Ω-packing assisted in finding an obstruction. For our proof, however, this is no longer the case; some of the odd st-walks in $L_4, \ldots, L_\tau$, as well as their mates, help us in finding either of the obstructions. This concludes our overview of the proof of Theorem 5.

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