Opposite elements in clutters

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Abstract

Let E be a finite set of elements, and let $\mathcal{L}$ be a clutter over ground set E. We say distinct elements e, f are *opposite* if every member and every minimal cover of $\mathcal{L}$ contains at most one of e, f. In this paper, we investigate opposite elements and reveal a rich theory lying underneath such a seemingly simple restriction. The clutter $\mathcal{C}$ obtained from $\mathcal{L}$ after identifying some opposite elements is called an *identification of* $\mathcal{L}$, and inversely, $\mathcal{L}$ is called a *split of* $\mathcal{C}$.

We will show that splitting preserves three clutter properties, namely, idealness, the max-flow min-cut property, as well as the packing property. We will also display several natural examples where a clutter does not have these properties but a split of them does. We will develop tools for recognizing when splitting is not a useful operation, and as well, we will characterize when identification preserves the three mentioned properties. Along the way, we will make connections to spanning arborescences, Steiner trees, comparability graphs, degenerate projective planes, binary clutters, matroids, as well as results of Menger and of Ford and Fulkerson, the Replication Conjecture and a conjecture on ideal, minimally non-packing clutters.

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1 Introduction

Finding a minimum weight spanning tree in an edge-weighted graph is a well-studied classical problem in the field of combinatorial optimization. While there are several approaches to solving the problem, the one that is most appealing to us is the following treatment by Edmonds in 1967. Let $G = (V, E)$ be a graph with non-negative weights $w \in \mathbb{R}^E_+$. The integer program

$$\begin{array}{lll} \min & w^\top x & \\ \text{s.t.} & \sum \big(x_e : e \in C\big) \geq 1 & \text{for each cut } C \\ & x_e \in \{0,1\} & \text{for each edge } e \end{array}$$

models our problem. Here a *cut* is a minimal subset of the form

$$\delta(U) := \big\{\{u,v\} \in E : u \in U, v \in V - U\big\}$$

where U is a non-empty, proper subset of V. Since integer programs in general are computationally difficult to solve, we resort to the following linear programming (LP) relaxation:

$$\begin{array}{lll} \min & w^\top x & \\ \text{s.t.} & \sum \big(x_e : e \in C\big) \geq 1 & \text{for each cut } C \\ & x \geq \mathbf{0}. & \end{array}$$

Unfortunately, this relaxation is quite weak in the sense that its optimal value can be smaller than the minimum weight of a spanning tree. For instance, if G is a triangle (i.e. a circuit on three edges) and each edge has weight one, then the minimum weight of a spanning tree is clearly 2, while the optimal value of the LP above is at most $\frac{3}{2}$ as $x = \big(\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\big)$ is feasible. One option to resolve this issue is to come up with a better LP relaxation, as is done in Nash-Williams [24] and Tutte [31], but we are reluctant to do so as the relaxation above is quite simple. A second option is a trick Edmonds employed in 1967 [11]. Let $D = (V, A)$ be the directed graph obtained from G after replacing each edge $\{u, v\}$ with two opposite arcs (u, v) and (v, u), each of the same weight as $\{u, v\}$. Let r be an arbitrary vertex of D. A *spanning r-arborescence* is a minimal subset of A that contains for each $v \in V$ a directed vr-path. The key observation now is that there is a weight-preserving bijection between the spanning trees in G and the spanning r-arborescences in D. Hence instead of looking for a minimum weight spanning tree in G, we can equivalently look for a minimum weight spanning r-arborescence in D, which can be modelled as the integer program

$$\begin{array}{lll} \min & \overrightarrow{w}^\top x & \\ \text{s.t.} & \sum \big(x_a : a \in L\big) \geq 1 & \text{for each } r\text{-cut } L \\ & x_a \in \{0,1\} & \text{for each arc } a. \end{array}$$

Here an *r-cut* is a minimal subset of the form

$$\delta^+(U) := \big\{(u,v) \in A : u \in U, v \in V - U\big\}$$

where U is a non-empty subset of $V - \{r\}$. This time around however, the natural LP relaxation

$$\begin{array}{lll} \min & \overrightarrow{w}^\top x & \\ \text{s.t.} & \sum \big(x_a : a \in L\big) \geq 1 & \text{for each } r\text{-cut } L \\ & x \geq \mathbf{0}. & \end{array}$$

is strong in the sense that it always has an integral optimal solution [11]![1]

To give an overview of what just happened, we need to define clutters. Let E be a finite set of *elements*, called a *ground set*, and let $\mathcal{L}$ be a family of subsets of E, called *members*. We say $\mathcal{L}$ is a *clutter* over ground set $E(\mathcal{L}) := E$ if no member is contained in, or is equal to, another member [12]. For instance, the family of cuts (resp. spanning trees) of a graph $G = (V, E)$ forms a clutter over E, and the family of r-cuts (resp. spanning r-arborescences) of a directed graph $D = (V, A)$ forms a clutter over A.

Our goal is to study and extend Edmonds' trick to the more general framework of clutters. Essentially what he does is bijectively map a weak clutter to a strong one, by means of replacing an element with two opposite elements. There are however two mysteries:

- What does it mean for two elements to be opposite in a clutter?
- What does it mean to replace an element with two elements in a clutter?

We therefore have our road paved for us, and we proceed in this order. In §2 we generalize the notion of opposite arcs, and in §3 we generalize the notion of bidirecting edges to clutters. In §4 we show that bidirecting in general preserves certain polyhedral attributes of the underlying clutter, and in §5 we study when bidirecting fails to be useful. In §6 we study the inverse operation of bidirecting and discover a beautiful theory lying underneath, and finally in §7, we conclude with open problems.

1.1 A glimpse of our results

Edmonds used his bidirecting trick to move from the clutter of spanning trees to the clutter of spanning arborescences. We find an analogue for vertex covers: we will show how to "bidirect" the clutter of vertex covers of any comparability graph to obtain the clutter of vertex covers of a bipartite graph. Essentially what this means is that comparability graphs are a distortion of bipartite graphs. Another application is about the relationship between st-path clutters and directed st-path clutters. There are two classical results in the field about these two clutters: Menger's theorem [23] stating that

> the minimum number of edges of an st-cut is equal to the maximum number of pairwise edge-disjoint st-paths,

as well as Ford and Fulkerson's result [14] stating that

> the minimum number of arcs of an st-cut is equal to the maximum number of pairwise arc-disjoint directed st-paths.

The machinery we develop implies the equivalence of these two results (that is to say, the machinery can be used in an abstract setting to prove either statements assuming the other one is true). We are also able to show that

[1] To be fair, Edmonds' linear program is different from the one stated here, but as pointed out by Fulkerson [15] (see his second paragraph of §4.4), Edmonds' proof can be modified to prove what is claimed here.

the fractional versions of these two results are equivalent. We make quite a natural connection to Conforti and Cornuéjols' so-called Replication Conjecture [7], and along the way, we prove a deep result that we believe to be in a sense the little sister of this conjecture. A connection to a conjecture by Cornuéjols, Guenin and Margot on ideal, minimally non-packing clutters [10] is also made.

1.2 Terminology

Let $\mathcal{L}$ be a clutter over ground set E. With clutter $\mathcal{L}$ we associate a $0,1$ matrix $M(\mathcal{L})$ whose columns are indexed by E and whose rows are the incidence vectors of the members of $\mathcal{L}$. (Note $M(\mathcal{L})$ is defined up to permutations of the rows.) Two clutters $\mathcal{L}_1, \mathcal{L}_2$ are *isomorphic*, denoted as $\mathcal{L}_1 \cong \mathcal{L}_2$, if relabeling the ground set of one clutter yields the other one. A *cover* is a subset of E that intersects every member of $\mathcal{L}$. The *blocker* of $\mathcal{L}$, denoted $b(\mathcal{L})$, is another clutter over the same ground set whose members are the *minimal* covers of $\mathcal{L}$. It is known that $b(b(\mathcal{L})) = \mathcal{L}$ [12, 17]. For instance, in a graph the blocker of the clutter of spanning trees is the clutter of cuts, and in a directed graph the blocker of the clutter of spanning r-arborescences is the clutter of r-cuts [15].

One can define two minor operations on $\mathcal{L}$. The *contraction* $\mathcal{L}/e$ of an element $e \in E$ is the clutter over ground set $E - \{e\}$ consisting of minimal sets in $\{L - \{e\} : L \in \mathcal{L}\}$. The *deletion* $\mathcal{L} \setminus e$ of an element $e \in E$ is the clutter over ground set $E - \{e\}$ with members $\{L : L \in \mathcal{L}, e \notin L\}$. Observe that $b(\mathcal{L} \setminus e) = b(\mathcal{L})/e$ and $b(\mathcal{L}/e) = b(\mathcal{L}) \setminus e$ [29]. A clutter $\mathcal{L}'$ obtained from $\mathcal{L}$ after contracting elements I and deleting elements J is called a *minor* of $\mathcal{L}$ (and $\mathcal{L}'$ does not depend on the order of the operations, i.e. minors operations commute). We denote $\mathcal{L}'$ by $\mathcal{L}/I \setminus J$. Clutter $\mathcal{L}'$ is a *proper* minor of $\mathcal{L}$ if $I \cup J \neq \emptyset$.

Take non-negative integral weights $w \in \mathbb{Z}_+^E$. Denote by $\tau(\mathcal{L}, w)$ the minimum weight of a cover, i.e.

$$\tau(\mathcal{L}, w) := \begin{array}{lll} \min & w^\top x & \\ \text{s.t.} & x(L) \geq 1 & L \in \mathcal{L} \\ & x \in \{0,1\}^E, & \end{array}$$

and denote by $\tau^\star(\mathcal{L}, w)$ the following lower bound:

$$\tau^\star(\mathcal{L}, w) := \begin{array}{lll} \min & w^\top x & \\ \text{s.t.} & x(L) \geq 1 & L \in \mathcal{L} \\ & x \geq \mathbf{0}. & \end{array}$$

Here $x(L)$ denotes $\sum(x_g : g \in L)$. Notice that covers of $\mathcal{L}$ are precisely the incidence vectors of solutions to the integer program. For this reason, we refer to solutions of the linear program as *fractional covers*. Denote by $Q(\mathcal{L})$ polyhedron of all fractional covers of $\mathcal{L}$. We say $\mathcal{L}$ is *ideal* if $Q(\mathcal{L})$ is an integral polyhedron, i.e. if each face contains an integral point. Since $Q(\mathcal{L})$ is contained in the non-negative orthant, it is a point polyhedron, so it is integral if and only if every extreme point of it is integral. Equivalently, $\mathcal{L}$ is ideal if for all weights $w \in \mathbb{Z}_+^E$, $\tau(\mathcal{L}, w) = \tau^\star(\mathcal{L}, w)$ [8]. If $\mathcal{L}$ is ideal, then so are $b(\mathcal{L})$ [15, 20] and every minor of $\mathcal{L}$ [30].

2 Opposite elements

Consider a directed graph with two distinguished opposite arcs. Then these two arcs are never used together in a spanning r-arborescence or in an r-cut. This is the key feature we will take advantage of.

Definition. *Let $\mathcal{L}$ be a clutter over ground set E. We say distinct elements $e, f \in E$ are* **opposite** *if no member or minimal cover contains both of them, i.e. if for each $L \in \mathcal{L} \cup b(\mathcal{L})$ we have $\{e, f\} \not\subseteq L$.*

2.1 A charaterization of opposite elements

Our definition of opposite elements in a clutter requires knowledge of the blocker. However, quite often it is the case that no good description of the blocker is known. Examples of such clutters include the clutter of edges of a graph (over vertices), maximal cliques of a graph (over vertices), Hamilton cycles of a graph (over edges), maximal common independent sets of two matroids, and many others. It is therefore quite natural to ask whether there is a characterization of opposite elements given no knowledge of the blocker, and the answer is yes!

Theorem 2.1. *Let $\mathcal{L}$ be a clutter and take distinct elements e, f. Then e, f are opposite if, and only if,*

$(\diamond)$ *for all $L_e, L_f \in \mathcal{L}$ with $e \in L_e$ and $f \in L_f$, there is a member contained in $(L_e \cup L_f) - \{e, f\}$.*

Proof. Suppose first that e, f are opposite. Choose $L_e, L_f \in \mathcal{L}$ with $e \in L_e, f \in L_f$. Suppose for a contradiction that there is no member of $\mathcal{L}$ contained in $(L_e \cup L_f) - \{e, f\}$. Then $K' := E(\mathcal{L}) - (L_e \cup L_f - \{e, f\})$ must be a cover. Let K be a minimal cover contained in K'; so $K \in b(\mathcal{L})$. In particular, $K \cap L_e \neq \emptyset$ and $K \cap L_f \neq \emptyset$. Since no member contains both e, f, it follows that $K' \cap L_e = \{e\}$ and $K' \cap L_f = \{f\}$, which in turn implies that $\{e, f\} \subseteq K$, a contradiction as no minimal cover contains both e, f.

Conversely suppose $(\diamond)$ holds. Note $(\diamond)$ implies (for equal L_e and L_f) that no member of $\mathcal{L}$ uses both e, f. To prove e, f are opposite, it remains to show that no minimal cover contains both e, f. Suppose otherwise. Let K be a minimal cover containing both e, f. Since $K - \{e\}$ and $K - \{f\}$ are not covers, there exist $L_e, L_f \in \mathcal{L}$ such that $K \cap L_e = \{e\}$ and $K \cap L_f = \{f\}$. Thus $K \cap (L_e \cup L_f - \{e, f\}) = \emptyset$, implying in particular that $L_e \cup L_f - \{e, f\}$ contains no member of $\mathcal{L}$, a contradiction. $\square$

It is worth pointing out that $(\diamond)$ is reminiscent of Lehman's weak circuit elimination axiom for matroids [19].[2] A consequence of this characterization is the following cute inequality:

Corollary 2.2. *Let $\mathcal{L}$ be a clutter with opposite elements e, f. Then for every extreme point $\bar{x}$ of $Q(\mathcal{L})$,*

$$\bar{x}_e + \bar{x}_f \leq 1.$$

Proof. If $\bar{x}_e = 0$ or $\bar{x}_f = 0$ then the result is clear. Otherwise $\bar{x}_e > 0$ and $\bar{x}_f > 0$. As $\bar{x}$ is an extreme point, there exist $L_e, L_f \in \mathcal{L}$ such that $e \in L_e, f \in L_f$ and $\bar{x}(L_e) = 1 = \bar{x}(L_f)$. By Theorem 2.1 there exists $L \in \mathcal{L}$

[2] A clutter of *circuits of a matroid* is one where for each element e and distinct members L_1, L_2 containing e, there is a member contained in $L_1 \cup L_2 - \{e\}$.

such that $L \subseteq (L_e \cup L_f) - \{e, f\}$. As $\bar{x} \in Q(\mathcal{L})$, we get

$$1 + \bar{x}_e + \bar{x}_f \le \bar{x}(L) + \bar{x}_e + \bar{x}_f \le \bar{x}(L_e) + \bar{x}(L_f) = 2$$

and the result follows. $\square$

2.2 Examples: directed Steiner trees, vertex covers and simple clutters

Directed Steiner trees and directed Steiner cuts: Let $D = (V, A)$ be a directed graph, $R \subseteq V$ a non-empty set of *terminals*, and $r \in R$ the *root*. A *directed Steiner tree* is a minimal subset of A that contains for each terminal v a directed vr-path. A *directed Steiner cut* is a minimal subset of A of the form $\delta^+(U)$, where $U \subseteq V - \{r\}$ and $U \cap R \neq \emptyset$. It is well-known that the clutter of directed Steiner trees and the clutter of directed Steiner cuts are blockers of one another [6]. It is now quite easy to see that opposite arcs are also opposite in these two blocking clutters.

Observe that by restricting $R = V$, one obtains the clutters of spanning r-arborescences and r-cuts, mentioned in the introduction, and by restricting $R = \{s, t\}$ and $r = t$, one obtains the clutters of *directed st-paths* and *directed st-cuts*.

Edges and vertex covers: Let $G = (V, E)$ be a graph without loops and parallel edges, and let $\mathcal{L}$ be the clutter of edges of G, over ground set V. (Edges are viewed as subsets of vertices.) A *vertex cover* is a subset of vertices intersecting every edge. By definition, the blocker of $\mathcal{L}$ is the clutter of minimal vertex covers. To identify opposite pairs for $\mathcal{L}$, we appeal to Theorem 2.1. Distinct vertices u, v are opposite if (i) they are not neighbours, and (ii) for each neighbour u' of u and each neighbour v' of v, u' and v' are neighbours. (See Figure 1.)

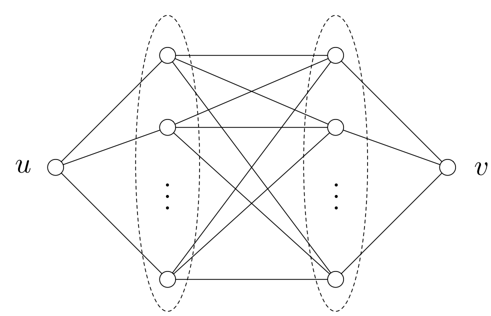


Figure 1: An illustration of opposite vertices u, v in the clutter of edges of a graph.

Simple and special clutters: Let $\mathcal{L}$ be a clutter and take distinct elements e, f. We say $\mathcal{L}$ is $\{e, f\}$-*simple* if there is a partition of $E(\mathcal{L}) - \{e, f\}$ into parts A, B such that

$$\{e\} \cup A, \{f\} \cup B \in \mathcal{L}$$

and there is no other member of $\mathcal{L}$ that uses either of e, f, and

$$\{e\} \cup B, \{f\} \cup A \in b(\mathcal{L})$$

and there is no other member of $b(\mathcal{L})$ that uses either of e, f. (Note $A, B \neq \emptyset$.) Observe that e, f are opposite in any $\{e, f\}$-simple clutter. Moreover by definition, if $\mathcal{L}$ is $\{e, f\}$-simple, then so is $b(\mathcal{L})$. The smallest simple clutter is

$$\mathbb{P}_4 := \big\{\{1,2\}, \{2,3\}, \{3,4\}\big\}$$

whose blocker is

$$b(\mathbb{P}_4) = \big\{\{1,3\}, \{3,2\}, \{2,4\}\big\}.$$

Observe that $\mathbb{P}_4$ is $\{e, f\}$-simple with $e = 1, f = 4, A = \{2\}$ and $B = \{3\}$.

As we will see in Proposition 2.4, simple clutters are the most basic clutters with (non-trivial) opposite elements. Next take an integer $n \geq 2$. A clutter $\mathcal{S}$ over ground set $\{e, f, 1, 2, \ldots, n\}$ is (e, f)-*special* if its incidence matrix is

$$M(\mathcal{S}) = \begin{pmatrix} 1 & & 1 & & & & & \\ & 1 & & 1 & 1 & \cdots & 1 & 1 \\ & & 1 & 1 & & & & \\ & & 1 & & 1 & & & \\ & & \vdots & & & \ddots & & \\ & & 1 & & & & 1 & \\ & & 1 & & & & & 1 \end{pmatrix}.$$

Formally,

$$\mathcal{S} = \big\{\{e,1\}, \{f,2,3,\ldots,n\}, \{1,2\}, \{1,3\}, \ldots, \{1,n\}\big\}.$$

Observe that

$$b(\mathcal{S}) = \big\{\{f,1\}, \{e,2,3,\ldots,n\}, \{1,2\}, \{1,3\}, \ldots, \{1,n\}\big\}$$

and its incidence matrix is

$$M\big(b(\mathcal{S})\big) = \begin{pmatrix} & 1 & 1 & & & & & \\ 1 & & & 1 & 1 & \cdots & 1 & 1 \\ & & 1 & 1 & & & & \\ & & 1 & & 1 & & & \\ & & \vdots & & & \ddots & & \\ & & 1 & & & & 1 & \\ & & 1 & & & & & 1 \end{pmatrix}.$$

Note that $b(\mathcal{S})$ is (f, e)-special. Observe further that e, f are opposite elements and $\mathcal{S}$ is $\{e, f\}$-simple. We will see in the next subsection that special clutters are a significant subclass of simple clutters.

2.3 Structural properties

The following remark is immediate by our definition of opposite elements and minors.

Remark 2.3. *Let $\mathcal{L}$ be a clutter with opposite elements e, f. Take disjoint $I, J \subseteq E(\mathcal{L}) - \{e, f\}$. Then e, f are opposite in $\mathcal{L}/I \setminus J$.*

Proposition 2.4. *Let $\mathcal{L}$ be a clutter with opposite elements e, f, where*

> ($\star$) *each of e, f is used in at least one member of the clutter.*

Choose a minor $\mathcal{L}'$ of $\mathcal{L}$ that is minimal subject to ($\star$). Then $\mathcal{L}'$ is $\{e, f\}$-simple.

Proof. An element is used by a member of a clutter if and only if it is used by a member of the blocker. Thus $b(\mathcal{L}')$ is a minor of $b(\mathcal{L})$ that is also minimal subject to

> ($\star\star$) each of e, f is used in at least one member of $b(\mathcal{L}')$.

Note that by Remark 2.3, elements e, f are opposite in $\mathcal{L}'$. Choose arbitrary $L_e, L_f \in \mathcal{L}'$ where $e \in L_e$ and $f \in L_f$. Let $I := L_e \cap L_f$ and $U := L_e \cup L_f$. Note $(I \cup \overline{U}) \cap \{e, f\} = \emptyset$ for $\overline{U} = E(\mathcal{L}') - U$. Observe that the clutter $\mathcal{L}'/I \setminus \overline{U}$ also satisfies ($\star$), as it contains members $L_e - I$ and $L_f - I$. Thus the minimality of $\mathcal{L}'$ implies that $I = \overline{U} = \emptyset$, so

$$L_e \cap L_f = \emptyset \quad \text{and} \quad L_e \cup L_f = E(\mathcal{L}').$$

Since the equalities hold for any arbitrary L_e and L_f, it follows that L_e (resp. L_f) is the unique member of $\mathcal{L}$ containing element e (resp. f). Similarly, as ($\star\star$) holds, there is a unique member K_e (resp. K_f) of $b(\mathcal{L}')$ containing e (resp. f) and

$$K_e \cap K_f = \emptyset \quad \text{and} \quad K_e \cup K_f = E(b(\mathcal{L}')) = E(\mathcal{L}').$$

Moreover, since K_e is a minimal cover, there must be a member of $\mathcal{L}$ intersecting K_e at precisely $\{e\}$; this member has to inevitably be the unique L_e. Hence, together with a similar argument for K_f and L_f, we get

$$L_e \cap K_e = \{e\} \quad \text{and} \quad L_f \cap K_f = \{f\}.$$

It now easily follows that $\mathcal{L}'$ is $\{e, f\}$-simple, where $A = K_e - \{e\}$ and $B = K_f - \{f\}$. □

We are now ready to state the main result of this subsection.

Theorem 2.5. *Let $\mathcal{L}$ be a clutter with opposite elements e, f. Then the following are equivalent:*

(i) *$\mathcal{L}$ has an (e, f)-special minor,*

(ii) *there exist $L_e \in \mathcal{L}$ and $K_f \in b(\mathcal{L})$ such that $e \in L_e, f \in K_f$ and $|L_e \cap K_f| = 1$.*

Proof. Suppose first that (i) holds. Choose disjoint $I, J \subseteq E(\mathcal{L}) - \{e, f\}$ such that $\mathcal{L}' := \mathcal{L}/I \setminus J$ is (e, f)-special. Choose the unique member L'_e (resp. K'_f) of $\mathcal{L}'$ (resp. $b(\mathcal{L}')$) containing e (resp. f). Note $|L'_e \cap K'_f| = 1$. Now choose $L_e \in \mathcal{L}$ and $K_f \in b(\mathcal{L})$ such that $L'_e \subseteq L_e \subseteq L'_e \cup I$ and $K'_f \subseteq K_f \subseteq K'_f \cup J$. Since $I \cap J = \emptyset$, it follows that $|L_e \cap K_f| = |L'_e \cap K'_f| = 1$. As $e \in L_e$ and $f \in K_f$, (ii) holds.

Suppose now that (ii) holds. Choose $g \in E(\mathcal{L}) - \{e, f\}$ such that $L_e \cap K_f = \{g\}$. Let $I := L_e - \{e, g\}$, $J := K_f - \{f, g\}$ and $\mathcal{L}' := \mathcal{L}/I \setminus J$.

Claim 1. *$\{e, g\}$ is the unique member of $\mathcal{L}'$ using e, and $\{f, g\}$ is the unique member of $b(\mathcal{L}')$ using f.*

Proof of Claim. Observe that $\{e, g\} \in \mathcal{L}'$ and $\{f, g\} \in b(\mathcal{L}')$, as $\{e, g\} \cup I = L_e$ and $\{f, g\} \cup J = K_f$. Let L'_e be any member of $\mathcal{L}'$ using e. Then $L'_e \cap \{f, g\} \neq \emptyset$ and as $f \notin L'_e$ by Remark 2.3, it follows that $g \in L'_e$ implying that $L'_e \supseteq \{e, g\}$ so $L'_e = \{e, g\}$. Hence, $\{e, g\}$ is the unique member of $\mathcal{L}'$ using e, and similarly, $\{f, g\}$ is the unique member of $b(\mathcal{L}')$ using f. $\diamondsuit$

Choose a minor $\mathcal{L}''$ of $\mathcal{L}'$ that is minimal subject to

each of e, f is used in at least one member of $\mathcal{L}''$.

Proposition 2.4 implies that $\mathcal{L}''$ is $\{e, f\}$-simple. In fact,

Claim 2. *$\mathcal{L}''$ is (e, f)-special.*

Proof of Claim. It follows from Claim 1 that $\{e, g\} \in \mathcal{L}''$ and $\{f, g\} \in b(\mathcal{L}'')$. Let $b_1, \ldots, b_{n-1}$ be the elements of $E(\mathcal{L}'') - \{e, f, g\}$. If $n = 2$ then $\mathcal{L}'' \cong \mathbb{P}_4$ and the result easily follows. Otherwise $n \geq 3$. To prove the claim, we need to show that

$$\mathcal{L}'' = \big\{\{e, g\}, \{f, b_1, \ldots, b_{n-1}\}, \{g, b_1\}, \{g, b_2\}, \ldots, \{g, b_{n-1}\}\big\}.$$

Let $B := \{b_1, \ldots, b_{n-1}\}$. As $\mathcal{L}''$ is $\{e, f\}$-simple, it follows that $\{f\} \cup B$ is the unique member of $\mathcal{L}''$ using f and $\{e\} \cup B = \{e, b_1, \ldots, b_{n-1}\}$ is the unique member of $b(\mathcal{L}'')$ using e. Since $\{e, b_1, \ldots, b_{n-1}\}$ is a minimal cover, for each $i \in [n-1]$ [3], there exists $L_i \in \mathcal{L}''$ such that $L_i \cap \{e, b_1, \ldots, b_{n-1}\} = \{b_i\}$. Since $n \geq 3$, it follows that $L_i \neq \{f, b_1, \ldots, b_{n-1}\}$, so $f \notin L_i$ implying inevitably that $L_i = \{g, b_i\}$. Thus,

$$\mathcal{L}'' \supseteq \big\{\{e, g\}, \{f, b_1, \ldots, b_{n-1}\}, \{g, b_1\}, \{g, b_2\}, \ldots, \{g, b_{n-1}\}\big\}.$$

Let L be any member of $\mathcal{L}''$. By uniqueness, if $e \in L$ then $L = \{e, g\}$ and if $f \in L$ then $L = \{f, b_1, \ldots, b_{n-1}\}$. Since $L \not\subseteq \{b_1, \ldots, b_{n-1}\}$, it follows that $g \in L$. As $L \neq \{g\}$, we must have that for some $i \in [n-1]$, $L \supseteq \{g, b_i\}$ and so $L = \{g, b_i\}$. Thus equality holds above, proving the claim. $\diamondsuit$

By Claim 2, $\mathcal{L}'$ has an (e, f)-special minor, and since $\mathcal{L}'$ itself is a minor of $\mathcal{L}$, (i) follows. $\square$

We will see an extension of this result for multiple opposite pairs in §6.2.

[3]For an integer $m \geq 1$, $[m] := \{1, \ldots, m\}$.

3 Generating opposite elements

Here we will see two procedures for generating opposite elements. To be precise, we show how starting with any clutter, one can obtain another clutter with opposite elements. We will then give the main motive for generating opposite elements, followed by several examples where this operation is particularly helpful, demonstrating our stance. We will wrap up this section after making connections to the extension complexity and the integrality gap of polyhedra.

3.1 Split and identify

Let $\mathcal{L}$ be a clutter with opposite elements e, f. The *single identification of* $\mathcal{L}$ *at* e *and* f is the family

$$\mathcal{C} = \big\{L : f \notin L \in \mathcal{L}\big\} \cup \big\{\big(L - \{f\}\big) \cup \{e\} : f \in L \in \mathcal{L}\big\}.$$

(Any family isomorphic to $\mathcal{C}$ is also considered a single identification of $\mathcal{L}$ at e, f.) Inversely, we call $\mathcal{L}$ a *single split of* $\mathcal{C}$ *at* e. When each of e, f is used in a member of $\mathcal{L}$, we say $\mathcal{L}$ is a *proper* single split of $\mathcal{C}$. (Remark 3.3 and Proposition 3.4 provide two stand-alone definitions of single splits.) For instance, the single identification of $\mathbb{P}_4 = \big\{\{1,2\},\{2,3\},\{3,4\}\big\}$ at opposite elements $1, 4$ is the clutter $\Delta_3 := \big\{\{1,2\},\{2,3\},\{3,1\}\big\}$, and inversely, $\mathbb{P}_4$ is a (proper) single split of Δ_3 at element 1.

Remark 3.1. *Take two clutters* $\mathcal{A}, \mathcal{B}$ *where every member of* $\mathcal{A}$ *contains one in* $\mathcal{B}$ *and vice versa. Then* $\mathcal{A} = \mathcal{B}$*.*

The following proposition proves preliminary results on single identifications.

Proposition 3.2. *Let* $\mathcal{L}$ *be a clutter with opposite elements* e, f*, and let* $\mathcal{C}$ *be its single identification at* e, f*. Then*

(i) $\mathcal{C}$ *is a clutter over ground set* $E(\mathcal{L}) - \{f\}$*, and*

(ii) $b(\mathcal{C})$ *is the single identification of* $b(\mathcal{L})$ *at* e, f*.*

Proof. (i) Suppose otherwise. Then there must exist $L_e, L_f \in \mathcal{L}$ such that $e \in L_e, f \in L_f$ and either $L_e - \{e\} \subseteq L_f - \{f\}$ or $L_f - \{f\} \subseteq L_e - \{e\}$. By Theorem 2.1, $(L_e \cup L_f) - \{e, f\}$ contains another member of $\mathcal{L}$, but $(L_e \cup L_f) - \{e, f\}$ is either $L_e - \{e\}$ or $L_f - \{f\}$, a contradiction.

(ii) Let $\mathcal{B}$ be the single identification of $b(\mathcal{L})$ at e, f:

$$\mathcal{B} = \big\{K : f \notin K \in b(\mathcal{L})\big\} \cup \big\{\big(K - \{f\}\big) \cup \{e\} : f \in K \in b(\mathcal{L})\big\}.$$

To prove $\mathcal{B} = b(\mathcal{C})$, we employ Remark 3.1. As every member of $b(\mathcal{L})$ is a cover for $\mathcal{L}$, it follows that every member of $\mathcal{B}$ is a cover for $\mathcal{C}$, so every member of $\mathcal{B}$ contains one in $b(\mathcal{C})$. Conversely, let $B \in b(\mathcal{C})$. If $e \notin B$, then B is also a cover of $\mathcal{L}$, so B contains a member of $b(\mathcal{L})$ and therefore of $\mathcal{B}$. Otherwise $e \in B$, and $B \cup \{f\}$ is a cover of $\mathcal{L}$, so $B \cup \{f\}$ contains a member K of $b(\mathcal{L})$, implying that B contains a member of $\mathcal{B}$ – this member is either K or $(K - \{f\}) \cup \{e\}$. Either way, every member of $b(\mathcal{C})$ contains one in $\mathcal{B}$. By Remark 3.1, $\mathcal{B} = b(\mathcal{C})$, finishing the proof.

□

More generally, an *identification* of clutter $\mathcal{L}$ is a clutter $\mathcal{C}$ (thanks to Proposition 3.2) obtained from $\mathcal{L}$ after applying a series of single identifications. Inversely, clutter $\mathcal{L}$ is a *split* of clutter $\mathcal{C}$. By the preceding proposition, $b(\mathcal{L})$ is a split of $b(\mathcal{C})$, and there is a one-to-one correspondence between the members of $\mathcal{L}$ and those of $\mathcal{C}$. Notice that a clutter may have many single splits at an element. It is therefore interesting to see how single splits at an element are created; we propose two approaches.

The first approach is an immediate corollary of Theorem 2.1. For a clutter $\mathcal{C}$ and an element e, denote by $\text{port}(\mathcal{C}, e)$ the family of members of $\mathcal{C}$ that contain element e.

Remark 3.3. *Let $\mathcal{C}$ be a clutter and take an element e. Then the following hold:*

(i) Let $\mathcal{L}$ be a single split of $\mathcal{C}$ at e. Then for $C_1 \in port(\mathcal{C}, e) \cap port(\mathcal{L}, e)$ and $C_2 \in port(\mathcal{C}, e) - port(\mathcal{L}, e)$, $(C_1 \cup C_2) - \{e\}$ contains another member of $\mathcal{C}$.

(ii) Let $port_e, port_f$ be a partition of $port(\mathcal{C}, e)$ such that for each $C_1 \in port_e$ and $C_2 \in port_f$, $(C_1 \cup C_2) - \{e\}$ contains another member of $\mathcal{C}$. Then there is a single split $\mathcal{L}$ of $\mathcal{C}$ at e such that $port_e = port(\mathcal{L}, e)$.

Therefore, to split $\mathcal{C}$ at element e, we should find a partition $\text{port}_e, \text{port}_f$ of $\text{port}(\mathcal{C}, e)$ so that (ii) is satisfied.

The second approach is also interesting. Take a clutter $\mathcal{C}$ and an element e. The *e-graph* is the bipartite graph on vertices $\text{port}(\mathcal{C}, e) \cup \text{port}(b(\mathcal{C}), e)$, where $C \in \text{port}(\mathcal{C}, e)$ and $B \in \text{port}(b(\mathcal{C}), e)$ are adjacent if $C \cap B = \{e\}$.

Proposition 3.4. *Let $\mathcal{C}$ be a clutter and take an element e. Then the following hold:*

(i) Let $\mathcal{L}$ be a single split of $\mathcal{C}$ at e. Then there is no edge of the e-graph of $\mathcal{C}$ with exactly one end in $port(\mathcal{L}, e) \cup port(b(\mathcal{L}), e)$.

(ii) Take non-empty subsets $U \subseteq port(\mathcal{C}, e)$ and $V \subseteq port(b(\mathcal{C}), e)$ such that there is no edge of the e-graph with exactly one end in $U \cup V$. Then there is a single split $\mathcal{L}$ of $\mathcal{C}$ at e such that $U = port(\mathcal{L}, e)$ and $V = port(b(\mathcal{L}), e)$.

Proof. (i) This follows immediately from the fact that whenever $C \cap B = \{e\}$, for some $C \in \mathcal{C}$ and $B \in b(\mathcal{C})$, then either

$$C \in \text{port}(\mathcal{L}, e) \quad \text{and} \quad B \in \text{port}(b(\mathcal{L}), e)$$

or

$$(C - \{e\}) \cup \{f\} \in \text{port}(\mathcal{L}, f) \quad \text{and} \quad (B - \{e\}) \cup \{f\} \in \text{port}(b(\mathcal{L}), f),$$

as $C \cap (B - \{e\} \cup \{f\}) = \emptyset$ and $(C - \{e\} \cup \{f\}) \cap B = \emptyset$.

(ii) Let

$$\mathcal{L} := \big\{C : e \notin C \in \mathcal{C}\big\} \cup \big\{C : C \in \text{port}(\mathcal{C}, e) \cap U\big\} \cup \big\{(C - \{e\}) \cup \{f\} : C \in \text{port}(\mathcal{C}, e) - U\big\},$$

where f is a new element. Note that by definition, $\text{port}(\mathcal{L}, e) = U$.

We claim that e and f are opposite in $\mathcal{L}$. By Theorem 2.1 it suffices to show, for all $C_1 \in U$ and $C_2 \in \text{port}(\mathcal{C}, e) - U$, that $(C_1 \cup C_2) - \{e\}$ contains another member of $\mathcal{C}$. Suppose not. Then elements of $\mathcal{C}$ not in $(C_1 \cup C_2) - \{e\}$ form a cover for $\mathcal{C}$, so there exists $B \in b(\mathcal{C})$ such that $B \cap C_1 = \{e\} = B \cap C_2$, so B is a common neighbour of C_1 and C_2 in the e-graph, contradicting our choice of $U \cup V$. Hence, since $\mathcal{C}$ is clearly the identification of $\mathcal{L}$ at e and f, $\mathcal{L}$ is a single split of $\mathcal{C}$ at e.

It remains to show that $\text{port}(b(\mathcal{L}), e) = V$. To show $\subseteq$, take $K \in \text{port}(b(\mathcal{L}), e)$. Note that $K \in \text{port}(b(\mathcal{C}), e)$. Since K intersects some member of $\text{port}(\mathcal{L}, e) = U$ at just $\{e\}$, it follows that there is an edge in the e-graph between K and some vertex of U, which due to our assumption implies that $K \in V$. Conversely, to show $\supseteq$, take $B \in V \subseteq \text{port}(b(\mathcal{C}), e)$. Then either B or $(B - \{e\}) \cup \{f\}$ is in $b(\mathcal{L})$. However, since B intersects each member in U at just $\{e\}$, it follows that $(B - \{e\}) \cup \{f\}$ is not a cover, so $B \in \text{port}(b(\mathcal{L}), e)$, as required. Hence, $\text{port}(b(\mathcal{L}), e) = V$, finishing the proof.

□

Thus, every disconnection in the e-graph of $\mathcal{C}$ gives rise to a proper single split of $\mathcal{C}$ at e.

3.2 Splits and covering parameters: why you should split!

Here we shed light on why one would want to split clutters. Let $\mathcal{C}$ be a clutter with weights $w \in \mathbb{Z}_+^{E(\mathcal{C})}$. Recall that $\tau(\mathcal{C}, w)$, the solution of a very general integer program, is the minimum weight of a cover while $\tau^\star(\mathcal{C}, w)$, its linear programming relaxation, is the minimum weight of a fractional cover. $\tau(\mathcal{C}, w)$ is a desirable quantity to compute, either exactly or approximately, and that is why the easier-to-find lower bound $\tau^\star(\mathcal{C}, w)$ has been the subject of extensive study [9, 27, 32].

The following proposition illustrates how splitting interacts with these parameters.

Proposition 3.5. *Let $\mathcal{C}$ be a clutter with weights $w \in \mathbb{Z}_+^{E(\mathcal{C})}$, and take an element e. Let $\mathcal{L}$ be a single split of $\mathcal{C}$ at e, and extend w to $E(\mathcal{L})$ by setting $w_f := w_e$. Then*

$$\tau(\mathcal{C}, w) = \tau(\mathcal{L}, w)$$

and

$$\tau^\star(\mathcal{L}, w) \geq \tau^\star(\mathcal{C}, w).$$

Proof. To prove $\tau(\mathcal{C}, w) = \tau(\mathcal{L}, w)$, let $B^\star$ and $K^\star$ be minimum weight covers for $\mathcal{C}$ and $\mathcal{L}$, respectively. By Proposition 3.2 (ii), $b(\mathcal{C})$ is the identification of $b(\mathcal{L})$ at e, f, so there is a one-to-one correspondence between the members of $b(\mathcal{C})$ and those of $b(\mathcal{L})$. In particular, let B be the corresponding member of $K^\star$ in $b(\mathcal{C})$, and let K be the corresponding member of $B^\star$ in $b(\mathcal{L})$. Then as $w_f = w_e$,

$$\tau(\mathcal{C}, w) = w(B^\star) = w(K) \geq \tau(\mathcal{L}, w) = w(K^\star) = w(B) \geq w(B^\star) = \tau(\mathcal{C}, w),$$

so equality holds throughout and $\tau(\mathcal{C}, w) = \tau(\mathcal{L}, w)$.

To prove $\tau^\star(\mathcal{L},w) \geq \tau^\star(\mathcal{C},w)$, let $\bar{x} \in \mathbb{R}_+^{E(\mathcal{L})}$ be a minimum weight fractional cover for $\mathcal{L}$. Define $x \in \mathbb{R}_+^{E(\mathcal{C})}$ as follows: for each $g \in E(\mathcal{C}) - \{e\}$, $x_g := \bar{x}_g$ and $x_e := \max\{\bar{x}_e, \bar{x}_f\}$. Observe that for each $L \in \mathcal{L}$ and its corresponding member C of $\mathcal{C}$, we have

$$x(C) \geq \bar{x}(L) \geq 1,$$

so x is a fractional cover for $\mathcal{C}$, implying that

$$\tau^\star(\mathcal{L},w) = \sum \left(w_g \bar{x}_g : g \in E(\mathcal{L})\right) \geq \sum \left(w_g x_g : g \in E(\mathcal{C})\right) \geq \tau^\star(\mathcal{C},w),$$

as required. □

Thus,

Corollary 3.6. *Let $\mathcal{C}$ be a clutter with weights $w \in \mathbb{Z}_+^{E(\mathcal{C})}$, and let $\mathcal{L}$ be a split of $\mathcal{C}$. Then one can extend w to $E(\mathcal{L})$ so that*

$$\tau(\mathcal{C},w) = \tau(\mathcal{L},w)$$

and

$$\tau^\star(\mathcal{L},w) \geq \tau^\star(\mathcal{C},w).$$

As a consequence, if our objective is approximating $\tau(\mathcal{C},w)$, then $\tau^\star(\mathcal{L},w)$ is a better lower bound than $\tau^\star(\mathcal{C},w)$. In fact, if the split $\mathcal{L}$ of $\mathcal{C}$ from the corollary happens to be ideal, then we are guaranteed to have that $\tau(\mathcal{C},w) = \tau^\star(\mathcal{L},w)$, thereby reducing the problem of finding $\tau(\mathcal{C},w)$ to solving a linear program. In the next subsection, we will see several examples of ideal splits.

3.3 Examples: deltas, Steiner trees and comparability graphs

Deltas (degenerate projective planes) and split deltas: Take an integer $n \geq 3$. A *delta of dimension* n is, up to isomorphism, the clutter Δ_n over ground set $\{1,2,\ldots,n\}$ whose incidence matrix is

$$M(\Delta_n) = \begin{pmatrix} 1 & 1 & & & & \\ 1 & & 1 & & & \\ \vdots & & & \ddots & & \\ 1 & & & & 1 & \\ 1 & & & & & 1 \\ & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

Formally,

$$\Delta_n = \big\{\{1,2\},\{1,3\},\ldots,\{1,n\},\{2,3,\ldots,n\}\big\}.$$

Observe that $b(\Delta_n) = \Delta_n$, and that Δ_n is a non-ideal clutter as

$$\left(\frac{n-2}{n-1}, \frac{1}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)^\top$$

is a fractional extreme point of $Q(\Delta_n)$. In the literature, a delta of dimension n is referred to as a degenerate projective plane of order $n-1$, but we will not be using this terminology (as deltas are not the only degenerate projective planes). Deltas play a crucial role in Lehman's seminal study of non-ideal clutters – we will see more on this in §5.1. Deltas and their splits will also be crucial to our work.

Let $k \in [n-2]$. A *split delta of order n and level k* is, up to isomorphism, the following clutter over ground set $\{1, 2, \ldots, n-k\} \cup \{e_1, \ldots, e_k, f_1, \ldots, f_k\}$:

$$\Delta_n^k := \big\{\{1,2\}, \{1,3\}, \ldots, \{1, n-k\}, \{1, e_1\}, \ldots, \{1, e_k\}, \{2, \ldots, n-k, f_1, \ldots, f_k\}\big\},$$

whose incidence matrix is

$$M\big(\Delta_n^k\big) = \begin{pmatrix} 1 & 1 & & & & & & & & & \\ 1 & & 1 & & & & & & & & \\ \vdots & & & \ddots & & & & & & & \\ 1 & & & & 1 & & & & & & \\ 1 & & & & & 1 & & & & & \\ \vdots & & & & & & \ddots & & & & \\ 1 & & & & & & & 1 & & & \\ & 1 & 1 & \cdots & 1 & & & & 1 & \cdots & 1 \end{pmatrix}.$$

The blocker of this clutter is

$$b\big(\Delta_n^k\big) = \big\{\{1,2\}, \{1,3\}, \ldots, \{1, n-k\}, \{1, f_1\}, \ldots, \{1, f_k\}, \{2, \ldots, n-k, e_1, \ldots, e_k\}\big\},$$

with incidence matrix

$$M\big(b\big(\Delta_n^k\big)\big) = \begin{pmatrix} 1 & 1 & & & & & & & & & \\ 1 & & 1 & & & & & & & & \\ \vdots & & & \ddots & & & & & & & \\ 1 & & & & 1 & & & & & & \\ 1 & & & & & & & & 1 & & \\ \vdots & & & & & & & & & \ddots & \\ 1 & & & & & & & & & & 1 \\ & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & & & \end{pmatrix},$$

which is also a split delta of the same order and level. Notice that for $i \in [k]$, elements e_i and f_i are opposite in Δ_n^k.

Remark 3.7. *Take integers $n \geq 3$ and $k \in [n-2]$. Let $\Delta_n^0 := \Delta_n$. Then*

(i) *Δ_n^k is the unique proper single split of Δ_n^{k-1},*

(ii) *Δ_n^{n-2} has no proper split,*

(iii) *if $n \geq 4$, we have $\Delta_n^k \setminus e_1/f_1 \cong \Delta_{n-1}^{k-1}$,*

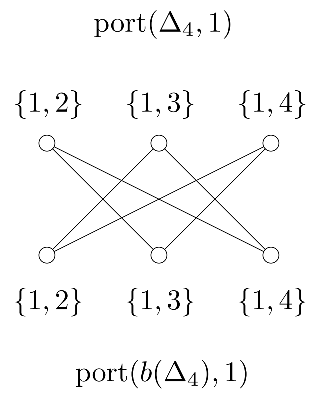


Figure 2: An illustration of the 1-graph of Δ_4.

(iv) if $n \geq 4$ and $k \in [n-3]$, Δ_n^k has a Δ_{n-k} minor.

Proof. **(i)** Observe that Δ_n^{k-1} is the identification of Δ_n^k over at e_1 and f_1, so Δ_n^k is a proper single split of Δ_n^{k-1}. In fact, for $i \in \{2, 3, \ldots, n-k+1\}$, any proper single split of Δ_n^{k-1} at element i is the split delta Δ_n^k. To prove uniqueness, it is therefore sufficient to show for $n \geq 4$ that Δ_n^{k-1} has no single split at element 1. This follows from Proposition 3.4 after observing that the 1-graph of Δ_n^{k-1} is connected – see Figure 2 for the case $n = 4$ and $k = 1$. An alternative reasoning is that $C \cup C' - \{1\}$ does not contain another member of Δ_n^{k-1}, for any $C, C' \in \mathrm{port}\big(\Delta_n^{k-1}, 1\big)$. We leave parts **(ii)** and **(iii)** to the reader. Notice that **(iv)** is an immediate corollary of (iii). $\square$

Thus, by part (i), a split delta of order n and level k is the k^{th} single split of Δ_n, with Δ_n^{n-2} being the ultimate split of it by part (ii). Let us analyze the idealness of deltas and their splits. As we mentioned already, for each $n \geq 3$, the delta Δ_n is non-ideal, but it can be readily checked that its split Δ_n^{n-2} is ideal. Hence, parts (i) and (ii) together with (iv) imply that,

Remark 3.8. *For $n \geq 3$, Δ_n^{n-2} is the unique ideal split of Δ_n.*

Another useful remark is the relationship between special clutters and deltas:

Remark 3.9. *Special clutters are precisely the proper single splits of deltas.*

Steiner trees: We have already defined directed Steiner trees as well as directed Steiner cuts. (Undirected) Steiner trees are defined in a similar fashion: let $G = (V, E)$ be a graph and let $R \subseteq V$ be a set of terminals. A *Steiner tree* is a minimal connected subset of E spanning the terminals, and a *Steiner cut* is a minimal edge subset of the form

$$\delta(U) := \big\{\{u, v\} \in E : u \in U, v \in V - U\big\}$$

for some $U \subseteq V$ such that $U \cap R \neq \emptyset$ and $(V - U) \cap R \neq \emptyset$. It is well-known that the clutter of Steiner trees and the clutter of Steiner cuts are blockers of one another [6]. Observe that when $R = V$, Steiner trees (resp.

Steiner cuts) are the same as spanning trees (resp. cuts), and when $R = \{s, t\}$, Steiner trees (resp. Steiner cuts) are the same as *st-paths* (resp. *st-cuts*).

The clutter of Steiner trees of a graph has a nice split that is easy to describe, namely the clutter of *directed* Steiner trees of the "bidirection" of the graph with an arbitrary root. To be formal,

Remark 3.10. *Let $G = (V, E)$ be a graph with terminals $R \subseteq V$. Let*

$$A = \big\{(u, v), (v, u) : \{u, v\} \in E\big\},$$

let $D = (V, A)$ and choose a root $r \in R$. Then the clutter $\mathcal{L}$ of directed Steiner trees (resp. cuts) of D is a split of the clutter $\mathcal{C}$ of Steiner trees (resp. cuts) of G.

Proof. As mentioned earlier, opposite arcs in D are also opposite in $\mathcal{L}$. Consider the clutter $\mathcal{C}'$ obtained after identifying all opposite pairs $(u, v), (v, u)$, for $\{u, v\} \in E$. We claim that $\mathcal{C}' \cong \mathcal{C}$. In other words, what we claim is that

> there is a unique way to orient the edges of a Steiner tree in G so as to obtain a directed Steiner tree of D.

This is clearly true. Hence, $\mathcal{C}$ is an identification of $\mathcal{L}$, and by Proposition 3.2, $b(\mathcal{C})$ is an identification of $b(\mathcal{L})$, finishing the proof. $\square$

Let us discuss idealness of such clutters. The clutter $\mathcal{C}$ of Steiner trees of a graph is quite often non-ideal. For instance, if $G = (V, E)$ is a triangle and $R = V$ then $\mathcal{C} \cong \Delta_3$, which is non-ideal. As a result, the clutter $\mathcal{C}$ is non-ideal as soon as the graph G has a minor isomorphic to a triangle with three terminals (the converse is proved to be true in Theorem 5.8). It is a pity that such a weak restriction makes $\mathcal{C}$ non-ideal. But what about splits of these clutters? The preceding remark points out a nice split $\mathcal{L}$ of $\mathcal{C}$, where $\mathcal{L}$ is a clutter of directed Steiner trees. This is good news, as the split $\mathcal{L}$ is guaranteed to be ideal when $R = \{s, t\}$ or $R = V$ or when G is a series-parallel graph [14, 11, 16, 25, 26].

Vertex covers of comparability graphs: Let V be a finite set and define a *(strict) partial order* $<$ on the elements of V satisfying the following:

for each $u \in V$, $u < u$ is not the case,

for $u, v \in V$, if $u < v$ then $v < u$ is not the case,

for $u, v, w \in V$, if $u < v$ and $v < w$ then $u < w$.

The pair $(V, <)$ is called a *partially ordered set*. A *comparability graph* is a graph whose vertices are V and whose edges are

$$\big\{\{u, v\} : u, v \in V, u < v\big\},$$

where $(V,<)$ is a partially ordered set. An easy fact we will use below about comparability graphs is that triangle-free comparability graphs are bipartite, that is, if a comparability graph does not have three vertices u,v,w such that $u<v<w$, then it must be bipartite.

Theorem 3.11. *Let $G=(V,E)$ be a comparability graph, and let $\mathcal{C}$ be the clutter of its minimal vertex covers. Then $\mathcal{C}$ has a split that is the clutter of minimal vertex covers of a bipartite graph. Moreover, this split is obtained after applying at most $|V|$ many single splits.*

Proof. We will appeal to Proposition 3.2 and instead prove that $b(\mathcal{C})=E$ has a split that is the clutter of *edges* of a bipartite graph. If G is triangle-free, then it is already bipartite, so we are done. Otherwise we construct another comparability graph with fewer triangles whose clutter of edges is a split of E, and we recurse until we reach a triangle-free comparability graph.

Suppose vertices u,v,w induce a triangle in G, so we may assume $u<v<w$. Let G' be obtained from G after splitting vertex v into two vertices v_1,v_2 and splitting the neighbourhood of v as follows: vertices smaller than v, including u, are now neighbours of v_1 and vertices larger than v, including w, are now neighbours of v_2. Formally, $G'=(V',E')$ where

$$V':=\big(V-\{v\}\big)\cup\{v_1,v_2\}$$

and

$$E':=\big\{\{x,y\}:x<y,v\notin\{x,y\}\big\}\cup\big\{\{x,v_1\}:x<v\big\}\cup\big\{\{v_2,y\}:v<y\big\}.$$

It is clear that the new graph G' has fewer triangles than before, as every triangle of G' is present in G and the triangle of G on vertices u,v,w no longer exists in G'.

To see why G' is still a comparability graph, we will modify the old partial order $<$ defined on V to a new partial order $<'$ defined on V', so that G' is the comparability graph of $<'$. Take $x,y\in V'$. Then $x<'y$ if

(i) $x,y\in V'-\{v_1,v_2\}$ and $x<y$, or

(ii) $x\in V'-\{v_1,v_2\}$, $y=v_1$ and $x<v$, or

(iii) $x=v_2$, $y\in V'-\{v_1,v_2\}$ and $v<y$.

One can easily verify that $<'$ is a partial order, and also that G' is the comparability graph of this partial order.

We next claim that v_1,v_2 are opposite in the clutter E'. Let $\{x,v_1\}$ and $\{v_2,y\}$ be edges in E'. Then by definition, $x<v$ and $v<y$, so $x<y$ and

$$\big(\{x,v_1\}\cup\{v_2,y\}\big)-\{v_1,v_2\}=\{x,y\}\in E'.$$

Since this is true for all such x,y, Theorem 2.1 implies that v_1,v_2 are opposite in E'. It is also clear that E is the identification of E' at v_1,v_2, so the new comparability graph G' is precisely what we were looking for.

Finally, notice that in our procedure, a vertex of G is never split more than once, implying that our algorithm terminates after applying at most $|V|$ many single splits. $\square$

Once again, let us discuss idealness of these clutters. It is well-known that the clutter $\mathcal{C}$ of minimal vertex covers of a graph is ideal if, and only if, the graph is bipartite [13, 18]. Hence, for comparability graphs, $\mathcal{C}$ is most often non-ideal. However, as the preceding proposition shows, $\mathcal{C}$ has a very nice split – the clutter of minimal vertex covers of a *bipartite* graph, which is always ideal.

3.4 Ideal splits and extension complexity

Splits can sometimes be used to upper-bound the so-called extension complexity of certain polyhedra. Let us elaborate. Let P be an arbitrary polyhedron in $\mathbb{R}^d$. An *extension of* P is a full-dimensional polyhedron Q in $\mathbb{R}^m$ for which there is an affine function $\phi : \mathbb{R}^m \to \mathbb{R}^d$ such that

$$\phi(Q) = P.$$

The *extension complexity of* P, denoted $\mathrm{xc}(P)$, is the minimum number of facets of an extension of it. Finding this parameter, or bounding it from above or below, is very desirable. The following proposition shows how ideal splits can help bound this parameter from above.

Let $\mathcal{C}$ be a clutter over ground set E. For a subset F of E, let $\chi_F \in \{0,1\}^E$ denote the characteristic vector of F. Denote by $\mathrm{conv}(\mathcal{C})$ the convex hull of

$$\left\{\chi_C \in \{0,1\}^{E(\mathcal{C})} : C \in \mathcal{C}\right\}$$

and denote by $\mathrm{conv}^+(\mathcal{C})$ its dominant, that is,

$$\mathrm{conv}^+(\mathcal{C}) = \left\{x + r : x \in \mathrm{conv}(\mathcal{C}), r \geq \mathbf{0}\right\}.$$

Proposition 3.12. *Let $\mathcal{C}$ be a clutter and let $\mathcal{L}$ be a split of it. Then the following hold:*

(1) $\mathrm{conv}^+(\mathcal{L})$ is an extension of $\mathrm{conv}^+(\mathcal{C})$,

(2) if $\mathcal{L}$ is ideal, then

$$\mathit{xc}\big(\mathit{conv}^+(\mathcal{C})\big) \leq |b(\mathcal{C})| + |E(\mathcal{L})|$$

and

$$\mathit{xc}\big(\mathit{conv}^+\big(b(\mathcal{C})\big)\big) \leq |\mathcal{C}| + |E(\mathcal{L})|.$$

Proof. (1) As composition of affine maps is also affine, it suffices to prove this part for single splits. To this end, assume that $\mathcal{L}$ is a single split of $\mathcal{C}$ at an element $e \in E(\mathcal{C})$. For each $x \in \mathbb{R}^{E(\mathcal{L})}$, define $\phi(x) \in \mathbb{R}^{E(\mathcal{C})}$ as follows: let $\phi(x)_e = x_e + x_f$ and for each $g \in E(\mathcal{C}) - \{e\}$, let $\phi(x)_g = x_g$. Clearly $\phi : \mathbb{R}^{E(\mathcal{L})} \to \mathbb{R}^{E(\mathcal{C})}$ is an affine map. Observe that ϕ maps (the characteristic vector of) each member of $\mathcal{L}$ to its corresponding member in $\mathcal{C}$. This, together with the affinity of ϕ, implies that

$$\phi\big(\mathrm{conv}(\mathcal{L})\big) = \mathrm{conv}(\mathcal{C}),$$

implying in turn that

$$\phi\big(\text{conv}^+(\mathcal{L})\big) = \text{conv}^+(\mathcal{C}),$$

finishing the proof of (1).

(2) Consider the polyhedron

$$Q(\mathcal{L}) = \Big\{x \in \mathbb{R}^{E(\mathcal{L})} : x(L) \geq 1, \forall L \in \mathcal{L}; x_g \geq 0, \forall g \in E(\mathcal{L})\Big\},$$

which is integral as $\mathcal{L}$ is assumed ideal. Thus, the extreme points of $Q(\mathcal{L})$ are precisely the members of $b(\mathcal{L})$, implying that $Q(\mathcal{L}) = \text{conv}^+(b(\mathcal{L}))$. So by (1), $Q(\mathcal{L})$ is an extension of $\text{conv}^+\big(b(\mathcal{C})\big)$ and

$$\text{xc}\big(\text{conv}^+\big(b(\mathcal{C})\big)\big) \leq |\mathcal{L}| + |E(\mathcal{L})| = |\mathcal{C}| + |E(\mathcal{L})|.$$

Since $\mathcal{L}$ is ideal, it follows that $b(\mathcal{L})$ is also ideal. But $b(\mathcal{L})$ is a split of $b(\mathcal{C})$, so what we just proved implies

$$\text{xc}\big(\text{conv}^+(\mathcal{C})\big) \leq |b(\mathcal{C})| + |E(\mathcal{L})|.$$

□

(Note that a crude bound on $|E(\mathcal{L})|$ is $|E(\mathcal{C})|$ times the maximum frequency of an element in $\mathcal{C}$.) Let us now apply this proposition to minimal vertex covers of comparability graphs:

Corollary 3.13. *Let $G = (V, E)$ be a comparability graph and let $\mathcal{C}$ be its clutter of minimal vertex covers. Then conv$^+(\mathcal{C})$ has an extension of dimension at most $2|V|$ and extension complexity at most $|E| + 2|V|$.*

Proof. Let $\mathcal{L}$ be the split of $\mathcal{C}$ from Theorem 3.11, for which $|E(\mathcal{L})| \leq |E(\mathcal{C})| + |V| = 2|V|$. Notice that $\mathcal{L}$ is an ideal clutter, so from Proposition 3.12 (2) it follows

$$\text{xc}\big(\text{conv}^+(\mathcal{C})\big) \leq |E| + |E(\mathcal{L})| \leq |E| + 2|V|,$$

as desired. □

It was known to Yannakakis [33] that in the world of comparability graphs, the convex hull of stable sets has extension complexity $O\left(|V|^2\right)$. As stable sets are complements of vertex covers, this work implies that for comparability graphs, the convex hull of (all) vertex covers has extension complexity $O\left(|V|^2\right)$.

3.5 Ideal splits and integrality gap

Splits can also be used sometimes to upper-bound an integrality gap parameter. Let $\mathcal{C}$ be a clutter over ground set E, and take a real number $k \geq 1$. We say $\mathcal{C}$ has *integrality gap at most k* if for each $w \in \mathbb{Z}_+^{E(\mathcal{C})}$, we have

$$k \geq \frac{\tau(\mathcal{C}, w)}{\tau^\star(\mathcal{C}, w)}.$$

Proposition 3.14. *Let $\mathcal{L}$ be an ideal clutter and let $\mathcal{C}$ be an identification of it. If each element of $\mathcal{C}$ has at most k elements of $\mathcal{L}$ identifying with it, then the integrality gap of $\mathcal{C}$ is at most k.*

Proof. Take weights $w \in \mathbb{Z}_+^{E(\mathcal{C})}$ and let $x^\star$ be a minimum weight fractional cover for $\mathcal{C}$. Extend $x^\star$ to a fractional cover of $\mathcal{L}$ as follows: for each element f of $\mathcal{L}$ that identifies with an element e of $\mathcal{C}$, let $x^\star_f := x^\star_e$. Also, extend w to $E(\mathcal{L})$ as follows: for each element f of $\mathcal{L}$ that identifies with an element e of $\mathcal{C}$, let $w_f := w_e$. As each element of $\mathcal{C}$ has at most k elements of $\mathcal{L}$ identifies with it, we have

$$\begin{aligned} k\,\tau^\star(\mathcal{C},w) = k\sum\big(w_g x^\star_g : g \in E(\mathcal{C})\big) &\geq \sum\big(w_g x^\star_g : g \in E(\mathcal{L})\big)\\ &\geq \tau^\star(\mathcal{L},w) \qquad \text{as } x^\star \in Q(\mathcal{L})\\ &= \tau(\mathcal{L},w) \qquad \text{as } \mathcal{L} \text{ is ideal}\\ &= \tau(\mathcal{C},w) \qquad \text{by Proposition 3.5.} \end{aligned}$$

Since this is true for any $w \in \mathbb{Z}_+^{E(\mathcal{C})}$, the result follows. $\square$

In other words, if a clutter has an ideal split where each element is split at most $k-1$ times, then its integrality gap is at most k. For instance, let $\mathcal{C}$ be the clutter of spanning trees of a graph. As we have seen, Edmonds found an ideal split of $\mathcal{C}$ where each element is split at most once. Therefore, we get the well-known fact that $\mathcal{C}$ has integrality gap at most 2 [32].

4 Splitting preserves many properties!

In the previous section, we saw several examples of non-ideal clutters with ideal splits. One may wonder whether there are ideal clutters with non-ideal splits. In this section, we answer this question negatively by showing that splitting preserves many nice properties, including idealness. In §6.1 we will see a neat application of the results here.

4.1 Idealness

Recall that a clutter $\mathcal{L}$ is ideal if every extreme point of

$$Q(\mathcal{L}) = \left\{x \in \mathbb{R}_+^{E(\mathcal{L})} : x(L) \geq 1, \forall L \in \mathcal{L}\right\}$$

is integral.

Theorem 4.1. *If a clutter is ideal, then so is any split of it.*

Proof. It suffices to prove this for single splits. Let $\mathcal{C}$ be an ideal clutter and let $\mathcal{L}$ be a proper single split of it at some element $e \in E(\mathcal{C})$. We may assume that every element in $E(\mathcal{C})$ is used in a member of $\mathcal{C}$, which in turn implies that every element in $E(\mathcal{L})$ is used in a member of $\mathcal{L}$. Suppose for a contradiction that $\mathcal{L}$ is non-ideal, and let $x^\star \in \mathbb{R}_+^{E(\mathcal{L})}$ be a non-integral extreme point of $Q(\mathcal{L})$. After relabeling e and f, if necessary, we may assume that $x^\star_e \geq x^\star_f$. Define $y \in \mathbb{R}_+^{E(\mathcal{C})}$ as follows: $y_e := \max\{x^\star_e, x^\star_f\} = x^\star_e$ and for each $g \in E(\mathcal{C}) - \{e\}$,

$y_g := x^\star_g$. For each $C \in \mathcal{C}$ and its corresponding member L in $\mathcal{L}$,

$$y(C) = \begin{cases} x^\star(L) & \text{if } f \notin L \\ x^\star(L) - x^\star_f + x^\star_e & \text{if } f \in L, \end{cases}$$

and since $x^\star(L) \geq 1$, we get $y(C) \geq 1$. As a result, $y \in Q(\mathcal{C})$ and since $Q(\mathcal{C})$ is integral, there exist members $B_1, \ldots, B_n$ of $b(\mathcal{C})$ so that y is at least as large as a convex combination of $\chi_{B_1}, \ldots, \chi_{B_n}$:

$$y \geq \sum_{i=1}^{n} \lambda_i \chi_{B_i}$$

for some $\lambda \in \mathbb{R}^n_+$ such that $\sum_{i=1}^n \lambda_i = 1$. For each $i \in [n]$, let K_i be the corresponding member of B_i in $b(\mathcal{L})$, and let

$$x := \sum_{i=1}^{n} \lambda_i \chi_{K_i} \in Q(\mathcal{L}).$$

Since $x^\star$ is a non-integral extreme point, the equation above implies that $x^\star$ does not dominate x. However, for each $g \in E(\mathcal{L}) - \{e, f\}$, $x^\star_g = y_g \geq x_g$ and

$$x^\star_e = y_e \geq x_e + x_f.$$

Hence, $x_f > x^\star_f$ and in particular, $x^\star_e > 0$. Therefore, since $x^\star$ is an extreme point of $Q(\mathcal{L})$, there exists $L_e \in \mathcal{L}$ with $e \in L_e$ such that $x^*(L_e) = 1$. But then, since $f \notin L_e$ and $x \in Q(\mathcal{L})$, it follows that

$$1 = x^\star(L_e) \geq x(L_e) \geq 1,$$

so equality holds throughout. In particular, $x^\star_e = x_e$, implying that $x_f = 0$, a contradiction with $x_f > x^\star_f$. $\square$

Let us point out that Corollary 3.6 does not imply this result.

4.2 The max-flow min-cut property

Let us go a step further and analyze a stronger property. Let $\mathcal{C}$ be a clutter over ground set E with weights $w \in \mathbb{Z}^E_+$. Recall the covering parameter $\tau(\mathcal{C}, w)$ computing the minimum weight of a cover, with lower bound $\tau^\star(\mathcal{C}, w)$ computing the minimum weight of a fractional cover. Paired with the covering parameter $\tau(\mathcal{C}, w)$ is the packing parameter:

$$\nu(\mathcal{C}, w) := \begin{array}{lll} \max & \sum (y_C : C \in \mathcal{C}) & \\ \text{s.t.} & \sum (y_C : C \in \mathcal{C}, g \in C) \leq w_g & g \in E \\ & y \in \mathbb{Z}^{\mathcal{C}}_+, & \end{array}$$

with the following upper bound:

$$\nu^\star(\mathcal{C}, w) := \begin{array}{lll} \max & \sum (y_C : C \in \mathcal{C}) & \\ \text{s.t.} & \sum (y_C : C \in \mathcal{C}, g \in C) \leq w_g & g \in E \\ & y \geq \mathbf{0}. & \end{array}$$

Solutions of this linear program are referred to as *fractional packings* (with respect to w). Notice that this linear program is the dual of the fractional covers linear program. Thus, it follows from linear programming duality that

$$\tau(\mathcal{C},w)\geq\tau^\star(\mathcal{C},w)=\nu^\star(\mathcal{C},w)\geq\nu(\mathcal{C},w).$$

Recall also that a clutter $\mathcal{C}$ is ideal if, and only if, for all $w\in\mathbb{Z}_+^{E(\mathcal{C})}$, $\tau(\mathcal{C},w)=\tau^\star(\mathcal{C},w)$. A clutter $\mathcal{C}$ has the *max-flow min-cut (MFMC)* property if for all weights $w\in\mathbb{Z}_+^E$, $\tau(\mathcal{C},w)=\nu(\mathcal{C},w)$. Observe that a clutter with the MFMC property is also ideal.

Proposition 4.2. *Let $\mathcal{C}$ be a clutter with weights $w\in\mathbb{Z}_+^{E(\mathcal{C})}$, and take an element e. Let $\mathcal{L}$ be a single split of $\mathcal{C}$ at e, and extend w to $E(\mathcal{L})$ by setting $w_f:=w_e$. Then*

$$\nu(\mathcal{L},w)\geq\nu(\mathcal{C},w).$$

Proof. Let $y^\star\in\mathbb{Z}_+^{\mathcal{C}}$ be a maximum integral packing for $\mathcal{C}$, and define $y\in\mathbb{Z}_+^{\mathcal{L}}$ as follows: for each $L\in\mathcal{L}$ and its corresponding member C in $\mathcal{C}$, let $y_L:=y^\star_C$. Then

$$\begin{aligned}w_e&\geq\sum\left(y^\star_C: C\in\mathcal{C}, e\in C\right)\geq\sum\left(y_L: L\in\mathcal{L}, e\in L\right)\\w_f=w_e&\geq\sum\left(y^\star_C: C\in\mathcal{C}, e\in C\right)\geq\sum\left(y_L: L\in\mathcal{L}, f\in L\right),\end{aligned}$$

and for each $g\in E(\mathcal{L})-\{e,f\}$,

$$w_g\geq\sum\left(y^\star_C: C\in\mathcal{C}, g\in C\right)=\sum\left(y_L: L\in\mathcal{L}, g\in L\right),$$

implying that y is an integral packing for $\mathcal{L}$. Moreover, note that

$$\nu(\mathcal{L},w)\geq\sum\left(y_L: L\in\mathcal{L}\right)=\sum\left(y^\star_C: C\in\mathcal{C}\right)=\nu(\mathcal{C},w),$$

as claimed. ☐

Theorem 4.3. *If a clutter has the max-flow min-cut property, then so does any split of it.*

Proof. Once again, it suffices to prove this for single splits. To this end, take a clutter $\mathcal{C}$ with the MFMC property and let $\mathcal{L}$ be a single split of it at some element $e\in E(\mathcal{C})$. Take weights $w\in\mathbb{Z}_+^{E(\mathcal{L})}$. We will prove by using induction on $|w_e-w_f|$ that $\tau(\mathcal{L},w)=\nu(\mathcal{L},w)$. If $w_e=w_f$ then

$$\begin{aligned}\nu(\mathcal{L},w)\leq\tau(\mathcal{L},w)&=\tau(\mathcal{C},w) &&\text{by Proposition 3.5}\\&=\nu(\mathcal{C},w)&&\text{since } \mathcal{C} \text{ has the MFMC property}\\&\leq\nu(\mathcal{L},w)&&\text{by Proposition 4.2,}\end{aligned}$$

so $\tau(\mathcal{L},w)=\nu(\mathcal{L},w)$. For the induction step, we may therefore assume that $w_e>w_f$. If element e does not appear in a minimum weight cover of $\mathcal{L}$, and w' is obtained from w after reducing w_e by 1, then $\tau(\mathcal{L},w)=\tau(\mathcal{L},w')$ and the induction hypothesis implies that

$$\tau(\mathcal{L},w)=\tau(\mathcal{L},w')=\nu(\mathcal{L},w')\leq\nu(\mathcal{L},w)\leq\tau(\mathcal{L},w),$$

so $\tau(\mathcal{L}, w) = \nu(\mathcal{L}, w)$. Otherwise, element e appears in a minimum weight cover, say K_e, of $\mathcal{L}$. In this case, since $w_e > w_f$ and $f \notin K_e$, we get that

$$\tau(\mathcal{L}, w) = \tau(\mathcal{C}, w)$$

and K_e is also a minimum weight cover of $\mathcal{C}$. (Note w_f is dropped on the right-hand-side of the equation.) Take a maximum integral packing $y^\star \in \mathbb{Z}_+^{\mathcal{C}}$ of $\mathcal{C}$. Since $\mathcal{C}$ has the MFMC property, we have

$$\sum (y^\star_C : C \in \mathcal{C}) = \tau(\mathcal{C}, w) := \tau.$$

Suppose $y^\star$ picks members $C_1, \ldots, C_\tau$ of $\mathcal{C}$ (many of the C_i's may be the same). For each $i \in [\tau]$, let L_i be the member corresponding to C_i in $\mathcal{L}$. Note that for each $g \in E(\mathcal{L}) - \{f\}$, the number of L_i's using element g is at most the number of C_i's using element g, which is bounded above by w_g. Moreover, since K_e has the same weight as the packing $C_1, \ldots, C_\tau$, it follows that for each $i \in [\tau]$, $|K_e \cap C_i| = 1$. So if for some $i \in [\tau]$, $e \in C_i$, then e must also belong to L_i as K_e is also a cover for $\mathcal{L}$. Thus, $f \notin L_1 \cup \cdots \cup L_\tau$, which in turn implies that $L_1, \ldots, L_\tau$ is a packing of $\mathcal{L}$ (with respect to w). Since $\tau(\mathcal{L}, w) = \tau$, it follows that

$$\tau(\mathcal{L}, w) = \nu(\mathcal{L}, w).$$

This completes the induction step, proving that $\tau(\mathcal{L}, w) = \nu(\mathcal{L}, w)$ for all $w \in \mathbb{Z}_+^{E(\mathcal{L})}$. Thus $\mathcal{L}$ has the MFMC property, finishing the proof. □

4.3 The packing property and the little sister of the Replication Conjecture

There is a notoriously difficult conjecture on the max-flow min-cut property that motivated us to analyze another property. For a clutter $\mathcal{C}$, let $\tau(\mathcal{C}) := \tau(\mathcal{C}, \mathbf{1})$ and $\nu(\mathcal{C}) := \nu(\mathcal{C}, \mathbf{1})$. We say that $\mathcal{C}$ *packs* if $\tau(\mathcal{C}) = \nu(\mathcal{C})$. In words, a clutter packs if the minimum size of a cover is equal to the maximum size of a packing (i.e. the maximum number of pairwise disjoint members). Notice that Propositions 3.5 and 4.2 imply that,

Corollary 4.4. *If a clutter packs, then so does any split of it.*

A clutter $\mathcal{C}$ has *the packing property* if every minor of it (including $\mathcal{C}$ itself) packs. Equivalently, a clutter $\mathcal{C}$ has the packing property if for all weights $w \in \{0, 1, \infty\}^{E(\mathcal{C})}$, $\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w)$ [9]. Clearly, clutters with the max-flow min-cut property have the packing property. The so-called Replication Conjecture of Conforti and Cornuéjols [7] predicts the converse is also true, that is,

(?) clutters with the packing property have the max-flow min-cut property (?)

Thus, if this conjecture were true, Theorem 4.3 would imply that if a clutter has the packing property, then so does every split of it. Here we prove this result without appealing to the conjecture. [4]

Remark 4.5. *For each $n \geq 3$, the delta Δ_n of dimension n does not pack.*

[4] Our to-be-proved result is in a sense the little sister of the Replication Conjecture, as their conjecture equivalently predicts that if a clutter has the packing property, then so does every replication of it – see §6.5 for a definition of replication.

Proof. This follows from the fact that $\tau(\Delta_n) = 2 > 1 = \nu(\Delta_n)$. $\square$

Lemma 4.6. *Take a clutter $\mathcal{C}$ with the packing property and let $\mathcal{L}$ be a single split of it at some element $e \in E(\mathcal{C})$. Then*

(1) $\mathcal{L} \setminus f$ packs,

(2) for all $L \in \mathcal{L}$ and $K \in b(\mathcal{L})$ such that $e \in L, f \in K$ or $f \in L, e \in K$, we have $|L \cap K| \geq 2$,

(3) if a minimum cover of $\mathcal{L}$ does not contain e, then $\mathcal{L}/e \setminus f$ packs,

(4) in fact, $\mathcal{L}/e \setminus f$ always packs, and

(5) $\mathcal{L}/e$ packs.

Proof. Observe by Corollary 4.4 that $\mathcal{L}$ packs, and by Proposition 3.5 that $\tau(\mathcal{L}) = \tau(\mathcal{C})$.

(1) Observe that $\tau(\mathcal{L}) - 1 \leq \tau(\mathcal{L} \setminus f) \leq \tau(\mathcal{L})$. If $\tau(\mathcal{L}) - 1 = \tau(\mathcal{L} \setminus f)$, then any maximum packing in $\mathcal{L}$ yields a packing in $\mathcal{L} \setminus f$ of size $\tau(\mathcal{L} \setminus f)$, implying that $\mathcal{L} \setminus f$ packs. Otherwise, we have that $\tau(\mathcal{L} \setminus f) = \tau(\mathcal{L})$. So no minimum cover of $\mathcal{L}$ uses f.

If there is no minimum cover of $\mathcal{L}$ using e either, then $\tau(\mathcal{L} \setminus \{e, f\}) = \tau(\mathcal{L})$ and since $\mathcal{L} \setminus \{e, f\} = \mathcal{C} \setminus e$ packs, it follows that $\mathcal{L} \setminus f$ packs.

Otherwise, there is a minimum cover K_e of $\mathcal{L}$ using e. Note that K_e is also a minimum cover of $\mathcal{C}$. Let $C_1, \ldots, C_\tau$ be a packing of $\mathcal{C}$, where $\tau = \tau(\mathcal{C}) = \tau(\mathcal{L}) = \tau(\mathcal{L} \setminus f)$. Let $L_1, \ldots, L_\tau$ be their corresponding members in $\mathcal{L}$. If for some $j \in [\tau]$, $f \in L_j$, then $e \in C_j$ so $K_e \cap C_j = \{e\}$, implying that $K_e \cap L_j = \emptyset$, which cannot be. Hence, $f \notin L_1 \cup \cdots \cup L_\tau$, so $L_1, \ldots, L_\tau$ yields a packing in $\mathcal{L} \setminus f$, so $\mathcal{L} \setminus f$ packs.

(2) Suppose for a contradiction that $|L \cap K| = 1$. By symmetry, we may assume that $e \in L, f \in K$. Then by Theorem 2.5, $\mathcal{L}$ has an (e, f)-special minor. However by Remark 3.9, this clutter corresponds to a delta minor in $\mathcal{C}$, which by Remark 4.5 does not pack, a contradiction as $\mathcal{C}$ has the packing property.

(3) Suppose there is a minimum cover of $\mathcal{L}$ avoiding element e. Then $\tau(\mathcal{L}/e) = \tau(\mathcal{L})$. If there is a minimum cover of $\mathcal{L}$ using f, then $\tau(\mathcal{L}/e \setminus f) = \tau(\mathcal{L}) - 1$, so a packing for $\mathcal{L}$ also yields a packing for $\mathcal{L}/e \setminus f$. We may therefore assume that no minimum cover of $\mathcal{L}$ uses f, so $\tau := \tau(\mathcal{L}/e \setminus f) = \tau(\mathcal{L})$.

If no minimum cover of $\mathcal{L}$ uses e either, then $\tau = \tau(\mathcal{L} \setminus \{e, f\})$, so a packing for $\mathcal{C} \setminus e = \mathcal{L} \setminus \{e, f\}$ yields one for $\mathcal{L}/e \setminus f$. Otherwise, there is a minimum cover K_e of $\mathcal{L}$ that uses e. Let $L_1, \ldots, L_\tau$ be a packing for $\mathcal{L}$. Note for each i, $|L_i \cap K_e| = 1$. So from (2) it follows that $f \notin L_1 \cup \ldots \cup L_\tau$, so $L_1, \ldots, L_\tau$ also yields a packing for $\mathcal{L}/e \setminus f$.

(4) We will need the following claim.

Claim. *Suppose L_e and L_f are disjoint members of $\mathcal{L}$ where $e \in L_e$ and $f \in L_f$. Then there exist disjoint $L, L' \in \mathcal{L}$ contained in $(L_e \cup L_f) - \{e, f\}$.*

Proof of Claim. By (2) every minimal cover using e (resp. f) intersects L_f (resp. L_e) at least twice. As a result, given that $F = \big(E(\mathcal{L}) - (L_e \cup L_f)\big) \cup \{e, f\}$, we have $\tau(\mathcal{L} \setminus F) \geq 2$. Since $\mathcal{L} \setminus F = \mathcal{C} \setminus F$ packs, the result follows. $\Diamond$

Let $L_1, \ldots, L_k, \ldots, L_{k+\ell}, \ldots, L_{k+\ell+r}$ be a packing for $\mathcal{L}/\{e, f\}$ where

$$\begin{aligned} k + \ell + r = \tau(\mathcal{L}/\{e, f\}) &\geq \tau(\mathcal{L}/e) \geq \tau(\mathcal{L}) \\ \{e\} \cup L_j \in \mathcal{L} & \quad j = 1, \ldots, k \\ \{f\} \cup L_j \in \mathcal{L} & \quad j = k+1, \ldots, k+\ell \\ L_j \in \mathcal{L} & \quad j = k+\ell+1, \ldots, k+\ell+r. \end{aligned}$$

By (3) we may assume there is a minimum cover K_e of $\mathcal{L}$ that contains e. Then $e \in K_e$ and by (2), K_e intersects each of $L_{k+1}, \ldots, L_{k+\ell}$ at least twice. As a result, $\tau(\mathcal{L}) = |K_e| \geq 1 + 2\ell + r$ which, together with $k + \ell + r \geq \tau(\mathcal{L})$, implies that $k > \ell$. By the claim above, for each $j \in [\ell]$, we can find disjoint $L_j^1, L_j^2 \in \mathcal{L}$ such that $L_j^1 \cup L_j^2 \subseteq L_j \cup L_{k+j}$. Observe now that

$$\begin{aligned} L_j^1, L_j^2 & \quad j = 1, \ldots, \ell \\ L_j & \quad j = \ell+1, \ldots, k \\ L_j & \quad k+\ell+1, \ldots, k+\ell+r \end{aligned}$$

is a packing of size $k + \ell + r$ in $\mathcal{L}/e \setminus f$. However, $\tau(\mathcal{L}/e \setminus f) \leq \tau(\mathcal{L}/\{e, f\}) = k + \ell + r$, implying that $\mathcal{L}/e \setminus f$ packs.

(5) Suppose for a contradiction that $\mathcal{L}/e$ does not pack. If there is a minimum cover of $\mathcal{L}$ that does not use e, then $\tau(\mathcal{L}/e) = \tau(\mathcal{L})$ so the packing in $\mathcal{L}$ gives a packing in $\mathcal{L}/e$, which is not the case. Hence, every minimum cover of $\mathcal{L}$ uses e, so $\tau(\mathcal{L}/e) \geq \tau(\mathcal{L}) + 1$.

If $\tau(\mathcal{L}/e \setminus f) = \tau(\mathcal{L}/e)$, then the packing in $\mathcal{L}/e \setminus f$ yields one in $\mathcal{L}/e$, which again cannot be the case. Hence, we have $r := \tau(\mathcal{L}/e \setminus f) = \tau(\mathcal{L}/e) - 1$. Together with the inequality above, we have $r \geq \tau(\mathcal{L})$.

Let $L_1, \ldots, L_r$ be a packing and let K be a cover of size r, in $\mathcal{L}/e \setminus f$. Note $K \cup \{f\}$ is a minimum cover of $\mathcal{L}/e$, and in particular, $K \cup \{f\} \in b(\mathcal{L})$. Since every minimum cover of $\mathcal{L}$ uses e and has size at most r, there exists $j \in [r]$ such that $L_j \cup \{e\} \in \mathcal{L}$. But then

$$\big|\big(L_j \cup \{e\}\big) \cap \big(K \cup \{f\}\big)\big| = |L_j \cap K| = 1,$$

contradicting (2).

$\square$

We are now ready to state and prove the main result of this subsection.

Theorem 4.7. *If a clutter has the packing property, then so does every split of it.*

Proof. As usual, we need only prove this for single splits. So let $\mathcal{C}$ be a clutter with the packing property and let $\mathcal{L}$ be a single split of it at some element $e \in E(\mathcal{C})$. We need to show that every minor of $\mathcal{L}$ packs. Take a minor $\mathcal{L}' := \mathcal{L}/I \setminus J$ of $\mathcal{L}$. If $\{e, f\} \subseteq I$ or $\{e, f\} \subseteq J$, then $\mathcal{L}'$ is also a minor of $\mathcal{C}$, so it packs. If $\{e, f\} \cap (I \cup J) = \emptyset$, then $\mathcal{L}'$ is a single split of a minor of $\mathcal{C}$, so by Corollary 4.4, $\mathcal{L}'$ packs. Otherwise, up to relabeling e and f, one of the following holds:

(i) $e \in I$ and $f \in J$: by applying Lemma 4.6 (4) to $\mathcal{C}/(I - \{e\}) \setminus (J - \{f\})$ and $\mathcal{L}/(I - \{e\}) \setminus (J - \{f\})$, we get that $\mathcal{L}'$ packs,

(ii) $e \notin I \cup J$ and $f \in J$: by applying Lemma 4.6 (1) to $\mathcal{C}/I \setminus (J - \{f\})$ and $\mathcal{L}/I \setminus (J - \{f\})$, we get that $\mathcal{L}'$ packs,

(iii) $e \in I$ and $f \notin I \cup J$: by applying Lemma 4.6 (5) to $\mathcal{C}/(I - \{e\}) \setminus J$ and $\mathcal{L}/(I - \{e\}) \setminus J$, we get that $\mathcal{L}'$ packs.

Thus, every minor of $\mathcal{L}$ packs, so $\mathcal{L}$ has the packing property. $\square$

5 When does splitting fail?

We have been analyzing three useful properties one would like to have a clutter satisfy at least one of: idealness, the packing and MFMC properties. We showed in the previous section that splitting preserves these properties, and we saw in §3.3 many examples that did not have one of these properties, but a split of them did. In a nutshell, we demonstrated that splitting can be used as a tool to modify clutters for the better. However, as the reader may expect, there are instances where splitting just does not help. In this section, we make an attempt to identify these bad instances. To be precise, if P denotes one of the three properties of interest, we address the following question:

> Suppose a clutter does not satisfy property P. When can we guarantee that every split of the clutter also does not satisfy P?

Property P is closed under taking minors, so it makes sense to look at clutters that minor-minimally do not satisfy the property: clutters that do not satisfy P but every proper minor of them does. As a first step to answer the question above, we *characterize* which of these clutters do not have a split that satisfies P, where P is either idealness or the packing property. (Using the forthcoming Remark 6.13, we can deduce similar results for the case when P has the MFMC property. However, these results do not add much value to the paper, so we drop them.)

5.1 The minimally non-ideal clutters with an ideal split

A clutter is *minimally non-ideal (mni)* if it is non-ideal but every proper minor of it is ideal. Note that the blocker of each mni clutter is also mni. For instance, each delta is mni. We saw in Remark 3.8 an ideal split of each delta. In this subsection, we prove that,

Theorem 5.1. *The only minimally non-ideal clutters with an ideal split are deltas.*

To prove Theorem 5.1, we appeal to Lehman's result on the structure of mni clutters that are different from deltas. Let us lay the groundwork to state his result. A square $0,1$ matrix is *r-regular* if each row and each column has exactly r ones. The *core* of a clutter $\mathcal{C}$, denoted $\overline{\mathcal{C}}$, is the clutter over the same ground set whose members are those of $\mathcal{C}$ with minimum cardinality.

Theorem 5.2 (Lehman [21] and Bridges and Ryser [5] – see Seymour [28])**.**
Suppose $\mathcal{C}$ is a minimally non-ideal clutter that is not a delta, and let $\mathcal{B} := b(\mathcal{C})$. Then

(1) $M(\overline{\mathcal{C}})$ and $M(\overline{\mathcal{B}})$ are square and non-singular matrices,

(2) for some integers $r \geq 2$ and $s \geq 2$: $M(\overline{\mathcal{C}})$ is r-regular and $M(\overline{\mathcal{B}})$ is s-regular,

(3) for $n := |E(\mathcal{C})|$, $rs \geq n+1$,

(4) there is a labeling $C_1, \ldots, C_n$ of the members of $\overline{\mathcal{C}}$ and a labeling $B_1, \ldots, B_n$ of the members of $\overline{\mathcal{B}}$ such that for all $i, j \in [n]$,

$$|C_i \cap B_j| \quad \begin{cases} = rs - n + 1 & \text{if } i = j \\ = 1 & \text{if } i \neq j, \end{cases}$$

(5) for all elements $g, h \in E(\mathcal{C})$:

$$\big|\{i \in [n] : g \in C_i, h \in B_i\}\big| \quad \begin{cases} = rs - n + 1 & \text{if } g = h \\ = 1 & \text{if } g \neq h. \end{cases}$$

Note (1) and (2) imply that $\left(\frac{1}{r}, \frac{1}{r}, \ldots, \frac{1}{r}\right)^\top$ is a fractional extreme point of $Q(\mathcal{C})$. Observe also that (5) implies that $\min\{r, s\} \geq rs - n + 1$.

As was shown in Lütolf and Margot [22], there are exactly two mni clutters that are different from deltas and have at most 5 elements. These two clutters are

$$\mathcal{C}_5 = \big\{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\big\}$$

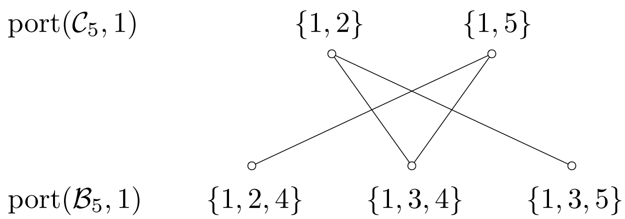
and its blocker

$$\mathcal{B}_5 = \big\{\{1,2,4\}, \{2,3,5\}, \{3,4,1\}, \{4,5,2\}, \{5,1,3\}\big\}.$$

Observe that $\mathcal{C}_5 = \overline{\mathcal{C}_5}$, $\mathcal{B}_5 = \overline{\mathcal{B}_5}$, $r = 2$ and $s = 3$.

Remark 5.3. *Clutters $\mathcal{C}_5, \mathcal{B}_5$ have no proper split.*

Proof. By Proposition 3.2, it suffices to show that $\mathcal{C}_5$ has no single split at element 1. This follows from Proposition 3.4 after observing that the 1-graph of $\mathcal{C}_5$, as portrayed below,



is connected. $\square$

The following is the key ingredient needed to prove Theorem 5.1.

Proposition 5.4. *Let $\mathcal{C}$ be a minimally non-ideal clutter that is not a delta, and let $\mathcal{L}$ be a single split of it at some element $e \in E(\mathcal{C})$. Then for some $g \in \{e, f\}$, $\mathcal{L} \setminus g$ is a minimally non-ideal clutter that is not a delta.*[5]

Proof. We may assume that $\mathcal{L}$ is a proper split of $\mathcal{C}$. Let n, r, s be the parameters from (2) and (3) and let $C_1, \ldots, C_n, B_1, \ldots, B_n$ be the labeling from (4) of Theorem 5.2. Let $d := rs - n \geq 1$. By (1) and (5), we may assume that

$$\begin{aligned} e \in C_1 \cap \cdots \cap C_r \quad &\text{and} \quad e \notin C_{r+1} \cup \cdots \cup C_n \\ e \in B_1 \cap \cdots \cap B_{d+1} \quad &\text{and} \quad e \notin B_{d+2} \cup \cdots \cup B_r. \end{aligned}$$

Note that either $C_1 \in \mathcal{L}$ or $C_1 \cup \{f\} - \{e\} \in \mathcal{L}$. After relabeling e and f, if necessary, we may assume that $C_1 \in \mathcal{L}$.

Claim 1. *For each $i \in [n]$, $C_i \in \mathcal{L}$ and $B_i \in b(\mathcal{L})$.*

Proof of Claim. We will first show that for each $i \in [n]$, $C_i \in \mathcal{L}$. As $C_1 \in \mathcal{L}$ and $e \notin C_{r+1} \cup \cdots \cup C_n$, it suffices to show that $C_2, \ldots, C_r \in \mathcal{L}$. By Proposition 3.4 it suffices to show that,

($\star$) $C_1, C_2, \ldots, C_r$ are in the same component of the e-graph of $\mathcal{C}$.

Notice that $C_1, \ldots, C_r$ and $B_1, \ldots, B_{d+1}$ are vertices of the e-graph. By Theorem 5.2 (4) we have

$$\begin{aligned} B_1 \cap C_i = \{e\} \quad \forall i \in [r] - \{1\} \\ B_2 \cap C_j = \{e\} \quad \forall j \in [r] - \{2\}, \end{aligned}$$

which in turn implies that vertices $\{B_1\} \cup \{C_i : i \in [r] - \{1\}\}$ are in the same component, and vertices $\{B_2\} \cup \{C_j : j \in [r] - \{2\}\}$ are in the same component. If $r \geq 3$ then C_3 is common to these vertex sets, so ($\star$) holds. We may therefore assume that $r = 2$. Therefore, since $\mathcal{C}$ is not Δ_3, we must have that $s \geq 3$, so there exists $B \in \{B_1, \ldots, B_n\} - \{B_1, B_2\}$ such that $e \in B$. Once again, (4) implies that $B \cap C_1 = B \cap C_2 = \{e\}$, implying that B, C_1, C_2 are in the same component, proving ($\star$). It remains to show for each $i \in [n]$ that $B_i \in b(\mathcal{L})$. Well,

[5] It is worth pointing out that deltas are the only mni clutters we found with a proper split. Also, we could not find any mni clutter with opposite elements.

since $B_2 \cap C_1 = \{e\}$, B_2 and C_1 are in the same component of the e-graph, so by Proposition 3.4, $B_2 \in b(\mathcal{L})$. Since $b(\mathcal{L})$ is also mni, a similar argument as for $(\star)$ implies that the members in $\{B_i : e \in B_i, i \in [n]\}$ belong to the same component of the e-graph of $\mathcal{C}$. As $B_2 \in b(\mathcal{L})$, we get that for each $i \in [n]$, $B_i \in b(\mathcal{L})$. $\Diamond$

Define $x^\star \in \mathbb{R}_+^{E(\mathcal{L})}$ as follows: $x^\star_f = 0$ and for each $g \in E(\mathcal{L}) - \{f\}$, $x^\star_g = \frac{1}{r}$.

Claim 2. *$x^\star$ is an extreme point of $Q(\mathcal{L})$.*

Proof of Claim. We first show that $x^\star \in Q(\mathcal{L})$. By definition, $x^\star \geq \mathbf{0}$. Let $L \in \mathcal{L}$. If L contains at least $r+1$ elements, then clearly $x^\star(L) \geq 1$. Otherwise, L has r elements, so its corresponding member C in $\mathcal{C}$ has r ones. This implies that $C \in \{C_1, \dots, C_n\}$, so by Claim 1, $C = L$. In particular, $e \in L$ and $f \notin L$, implying that $x^\star(L) \geq 1$. As this is true for each $L \in \mathcal{L}$, $x^\star \in Q(\mathcal{L})$. Since $x^\star(C_1) = \cdots = x^\star(C_n) = 1$ and $x^\star_f = 0$, Theorem 5.2 (1) implies that $x^\star$ is an extreme point of $Q(\mathcal{L})$, as required. $\Diamond$

In particular, $\mathcal{L}$ is non-ideal. In fact,

Claim 3. *$\mathcal{L} \setminus f$ is mni and $\overline{\mathcal{L} \setminus f} = \{C_1, \dots, C_n\}$.*

Proof of Claim. Since $x^\star_f = 0$, $\mathcal{L} \setminus f$ is non-ideal. To prove that it is mni, we need to show for each $g \in E(\mathcal{L}) - \{f\}$ that both $\mathcal{L} \setminus f \setminus g$, $\mathcal{L} \setminus f/g$ are ideal. Suppose first that $g \neq e$. Since $\mathcal{C}$ is mni, $\mathcal{C} \setminus g$ and $\mathcal{C}/g$ are ideal, so by Theorem 4.1 their splits $\mathcal{L} \setminus g, \mathcal{L}/g$ are ideal. In particular, the minors $\mathcal{L} \setminus g \setminus f$, $\mathcal{L}/g \setminus f$ are ideal, as needed. Suppose now that $g = e$. Notice that $\mathcal{L} \setminus f \setminus e = \mathcal{C} \setminus e$, so $\mathcal{L} \setminus f \setminus e$ is ideal. It remains to show that $\mathcal{L} \setminus f/e$ is ideal. Suppose not. Then $\mathcal{L} \setminus f/e$ must be mni. If this minor is a delta, then $M(\mathcal{L} \setminus f/e)$ has $n-2$ rows with $r-1 = 2$ ones in them. Each one of these rows must correspond to one of $C_1, \dots, C_r$ in $\mathcal{L}$, so $3 = r \geq n-2$, implying that $n = 5$. Since $r = 3$ it follows that $\mathcal{C} \cong \mathcal{B}_5$, but by Remark 5.3 $\mathcal{B}_5$ has no proper split. Hence, $\mathcal{L} \setminus f/e$ is not a delta. Then by Theorem 5.2 (1), $M(\mathcal{L} \setminus f/e)$ has precisely $n-1$ rows with $r-1$ many ones in them. Once again, all these rows must correspond to $C_1, \dots, C_r$ in $\mathcal{L}$, so $r \geq n-1$. We leave it as an exercise for the reader to verify that this implies $n = 3$, which in turn is a contradiction. Hence, $\mathcal{L} \setminus f$ is mni. Since each member of $\mathcal{L} \setminus f$ has at least r members and the members $C_1, \dots, C_n$ have precisely r members, it follows that $\overline{\mathcal{L} \setminus f} = \{C_1, \dots, C_n\}$. $\Diamond$

Now since $n \geq 5$, the preceding claim implies that $\mathcal{L} \setminus f$ is an mni clutter that is not a delta, finishing the proof. $\Box$

We are now ready to prove Theorem 5.1, stating that the only mni clutters with an ideal split are deltas.

Proof of Theorem 5.1. We know by Remark 3.8 that each delta has an ideal split. For the converse, let $\mathcal{C}$ be an mni clutter that is not a delta and let $\mathcal{L}$ be a split of it. By a repeated application of Proposition 5.4 it follows that $\mathcal{L}$ has as minor an mni clutter that is not a delta. In particular $\mathcal{L}$ is non-ideal, finishing the proof. $\Box$

5.2 Corollaries: binary clutters, odd holes, Steiner trees and matroids

Since an ideal split of a clutter $\mathcal{C}$ yields an ideal split for every minor of $\mathcal{C}$, Theorem 5.1 implies that,

Corollary 5.5. *If a clutter has a minimally non-ideal minor that is not a delta, then it has no ideal split.*

This corollary can be used as a weapon to recognize clutters that do not allow for an ideal split – we will see two nice examples. However, it can also be used to find examples where an ideal split may exist – we will see two examples for this as well.

A clutter $\mathcal{C}$ is *binary* if for all $C \in \mathcal{C}$ and $B \in b(\mathcal{C})$, $|C \cap B|$ is odd. It can be readily checked that if a clutter is binary, then so is each minor of it [29]. A delta Δ_n is not binary as for $\{1,2\} \in \Delta_n = b(\Delta_n)$, $|\{1,2\} \cap \{1,2\}| = 2$. Hence, binary clutters do not have a delta minor, so by Corollary 5.5,

Corollary 5.6. *A non-ideal binary clutter has no ideal split.*

For another consequence, consider the clutter of minimal vertex covers of a graph $G = (V, E)$. We saw in §3.3 an ideal split of this clutter when G was a comparability graph. But what about other graphs? We say G has an *odd hole* if it has a circuit of odd length at least five as an induced subgraph. In this case, if $n \geq 5$ is the length of the odd circuit, the clutter E of edges of G has the clutter

$$\mathcal{C}_n^2 := \big\{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\big\}$$

as a (deletion) minor, so the clutter $b(E)$ of minimal vertex covers of G has the clutter $b(\mathcal{C}_n^2)$ as a minor. Well, $\mathcal{C}_n^2$ and its blocker $b(\mathcal{C}_n^2)$ are mni clutters that are different from deltas [20]. Thus by Corollary 5.5,

Corollary 5.7. *If a graph has an odd hole, then its clutter of minimal vertex covers has no ideal split.*

Note that comparability graphs are exempt from this corollary, as they are perfect and therefore have no odd hole [4].

We also saw in §3.3 that under various conditions on the input, a clutter of Steiner trees has an ideal split. It would therefore be interesting to see what Corollary 5.5 says for these clutters in general. (The following proof employs similar techniques as in [3].)

Theorem 5.8. *Let $\mathcal{T}$ be a clutter of Steiner trees. Then every minimally non-ideal minor of $\mathcal{T}$, if any, is Δ_3.*

Proof. The class of Steiner tree clutters is closed under taking minors. We may therefore assume that $\mathcal{T}$ itself is mni. Suppose $\mathcal{T}$ is the clutter of Steiner trees of graph $G = (V, E)$ with terminals $R \subseteq V$. Let $\mathcal{C}$ be the clutter of Steiner cuts of G, and note that $\mathcal{C}$ is the blocker of $\mathcal{T}$. If $\mathcal{C}$ is a delta let $\overline{\mathcal{C}} := \mathcal{C}$ and $\overline{\mathcal{T}} := \mathcal{T}$, and otherwise let $\overline{\mathcal{C}}, \overline{\mathcal{T}}$ be the cores of $\mathcal{C}, \mathcal{T}$, respectively. (We are aware of the abuse of notation here.) Note that either way, $M(\overline{\mathcal{C}})$ and $M(\overline{\mathcal{T}})$ are non-singular by Theorem 5.2 (1).

Let $n := |E(\mathcal{C})| = |E|$ and suppose $\overline{\mathcal{C}} = \{C_1, \ldots, C_n\}$. Fix an $r \in R$ and for each $i \in [n]$, let U_j be a minimal subset of $V - \{r\}$ such that $U_j \cap R \neq \emptyset$ and $\delta(U_j) = C_j$.

Claim 1. *For distinct $i, j \in [n]$, either $U_i \cap U_j = \emptyset$, $U_i \subsetneq U_j$ or $U_j \subsetneq U_i$.*

Proof of Claim. Suppose otherwise. If $U_i \cap U_j \cap R \neq \emptyset$ let $U := U_i \cap U_j$ and $W := U_i \cup U_j$, and otherwise let $U := U_i - U_j$ and $W := U_j - U_i$. Suppose that $\mathcal{C}$ is not a delta. (We leave the other case to the reader.) Assume that $\overline{\mathcal{C}}$ is c-regular. By submodularity, we have $\delta(U) \cap \delta(W) \subseteq \delta(U_i) \cap \delta(U_j)$ and $\delta(U) \cup \delta(W) \subseteq \delta(U_i) \cup \delta(U_j)$. In particular,

$$2c = |\delta(U_i)| + |\delta(U_j)| \geq |\delta(U)| + |\delta(W)| \geq 2c,$$

so equality holds throughout and $\delta(U) \cap \delta(W) = \delta(U_i) \cap \delta(U_j)$ and $\delta(U) \cup \delta(W) = \delta(U_i) \cup \delta(U_j)$. Hence, $\delta(U), \delta(W)$ belong to $\overline{\mathcal{C}}$ and their corresponding rows in $M(\overline{\mathcal{C}})$ have the same sum as the rows corresponding to $\delta(U_i), \delta(U_j)$. Non-singularity of this matrix implies that $\{\delta(U), \delta(W)\} = \{\delta(U_i), \delta(U_j)\}$, contradicting our minimal choice for one of U_i, U_j. $\Diamond$

Claim 2. *Fix $i \in [n]$. Then for each $u \in U_i \cap R$ and $w \in U_i$, there is a uw-path in $G[U_i]$.*

Proof of Claim. Let U_i' be the set of all vertices of U_i reachable from u inside $G[U_i]$. Then $\delta(U_i')$ contains a Steiner cut and is contained in $C_i = \delta(U_i)$. The minimality of U_i and C_i implies that $U_i = U_i'$, proving the claim. $\Diamond$

Assume that $\overline{\mathcal{T}} = \{T_1, \ldots, T_n\}$.

Claim 3. *$G \setminus r$ is connected.*

Proof of Claim. Suppose otherwise, and let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two induced subgraphs of G where $V_1 \cap V_2 = \{r\}$ and E_1, E_2 partition E. Since $E_1, E_2 \neq \emptyset$ and each edge of $E_1 \cup E_2$ appears in a member of $\mathcal{T}$, it follows that $(V_1 - \{r\}) \cap R, (V_2 - \{r\}) \cap R \neq \emptyset$. Thus, every Steiner tree contains an edge from E_1 and an edge from E_2. This immediately implies that $\mathcal{T}$ is not a delta, as such a partition does not exist for deltas. For each $i \in [n]$ let $X_i := T_i \cap E_1$ and $Y_i := T_i \cap E_2$. Assume $\overline{\mathcal{T}}$ is t-regular. Then for each $i \in [t]$, $|X_i| + |Y_i| = t$, and since for each $j \in [t]$, $X_i \cup Y_j$ is also a Steiner tree, it follows that $|X_i| + |Y_j| \geq t$. This in turn implies that $|X_1| = \cdots = |X_n|$ and $|Y_1| = \cdots = |Y_n|$, and so for all $i, j \in [n]$, $X_i \cup Y_j \in \overline{\mathcal{T}}$. This contradicts the non-singularity of $M(\overline{\mathcal{T}})$. $\Diamond$

Let e, f be distinct edges incident to r, and let u', w' be their other ends, respectively. By Theorem 5.2 (2), there are $i, j \in [n]$ such that $e \in C_i \not\ni f$ and $f \in C_j \not\ni e$. Then Claim 1 implies that $U_i \cap U_j = \emptyset$. Now take $u \in U_i \cap R$ and $w \in U_j \cap R$. By Claim 2 there is a uu'-path P_u in $G[U_i]$ and there is ww'-path P_w in $G[U_j]$. Also, Claim 3 yields a uw-path P_{uw} in $G \setminus r$. Now delete all edges outside $P_u \cup P_w \cup P_{uw} \cup \{e, f\}$, contract paths P_u, P_w and contract P_{uw} to a single edge between u, w. The resulting minor is a triangle on terminals u, w, r, so $\mathcal{T}$ has a Δ_3 minor. The minimality of $\mathcal{T}$ implies that it is Δ_3, as required. $\square$

Therefore, Corollary 5.5 does not say anything about clutters of Steiner trees. So when does a clutter of Steiner trees have an ideal split? We leave this as an open-ended question.

As mentioned earlier, Edmonds proved that the clutter of spanning r-arborescences of a directed graph is always ideal. In particular, the clutter of spanning trees of a graph has an ideal split. Well, can we generalize this statement to matroids? To be more specific, is there an ideal split for the clutter of bases of a matroid?

If $\mathcal{C}$ is the clutter of bases of a matroid, then $b(\mathcal{C})$ is the clutter of circuits of the dual matroid. So the question above may be rephrased as follows: is there an ideal split for the clutter of circuits of a matroid?

Theorem 5.9. *Let $\mathcal{C}$ be the clutter of circuits of a matroid. Then every minimally non-ideal minor of $\mathcal{C}$, if any, is Δ_3.*

Before proving this result, let us point out that if a minimally non-ideal clutter, other than deltas, has a member of size two, then its core is an odd hole [9]:

$$\mathcal{C}_n^2 = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\} \quad n = 5,7,9,\ldots$$

Proof. Since the class of matroid circuit clutters is closed under taking minors, we may assume that $\mathcal{C}$ itself is mni. Suppose $\mathcal{C}$ is the clutter of circuits of matroid M, and let $\mathcal{B}$ be the clutter of bases of the dual matroid $M^\star$. Notice that $\mathcal{B}$ is the blocker of $\mathcal{C}$. Since the members of $\mathcal{B}$ have the same size, it follows that if $\mathcal{B}$ is a delta, then it is Δ_3. It therefore suffices to show that $\mathcal{B}$ is a delta. Suppose otherwise. We will apply Theorem 5.2. Let n, r, s be the parameters from (2) and (3) and let $C_1,\ldots,C_n,B_1,\ldots,B_n$ be the labeling from (4) of Theorem 5.2. Let $q := rs - n + 1 \in \{2,\ldots,\min\{r,s\}\}$. Since bases of a matroid have the same cardinality, it follows that $\mathcal{B}$ is equal to its core.

We claim that $s = 2$. Suppose otherwise. Take $e \in E$ and rearrange $B_1,\ldots,B_n$ and $C_1,\ldots,C_n$ so that $e \in B_j \cap C_j$ for all $1 \le j \le q$, and $B_1 - \{e\},\ldots,B_q - \{e\}, C_1 - \{e\},\ldots,C_q - \{e\}$ are pairwise disjoint except for the pairs $B_j - \{e\}, C_j - \{e\}$, $1 \le j \le q$. Since $s \ge 3$ there exists $f \in B_1 - \{e\}$ such that $C_1 \cap B_1 - \{e,f\} \ne \emptyset$. Consider B_2. By the basis exchange property for matroids, there exists $g \in B_2 - \{e\}$ such that $B_1 - \{f\} \cup \{g\} = B_j$, for some $2 \le j \le n$. But then $|C_1 \cap B_j| \ge 2$, a contradiction as $j \ne 1$. Hence $s = 2$. This implies that $\mathcal{B}$ is an odd hole. However, odd holes do not satisfy the basis exchange property of matroids, a contradiction. $\square$

So Corollary 5.5 does not rule out any such clutter from possessing an ideal split.

5.3 The minimally non-packing clutters that have a split with the packing property

Here we will play the same game with the packing property, but first let us discuss a relevant application of Theorem 5.2. Let $\mathcal{C}$ be an mni clutter that is not a delta, let r be the minimum size of a member, let s be the minimum size of a cover, and let $n := |E(\mathcal{C})|$. By definition $\tau(\mathcal{C},\mathbf{1}) = s$. Also by definition, $x^\star := \left(\frac{1}{r},\frac{1}{r},\ldots,\frac{1}{r}\right)^\top$ is a point in $Q(\mathcal{C})$. Thus by Theorem 5.2 (3),

$$\nu(\mathcal{C},\mathbf{1}) \le \nu^\star(\mathcal{C},\mathbf{1}) = \tau^\star(\mathcal{C},\mathbf{1}) \le \sum_{g \in E(\mathcal{C})} x_g^\star = \frac{n}{r} \le \frac{rs-1}{r} < s = \tau(\mathcal{C},\mathbf{1}).$$

In particular, $\mathcal{C}$ does not pack. Since deltas do not pack either,

Corollary 5.10 ([21])**.** *A clutter with the packing property is ideal.*

We say a clutter is *minimally non-packing (mnp)* if it does not pack but every proper minor of it does. For instance, each delta is mnp. We saw in Remark 3.7 (ii) that for each $n \geq 3$, the delta Δ_n splits to Δ_n^{n-2}, which can easily be checked to have the packing property. In fact,

Theorem 5.11. *Deltas are the only minimally non-packing clutters that have a split with the packing property.*

By Corollary 5.10, an mnp clutter is either ideal or mni. Quite naturally then, we prove Theorem 5.11 in two stages. Let us deal with the first stage, and the second stage will follow as a corollary of Theorem 5.1 and Corollary 5.10.

Proposition 5.12 (Cornuéjols et al. [10])**.**
Let $\mathcal{C}$ be an ideal, minimally non-packing clutter. Then for each $g \in E(\mathcal{C})$ the following hold:

(1) g appears in a minimum cover,

(2) there is a member of $\mathcal{C}$ containing g that intersects every minimum cover exactly once, and

(3) there is a minimum cover that does not contain g.

Proof. (1) By definition $\mathcal{C} \setminus g$ packs, so since $\mathcal{C}$ does not pack, it must be the case that $\tau(\mathcal{C} \setminus g) < \tau(\mathcal{C})$. In other words, element g appears in a minimum cover B_g of $\mathcal{C}$.

(2) Since $\mathcal{C}$ is ideal, it follows that $\bar{x} := \chi_{B_g}$ is in fact a minimum fractional cover, with respect to weights $\mathbf{1}$. Let $\bar{y} \in \mathbb{R}_+^{\mathcal{C}}$ be a maximum fractional packing, with respect to the same weights. By the complementary slackness conditions, since $\bar{x}_g = 1 > 0$ it must be that

$$\sum \left(\bar{y}_C : C \in \mathcal{C}, g \in C\right) = 1.$$

In particular, there exists $C_g \in \mathcal{C}$ such that $g \in C_g$ and $\bar{y}_{C_g} > 0$. We claim that C_g is the desired member. Let B be an arbitrary minimum cover. We need to show that $|B \cap C_g| = 1$. Then $x := \chi_B$ is a minimum fractional cover, with weights $\mathbf{1}$. Since $\bar{y}_{C_g} > 0$, the complementary slackness conditions once again imply that $x(C_g) = 1$, i.e. $|B \cap C_g| = 1$, as needed.

(3) Take an element $h \in C_g - \{g\}$. By (1) there is a minimum cover B_h containing h. Then $h \in B_h \cap C_g$ and since C_g intersects B_h exactly once, it follows that $g \notin B_h$.

□

Proposition 5.13. *Let $\mathcal{C}$ be an ideal, minimally non-packing clutter and let $\mathcal{L}$ be a single split of it at some element $e \in E(\mathcal{C})$. Then*

(i) for all $L \in \mathcal{L}$ and $K \in b(\mathcal{L})$ such that $e \in L, f \in K$ or $f \in L, e \in K$, we have $|L \cap K| \geq 2$,

(ii) one of $\mathcal{L} \setminus e, \mathcal{L} \setminus f$ is ideal and minimally non-packing.

Proof. (i) Suppose otherwise. Then by Theorem 2.5, $\mathcal{L}$ has an $\{e, f\}$-special minor, which corresponds to a delta minor in $\mathcal{C}$. However, $\mathcal{C}$ is ideal while its delta minor is not, a contradiction.

(ii) By Proposition 5.12 (2), there is a member $C_e \in \mathcal{C}$ containing e that intersects every minimum cover exactly once. After relabeling e and f, if necessary, we may assume that the member corresponding to C_e in $\mathcal{L}$ uses e; call this member L_e.

We claim that f never appears in a minimum cover of $\mathcal{L}$. Suppose for a contradiction that K_f is a minimum cover of $\mathcal{L}$ that uses f, and let B be its corresponding member in $b(\mathcal{C})$. Since $|L_e \cap K_f| \geq 1$, it follows that $|C_e \cap B| \geq 2$. However, B is also a minimum cover of $\mathcal{C}$, contradicting our choice of C_e.

Since f does not appear in a minimum cover, it follows that $\tau(\mathcal{L} \setminus f) = \tau(\mathcal{L}) = \tau(\mathcal{C})$. Thus, if $\mathcal{L} \setminus f$ were to pack then so would $\mathcal{C}$, which is not the case. So $\mathcal{L} \setminus f$ does not pack. (This is sufficient to prove Theorem 5.11, but we will go further.) We claim that $\mathcal{L} \setminus f$ is in fact ideal and mnp. Since $\mathcal{C}$ is ideal, Theorem 4.1 implies that $\mathcal{L}$ is ideal, so $\mathcal{L} \setminus f$ is also ideal. To prove $\mathcal{L} \setminus f$ is mnp, we need to prove that for each $g \in E(\mathcal{L}) - \{f\}$, clutters $\mathcal{L} \setminus f \setminus g$ and $\mathcal{L} \setminus f/g$ pack. If $g \neq e$ then since $\mathcal{C} \setminus g, \mathcal{C}/g$ have the packing property, Theorem 4.7 implies that $\mathcal{L} \setminus f \setminus g$ and $\mathcal{L} \setminus f/g$ pack. Moreover, $\mathcal{L} \setminus f \setminus e = \mathcal{C} \setminus e$ packs by definition. Thus, it remains to show that $\mathcal{L} \setminus f/e$ packs.

Note that

$$\tau(\mathcal{L} \setminus f/e) \geq \tau(\mathcal{L} \setminus f) = \tau(\mathcal{L}).$$

By Proposition 5.12 (3), there is a minimum cover of $\mathcal{C}$ avoiding e, so there is a minimum cover of $\mathcal{L}$ avoiding e, which implies that equality holds above, so

$$\tau(\mathcal{L} \setminus f/e) = \tau(\mathcal{L}) = \tau(\mathcal{C}).$$

So we need to find a packing of $\tau(\mathcal{C})$ members in $\mathcal{L} \setminus f/e$. The rest of the proof is similar to that of Proposition 4.6 (4).

Claim. *Suppose L_e and L_f are disjoint members of $\mathcal{L}$ where $e \in L_e$ and $f \in L_f$. Then there exist disjoint $L, L' \in \mathcal{L}$ contained in $(L_e \cup L_f) - \{e, f\}$.*

Proof of Claim. By (i) every minimal cover using e (resp. f) intersects L_f (resp. L_e) at least twice. As a result, given that $F = \big(E(\mathcal{L}) - (L_e \cup L_f)\big) \cup \{e, f\}$, we have $\tau(\mathcal{L} \setminus F) \geq 2$. Since $\mathcal{L} \setminus F = \mathcal{C} \setminus F$ packs, the result follows. $\Diamond$

Let $L_1, \ldots, L_k, \ldots, L_{k+\ell}, \ldots, L_{k+\ell+r}$ be a packing for $\mathcal{L}/\{e, f\}$ where

$$\begin{aligned} k + \ell + r = \tau(\mathcal{L}/\{e,f\}) &\geq \tau(\mathcal{L}/e) = \tau(\mathcal{L}) \\ \{e\} \cup L_j \in \mathcal{L} & \quad j = 1, \ldots, k \\ \{f\} \cup L_j \in \mathcal{L} & \quad j = k+1, \ldots, k+\ell \\ L_j \in \mathcal{L} & \quad j = k+\ell+1, \ldots, k+\ell+r. \end{aligned}$$

Let K_e be a minimum cover of $\mathcal{L}$ using element e. Then by (i), K_e intersects each of $L_{k+1}, \ldots, L_{k+\ell}$ at least twice. As a result, $\tau(\mathcal{L}) = |K_e| \geq 1 + 2\ell + r$ which, together with $k + \ell + r \geq \tau(\mathcal{L})$, implies that $k > \ell$.

By the claim above, for each $j \in [\ell]$, we can find disjoint $L_j^1, L_j^2 \in \mathcal{L}$ such that $L_j^1 \cup L_j^2 \subseteq L_j \cup L_{k+j}$. Observe now that

$$\begin{array}{rl} L_j^1, L_j^2 & j = 1, \ldots, \ell \\ L_j & j = \ell+1, \ldots, k \\ L_j & k+\ell+1, \ldots, k+\ell+r \end{array}$$

is a packing of size $k + \ell + r \geq \tau(\mathcal{L}) = \tau(\mathcal{C})$ in $\mathcal{L} \setminus f/e$, as required.

□

We are now ready to prove Theorem 5.11, stating that deltas are the only mnp clutters that have a split with the packing property.

Proof of Theorem 5.11. We know by Remark 6.9 that deltas do have a split with the packing property. Now let $\mathcal{C}$ be an mnp clutter that is not a delta. If $\mathcal{C}$ is ideal, then by a repeated application of Proposition 5.13 (ii), any split of $\mathcal{C}$ has as minor an ideal mnp clutter, which implies that no split of $\mathcal{C}$ has the packing property. Otherwise, when $\mathcal{C}$ is mni, then Theorem 5.1 implies that any split of $\mathcal{C}$ is non-ideal, and using Corollary 5.10, this implies that no split of $\mathcal{C}$ has the packing property, as required. □

5.4 Corollaries and a connection to ideal, minimally non-packing clutters

Since a packing split of a clutter $\mathcal{C}$ yields a packing split for every minor of $\mathcal{C}$, Theorem 5.11 implies that,

Corollary 5.14. *If a clutter has a minimally non-packing minor that is not a delta, then it has no split with the packing property.*

This result, together with Corollary 5.10, imply the following interesting result, that will be useful later:

Corollary 5.15. *Let $\mathcal{L}$ be a clutter with the packing property and let $\mathcal{C}$ be an identification of it. Then $\mathcal{C}$ has the packing property if and only if it is ideal.*

Proof. If $\mathcal{C}$ has the packing property, then by Corollary 5.10 it is also ideal. Conversely, if $\mathcal{C}$ does not have the packing property, then it has an mnp minor which by Corollary 5.14 is a delta, which implies in particular that $\mathcal{C}$ is non-ideal, finishing the proof. □

Before moving on, we should point out that the corollaries in §5.2 have analogues for the case of the packing property – since the statements are as one would expect and the proofs are exactly the same, we refrain from mentioning these corollaries. Let us instead discuss an intriguing connection. We first need the following result:

Proposition 5.16. *Let $\mathcal{C}$ be an ideal, minimally non-packing clutter. If $\mathcal{C}$ has a proper split or a pair of opposite elements, then $\tau(\mathcal{C}) \geq 3$.*

Proof. Suppose first that $\mathcal{C}$ has a proper single split $\mathcal{L}$ at some element $e \in E(\mathcal{C})$. By Proposition 5.12 (1) element e is in a minimum cover of $\mathcal{C}$, so after possibly relabeling e and f, we may assume that e is in a minimum cover K_e of $\mathcal{L}$. Let L_f be a member of $\mathcal{L}$ that uses f; this member exists as $\mathcal{L}$ is a proper split. Then by Proposition 5.13 (i),

$$\tau(\mathcal{C}) = \tau(\mathcal{L}) = |K_e| \geq 1 + |K_e \cap L_f| \geq 3,$$

as claimed.

Suppose next that $\mathcal{C}$ has opposite elements e, f. Assume for a contradiction $\tau(\mathcal{C}) = 2$. By Proposition 5.12 (1), there is a minimum cover B_e such that $e \in B_e$. Let g be the other element of B_e, and note that $g \neq f$. Let L_f be a member of $\mathcal{L}$ such that $f \in L_f$. Then $B_e \cap L_f \neq \emptyset$ and since $e \notin B_e \cap L_f$, we have $g \in L_f$. Thus, every member of $\mathcal{L}$ that uses f also uses g. Hence, since $\mathcal{L}$ does not pack, it follows that $\mathcal{L}/f$ does not pack, a contradiction. □

It is conjectured by Cornuéjols, Guenin and Margot [10] that

(?) for every ideal, minimally non-packing clutter $\mathcal{C}$, we have $\tau(\mathcal{C}) = 2$ (?)

Together with Proposition 5.16, this seems to imply that

(?) no ideal, minimally non-packing clutter has a proper split or a pair of opposite elements (?)

We were not able to prove (or disprove) any of these statements. The strongest property on such clutters that we have been able to show is Proposition 5.13 (ii).

6 Identifications: another glimpse of heaven

Here we characterize when identifications of ideal clutters, clutters with the packing property, and clutters with the MFMC property are respectively non-ideal, do not have the packing property, and do not have the MFMC property. We start and end this section with applications to st-path and directed st-path clutters and glean some insights.

6.1 From st-paths to directed st-paths

Let $G = (V, E)$ be a graph with distinguished vertices s, t. Consider the clutter $\mathcal{C}$ of st-paths of G, taken over ground set E. Observe that Theorem 4.1 implies that if $\mathcal{C}$ is ideal, then so is *any* split of it. Well, we *do* know that $\mathcal{C}$ is ideal. Also, we know by Remark 3.10 that the clutter of directed st-paths of the bidirection of G is a split of $\mathcal{C}$, so this new clutter is ideal too. Hence, from a universal point of view, the idealness of st-path clutters implies the idealness of bidirected st-path clutters. Moreover, any directed st-path clutter is a (deletion) minor of an appropriate bidirected st-path clutter. So since idealness is a minor-closed property, the idealness of st-path clutters implies the idealness of directed st-path clutters. Furthermore, we also know that clutter $\mathcal{C}$ has the packing and MFMC properties. These properties are also minor-closed, so by Theorems 4.7 and 4.3 we get that,

Corollary 6.1. *The idealness (resp. packing property, max-flow min-cut property) of st-path clutters implies the idealness (resp. packing property, max-flow min-cut property) of directed st-path clutters.*

Let us emphasize that the idealness, the packing and MFMC properties of such clutters (st-paths and directed st-paths) are well-known results [23, 14], and that this corollary is *not* an attempt to reprove these tools. What the corollary *does* state is the implication of one result from the other, using our results.

A natural question arises: does the idealness (resp. packing property, MFMC property) of directed st-path clutters imply the idealness of st-path clutters? In short, we are asking whether the converse of Corollary 6.1 holds. This question motivates us to study when identification does not preserve these properties.

6.2 Double deltas and their splits

Here we provide a preliminary for the forthcoming subsections. The result we prove is a natural extension of Theorem 2.5 for multiple opposite pairs.

Take integers $m \geq 1$ and $n \geq 1$. A *double delta of order* (m, n) is, up to isomorphism, the clutter $\Delta_{m,n}$ over ground set $\{e_1, \ldots, e_m, g, g_1, \ldots, g_n\}$ whose members are

$$\begin{aligned} &\{e_1, \ldots, e_m, g\} \\ &\{e_i, g_1, \ldots, g_n\} \quad i \in [m] \\ &\{g, g_j\} \quad j \in [n], \end{aligned}$$

and its incidence matrix is

$$M(\Delta_{m,n}) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & & & & \\ 1 & & & & & 1 & 1 & \cdots & 1 \\ & 1 & & & & 1 & 1 & \cdots & 1 \\ & & \ddots & & & \vdots & \vdots & \ddots & \vdots \\ & & & 1 & & 1 & 1 & \cdots & 1 \\ & & & & 1 & 1 & & & \\ & & & & 1 & & 1 & & \\ & & & & \vdots & & & \ddots & \\ & & & & 1 & & & & 1 \end{pmatrix}.$$

Observe that $b(\Delta_{m,n}) = \Delta_{m,n}$. Moreover, $\Delta_{m,1} \cong \Delta_{m+2}$ and $\Delta_{1,n} \cong \Delta_{n+2}$. Also, if $m \geq 2$ and $n \geq 2$, then $\Delta_{m,n} \setminus \{e_1, \ldots, e_{m-1}\}/e_m \cong \Delta_{n+1}$. Thus,

Remark 6.2. *A double delta has a delta minor.*

A *split double delta with opposite pairs* $(e_1, f_1), \ldots, (e_m, f_m)$ is, up to isomorphism, the clutter $\mathcal{S}_{m,n}$ over ground set $\{e_1, \ldots, e_m, f_1, \ldots, f_m, g, g_1, \ldots, g_n\}$ whose members are

$$\begin{aligned} &\{e_1, \ldots, e_m, g\} \\ &\{f_i, g_1, \ldots, g_n\} \quad i \in [m] \\ &\{g, g_j\} \quad j \in [n]. \end{aligned}$$

Its blocker has members

$$\begin{aligned} &\{f_1,\dots,f_m,g\} \\ &\{e_i,g_1,\dots,g_n\} \quad i\in[m] \\ &\{g,g_j\} \quad j\in[n], \end{aligned}$$

which itself is a split double delta with opposite pairs $(f_1,e_1),\dots,(f_m,e_m)$. It is now clear that for each $i\in[n]$, e_i and f_i are opposite for $\mathcal{S}_{m,n}$.

Remark 6.3. *The following hold:*

(1) $\mathcal{S}_{m,n}$ is a split of $\Delta_{m,n}$,

(2) $\mathcal{S}_{m,1}\cong\Delta_{m+2}^m$,

(3) Δ_{m+2}^m has the max-flow min-cut property,

(4) $\mathcal{S}_{m,n}$ has a split with the max-flow min-cut property (hence the split is ideal and has the packing property).

Proof. (1) The clutter obtained from $\mathcal{S}_{m,n}$ after identifying opposite pairs $(e_i,f_i), i\in[m]$ is $\Delta_{m,n}$, so the result follows. We leave (2) and (3) to the reader. (4) If $n=1$, then the result follows from (2) and (3). Otherwise, $n\geq 2$. In this case, consider the clutter over ground set

$$\{e_1,\dots,e_m,f_1,\dots,f_m,g,g_1,\dots,g_n,h_2,\dots,h_n\}$$

whose members are

$$\begin{aligned} &\{e_1,\dots,e_m,g\} \\ &\{f_i,g_1,\dots,g_n\} \quad i\in[m] \\ &\{g,g_1\} \\ &\{g,h_j\} \quad j\in\{2,\dots,n\}. \end{aligned}$$

For each $j\in\{2,\dots,n\}$ and $i\in[m]$, $\{g,g_1\}$ is contained in $\{f_i,g_1,\dots,g_n\}\cup\{g,h_j\}-\{g_j,h_j\}$, so by Theorem 2.1 we get that g_j and h_j are opposite. It is now clear that this clutter is a split of $\mathcal{S}_{m,n}$. It is left to the reader to check that this clutter has the MFMC property. □

We need the following ingredient:

Remark 6.4. *Let $\mathcal{C},\mathcal{C}'$ be two clutters. If $\mathcal{C}\subseteq\mathcal{C}'$ and $b(\mathcal{C})\subseteq b(\mathcal{C}')$ then $\mathcal{C}=\mathcal{C}'$.*

Note that (e,f)-special clutters are precisely split double deltas with a single opposite pair (e,f). Thus, the following may be viewed as an extension of Theorem 2.5. We say two pairs are *disjoint* if no component of the first pair is equal to a component of the second pair.

Theorem 6.5. *Let $\mathcal{L}$ be a clutter with pairwise disjoint opposite pairs $(e_1, f_1), \ldots, (e_m, f_m)$, for which there are $L \in \mathcal{L}$ and $K \in b(\mathcal{L})$ such that $\{e_1, \ldots, e_m\} \subset L$, $\{f_1, \ldots, f_m\} \subset K$ and $|L \cap K| = 1$. Then for some non-empty $J \subseteq [m]$, the identification of $\mathcal{L}$ at opposite pairs $(e_i, f_i), i \in [m] - J$ has as minor a split double delta with opposite pairs $(e_j, f_j), j \in J$.*

Proof. We say $\mathcal{C}$ is a *projective minor* of $\mathcal{L}$ if for some non-empty $J \subseteq [m]$:

(i) $\mathcal{C}$ is a minor of the identification of $\mathcal{L}$ at $(e_i, f_i), i \in [m] - J$, and

(ii) there are $C \in \mathcal{C}$ and $B \in b(\mathcal{C})$ such that $\{e_j : j \in J\} \subset C, \{f_j : j \in J\} \subset B$ and $|C \cap B| = 1$.

Notice that $\mathcal{L}$ is a projective minor of itself. After possibly replacing $\mathcal{L}$ with a projective minor, we may assume that $\mathcal{L}$ is the only projective minor of itself. Observe also that $b(\mathcal{L})$ is the only projective minor of $b(\mathcal{L})$. Suppose $L \cap K = \{g\}$. We will prove that $\mathcal{L}$ is a split double delta with opposite pairs $(e_1, f_1), \ldots, (e_m, f_m)$, finishing the proof.

If $m = 1$ then by Theorem 2.5, $\mathcal{L}$ is a split double delta with opposite pair (e_1, f_1), as desired. We may therefore assume that $m \geq 2$.

Claim 1. *Fix $i \in [m]$. If $L' \in \mathcal{L}$ satisfies $L' \cap K = \{f_i\}$, then*

$$L' \cap \{e_1, \ldots, e_m\} = \emptyset.$$

Similarly, if $K' \in b(\mathcal{L})$ satisfies $L \cap K' = \{e_i\}$, then

$$K' \cap \{f_1, \ldots, f_m\} = \emptyset.$$

Proof of Claim. We may assume that $i = 1$. We prove the first part, and the second part follows by symmetry. Since e_1, f_1 are opposite and $f_1 \in L'$, it follows that $e_1 \notin L'$. Let $J := \{j \in [m] : e_j \in L'\} \subseteq [m] - \{1\}$. Suppose for a contradiction that $J \neq \emptyset$. Let $\mathcal{C}$ be the identification of $\mathcal{L}$ at $(e_i, f_i), i \in [m] - J$, let C be the member corresponding to L' in $\mathcal{C}$, and let B be the member corresponding to K in $b(\mathcal{C})$. Since $L' \cap K = \{f_1\}$ we get that $|C \cap B| = 1$, so both (i) and (ii) are satisfied. Thus, $\mathcal{C}$ is a proper projective minor of $\mathcal{L}$, a contradiction to our choice of $\mathcal{L}$. ◊

Claim 2. *$L = \{e_1, \ldots, e_m, g\}$ and $K = \{f_1, \ldots, f_m, g\}$.*

Proof of Claim. The clutter obtained after contracting $L - \{e_1, \ldots, e_m, g\}$ and deleting $K - \{f_1, \ldots, f_m, g\}$ is a projective minor of $\mathcal{L}$, so the claim follows as $\mathcal{L}$ is the unique projective minor. ◊

Label $E(\mathcal{L}) - (L \cup K)$ so that for some $n \geq 0$, $E(\mathcal{L}) - (L \cup K) = \{g_1, \ldots, g_n\}$.

Claim 3. *For each $i \in [m]$, $\{f_i, g_1, \ldots, g_n\} \in \mathcal{L}$ and similarly, $\{e_i, g_1, \ldots, g_n\} \in b(\mathcal{L})$.*

Proof of Claim. We may assume that $i = 1$. We prove the first part, and the second part follows by symmetry. Choose $L_1' \in \mathcal{L}$ such that $L_1' \cap K = \{f_1\}$. By Claim 1, $L_1' \subseteq \{f_1, g_1, \dots, g_m\}$; denote by E_d the difference. We claim that $E_d = \emptyset$, thereby finishing the proof. Suppose otherwise. Let $\mathcal{L}' := \mathcal{L} \setminus E_d$ and let K' be a member of $b(\mathcal{L}')$ contained in K. Since $K' \cap L \neq \emptyset$ and $K' \cap L_1' \neq \emptyset$, it follows that $\{g, f_1\} \subseteq K'$. Let $J := \{j \in [m] : f_j \in K'\} \ni 1$. Let $\mathcal{C}$ be the identification of $\mathcal{L}$ at $(e_i, f_i), i \in [m] - J$, let C be the member corresponding to L in $\mathcal{C}$, and let B be the member corresponding to K' in $b(\mathcal{C})$. Since $L \cap K = \{g\}$ we have that $C \cap B = \{g\}$, so both (i) and (ii) are satisfied. Thus, $\mathcal{C}$ is a projective minor of $\mathcal{L}$, which is different from $\mathcal{L}$ as $E_d \neq \emptyset$, a contradiction. $\Diamond$

Since every member and cover intersect, it follows that $n \geq 1$.

Claim 4. *For each $i \in [n]$, $\{g, g_i\} \in \mathcal{L}$ and similarly, $\{g, g_i\} \in b(\mathcal{L})$.*

Proof of Claim. We may assume that $i = 1$. Once again, we only prove the first statement and the second one will follow by symmetry. Choose $L_1' \in \mathcal{L}$ such that $L_1' \cap \{e_1, g_1, \dots, g_n\} = \{g_1\}$. We will see that $L_1' = \{g, g_1\}$.

We first claim that $L_1' \cap \{f_1, \dots, f_m\} = \emptyset$. Suppose for a contradiction that for some $i \in [m]$, $f_i \in L_1'$. Let $\mathcal{C}$ be the identification of $\mathcal{L}$ at $(e_j, f_j), j \in [m] - \{i\}$, let C be the member corresponding to L_1' in $\mathcal{C}$, and let $B := \{e_i, g_1, \dots, g_n\} \in b(\mathcal{C})$. Then $C \cap B = \{g_1\}$, so both (i) and (ii) are satisfied for $J := \{i\}$. Thus, $\mathcal{C}$ is a projective minor of $\mathcal{L}$, which is different from $\mathcal{L}$ as $m \geq 2$, a contradiction.

Since $L_1' \cap K \neq \emptyset$ it follows that $g \in L_1'$. To show that $L_1' = \{g, g_1\}$, it therefore remains to show that $L_1' \cap \{e_1, \dots, e_m\} = \emptyset$. Let $J := \{j \in [m] : e_j \in L_1'\}$ and suppose for a contradiction that $J \neq \emptyset$. Note that $1 \notin J$. Let $\mathcal{C}$ be the identification of $\mathcal{L}$ at $(e_j, f_j), j \in [m] - J$, let C be the member corresponding to L_1' in $\mathcal{C}$, and let B be the member corresponding to K in $b(\mathcal{C})$. Then $C \cap B = \{g\}$, so both (i) and (ii) are satisfied. Thus, $\mathcal{C}$ is a proper projective minor of $\mathcal{L}$, a contradiction. $\Diamond$

It therefore follows from Remark 6.4 and Claims 2, 3 and 4 that $\mathcal{L}$ is a split double delta, finishing the proof. $\square$

We get the following very useful corollary:

Corollary 6.6. *Let $\mathcal{L}$ be a clutter and let $\mathcal{C}$ be an identification of it. Suppose for some opposite elements e, f of $\mathcal{L}$ that are identified in $\mathcal{C}$, there exist $L_e \in \mathcal{L}$ and $K_f \in b(\mathcal{L})$ such that $e \in L_e$, $f \in K_f$ and $|L_e \cap K_f| = 1$. Then $\mathcal{C}$ has a delta minor.*

Proof. Observe that each element g of $\mathcal{C}$ corresponds to a subset E_g of elements of $\mathcal{L}$, and that $\big(E_g : g \in E(\mathcal{C})\big)$ is a partition of $E(\mathcal{L})$. Note that e, f belong to the same part. Note further that each member of $\mathcal{L}$ (resp. $b(\mathcal{L})$) picks at most one element from each part. Let $\mathcal{P}$ be the set of all pairs (e', f') such that

(i) $e', f' \in E(\mathcal{L}')$ are distinct and belong to the same part, and

(ii) $e' \in L_e$ and $f' \in K_f$.

Note that $(e,f) \in \mathcal{P}$ and any two pairs in $\mathcal{P}$ are disjoint. Thus for some $m \geq 1$, $\mathcal{P} = \{(e_1, f_1), \ldots, (e_m, f_m)\}$. Let $\mathcal{L}'$ be an identification of $\mathcal{L}$ that satisfies the following:

> for each $i \in [m]$, e_i and f_i are opposite in $\mathcal{L}'$, and the identification of $\mathcal{L}'$ at opposite pairs $(e_1, f_1), \ldots, (e_m, f_m)$ is $\mathcal{C}$.

Let L' be the member corresponding to L_e in $\mathcal{L}'$, and let K' be the member corresponding to K_f in $b(\mathcal{L}')$. Then $\{e_1, \ldots, e_m\} \subset L'$, $\{f_1, \ldots, f_m\} \subset K'$ and $|L' \cap K'| = |L \cap K| = 1$. Thus, it follows from Theorem 6.5 that $\mathcal{C}$ has a double delta minor. So by Remark 6.2, $\mathcal{C}$ has a delta minor, as required. $\square$

6.3 When identification does not preserve: idealness

We start with the following:

Proposition 6.7. *Let $\mathcal{L}$ be an ideal clutter with opposite elements e, f and let $\mathcal{C}$ be its identification at the two elements. Then the following are equivalent:*

(i) $\mathcal{C}$ is non-ideal,

(ii) there exist $L_e \in \mathcal{L}$ and $K_f \in b(\mathcal{L})$ such that $e \in L_e$, $f \in K_f$ and $|L_e \cap K_f| = 1$,

(iii) $\mathcal{C}$ has a Δ_3 minor.

Proof. Suppose first that (ii) holds. Then by Theorem 2.5, $\mathcal{L}$ has an (e,f)-special minor. By Remark 3.9, this clutter corresponds to a Δ_n minor in $\mathcal{C}$, for some $n \geq 3$. Since $\mathcal{C}$ has an ideal single split, it follows that Δ_n has an ideal single split, so by Remark 3.8 we get that $n = 3$, and (iii) holds.

Suppose next that (iii) holds. Then since Δ_3 is non-ideal, it follows that $\mathcal{C}$ is also non-ideal, so (i) holds.

Suppose finally that (i) holds. We may assume that each element of $\mathcal{C}$ is used in a member, so $Q(\mathcal{C})$ is a full-dimensional and pointed polyhedron. Let $y^\star \in \mathbb{R}_+^{E(\mathcal{C})}$ be a fractional extreme point of $Q(\mathcal{C})$, and consider the constraints that are tight at $y^\star$; collect their coefficient vectors into a family $\mathcal{T}$. Then each vector a of $\mathcal{T}$ could be one of two types: (1) $a = \chi_C$ for some $C \in \mathcal{C}$ such that $y^\star(C) = 1$, or (2) $a = \chi_{\{g\}}$ for some $g \in E(\mathcal{C})$ such that $y^\star_g = 0$. Let $m := |E(\mathcal{C})|$. Then $\mathcal{T}$ spans a linear space of dimension m, i.e. $\text{rank}(\mathcal{T}) = m$.

Define $x \in \mathbb{R}_+^{E(\mathcal{L})}$ as follows: for each $g \in E(\mathcal{L}) - \{f\}$, $x_g := y^\star_g$ and $x_f := x_e$. Then $x \in Q(\mathcal{L})$ and consider the constraints that are tight at x; collect their coefficient vectors into a family $\mathcal{T}'$. Then a vector a is in $\mathcal{T}'$ if and only if one of the following holds: (1) for some $C \in \mathcal{C}$ and its corresponding member L in $\mathcal{L}$ such that $\chi_C \in \mathcal{T}$, $a = \chi_L$, (2) for some $g \in E(\mathcal{C})$ such that $\chi_{\{g\}} \in \mathcal{T}$, $a = \chi_{\{g\}}$, or (3) if $\chi_{\{e\}} \in \mathcal{T}$ then $a = \chi_{\{f\}}$. Notice that $m + 1 \geq \text{rank}(\mathcal{T}') \geq m$, and since $Q(\mathcal{L})$ is an integral polyhedron while x is fractional, we have that $\text{rank}(\mathcal{T}') = m$.

Claim. *The following hold:*

(1) for each $g \in E(\mathcal{L})$, x_g appears in a tight constraint of $Q(\mathcal{L})$, and

(2) $x_e = x_f > 0$.

Proof of Claim. (1) This is clearly true for each $g \in E(\mathcal{L}) - \{e, f\}$. Suppose for a contradiction that for some $g \in \{e, f\}$, x_g does not appear in a tight constraint. After relabeling e and f, if necessary, we may assume $g = f$. Decrease component x_f of x until a new constraint of $Q(\mathcal{L})$ becomes tight; call this new point $x^\star$. Note that the rank of the tight constraints at $x^\star$ is that of x plus 1, so the rank is $m + 1$ and as a result, $x^\star$ is an extreme point of $Q(\mathcal{L})$. However, since at least two components of $y^\star$ are fractional, it follows that $x^\star$ is also fractional, a contradiction as $Q(\mathcal{L})$ is an integral polyhedron.

(2) For if not, rank$(\mathcal{T}') = m + 1$, which is not the case.

$\Diamond$

In particular, there exists $L_e \in \mathcal{L}$ such that $e \in L_e$ and $\chi_{L_e} \in \mathcal{T}'$. Since rank$(\mathcal{T}') = m$, point x lies on an edge of $Q(\mathcal{L})$, and by part (1) of the claim, this edge is not an extreme ray. Thus, this edge contains two extreme points χ_K and $\chi_{K'}$, for some $K, K' \in b(\mathcal{L})$. Moreover, for some $\lambda \in (0, 1)$, we have

$$x = \lambda \chi_K + (1 - \lambda)\chi_{K'}.$$

Since $x_e = x_f > 0$, it follows that $\lambda = \frac{1}{2}$ and after relabeling K and K', if necessary, $f \in K$ and $e \in K'$. (In particular, $y^\star$ is half-integral and $y^\star_e = \frac{1}{2}$.) However, note that the constraints corresponding to $\mathcal{T}'$ are also tight for K, K', so in particular,

$$|L_e \cap K| = 1,$$

implying (ii) holds. $\square$

We are now ready for the main result of this subsection.

Theorem 6.8. *Let $\mathcal{L}$ be an ideal clutter and let $\mathcal{C}$ be an identification of it. Then the following are equivalent:*

(i) $\mathcal{C}$ is non-ideal,

(ii) for some opposite elements e, f of $\mathcal{L}$ that are identified in $\mathcal{C}$, there exist $L_e \in \mathcal{L}$ and $K_f \in b(\mathcal{L})$ such that $e \in L_e, f \in K_f$ and $|L_e \cap K_f| = 1$,

(iii) $\mathcal{C}$ has a delta minor.

Proof. Corollary 6.6 implies (ii) $\Rightarrow$ (iii), and since deltas are non-ideal, (iii) $\Rightarrow$ (i). Suppose now that (i) holds. Let $\mathcal{L} = \mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{k-1}, \mathcal{L}_k = \mathcal{C}$ be a sequence of clutters where for each $i \in [k]$, $\mathcal{L}_i$ is a single identification of $\mathcal{L}_{i-1}$. Let $j \in [k]$ be the smallest index such that $\mathcal{L}_j$ is non-ideal. Assume that opposite elements e, f of $\mathcal{L}_{j-1}$ are identified to obtain $\mathcal{L}_j$. Then by Proposition 6.7, there exist $L'_e \in \mathcal{L}_{j-1}$ and $K'_f \in b(\mathcal{L}_{j-1})$ such that $e \in L'_e, f \in K'_f$ and $|L'_e \cap K'_f| = 1$. Let L_e (resp. K_f) be the corresponding member of L'_e (resp. K'_f) in $\mathcal{L}$. Then

$$1 \leq |L_e \cap K_f| \leq |L'_e \cap K'_f| = 1,$$

so $|L_e \cap K_f| = 1$, so (ii) holds. $\square$

There is another way to prove the equivalence of (i) and (iii), by using Lehman's heavy machinery instead of Corollary 6.6. It is clear (iii) $\Rightarrow$ (i). To show (i) $\Rightarrow$ (iii), assume $\mathcal{C}$ is non-ideal, so it has mni minors. Since $\mathcal{C}$ has an ideal split, it follows from Corollary 5.5 that each mni minor of it is a delta, proving (iii) in particular.

6.4 When identification does not preserve: the packing property

We start with the following two ingredients:

Remark 6.9. *For $n \geq 3$, Δ_n^{n-2} is the unique split of Δ_n with the packing property.*

Proof. This follows immediately from Remark 3.7 (iv) and Remark 4.5. □

Proposition 6.10. *Take a clutter $\mathcal{L}$ with the packing property with opposite elements e, f and let $\mathcal{C}$ be its identification at the two elements. Then the following are equivalent:*

(i) $\mathcal{C}$ does not have the packing property,

(ii) there exist $L_e \in \mathcal{L}$ and $K_f \in b(\mathcal{L})$ such that $e \in L_e, f \in K_f$ and $|L_e \cap K_f| = 1$,

(iii) $\mathcal{C}$ has a Δ_3 minor.

Proof. Suppose first that (ii) holds. Then by Theorem 2.5, $\mathcal{L}$ has an (e, f)-special minor. By Remark 3.9, this clutter corresponds to a Δ_n minor in $\mathcal{C}$. Since $\mathcal{C}$ has a single split with the packing property, so does Δ_n and by Remark 6.9, we get that $n = 3$, so (iii) holds.

Suppose next that (iii) holds. Since Δ_3 does not pack, it follows that $\mathcal{C}$ does not have the packing property, so (i) holds.

Suppose finally that (i) holds. Take disjoint subsets $I, J \subseteq E(\mathcal{C})$ such that $\mathcal{C}' := \mathcal{C} \setminus I/J$ is mnp. Since $\mathcal{L}$ has the packing property, $\mathcal{C}'$ is not a minor of $\mathcal{L}$, so $e \notin I \cup J$. Consider the minor $\mathcal{L}' := \mathcal{L} \setminus I/J$ of $\mathcal{L}$, which is a single split of $\mathcal{C}'$. Let $k := \tau(\mathcal{L}') = \tau(\mathcal{C}')$ and let $L'_1, \ldots, L'_k$ be pairwise disjoint members of $\mathcal{L}'$. Since $\mathcal{C}'$ does not pack, we may assume that $e \in L'_1$ and $f \in L'_2$. Moreover, since pairwise disjoint members in $\mathcal{L}' \setminus e$ and $\mathcal{L}' \setminus f$ correspond to pairwise disjoint members in $\mathcal{C}'$, it follows that each of e, f appears in a minimum cover of $\mathcal{L}'$. Let K'_f be a minimum cover of $\mathcal{L}'$ that contains element f. Then $|K'_f \cap L'_1| = 1$. Now let L_e be a member of $\mathcal{L}$ containing L'_1 and contained in $L'_1 \cup J$, and let K_f be a minimal cover of $\mathcal{L}$ containing K'_f and contained in $K'_f \cup I$. Then

$$|L_e \cap K_f| = |L'_1 \cap K'_f| = 1,$$

so (ii) holds. □

We are now ready for the main result of this subsection.

Theorem 6.11. *Take a clutter $\mathcal{L}$ with the packing property and let $\mathcal{C}$ be an identification of it. Then the following are equivalent:*

(i) $\mathcal{C}$ does not have the packing property,

(ii) for some opposite elements e, f of $\mathcal{L}$ that are identified in $\mathcal{C}$, there exist $L_e \in \mathcal{L}$ and $K_f \in b(\mathcal{L})$ such that $e \in L_e, f \in K_f$ and $|L_e \cap K_f| = 1$,

(iii) $\mathcal{C}$ has a delta minor.

The following are also equivalent:

(i') $\mathcal{C}$ has the packing property,

(ii') for each pair e, f of opposite elements of $\mathcal{L}$ that are identified in $\mathcal{C}$: for all $L_e, L_f \in \mathcal{L}$ such that $e \in L_e, f \in L_f$, there exist $L, L' \in \mathcal{L}$ such that $L \cup L' \subseteq (L_e \cup L_f) - \{e, f\}$ and $L \cap L' \subseteq L_e \cap L_f$.

Proof. We first prove the equivalence of (i)-(iii).

Corollary 6.6 implies (ii) $\Rightarrow$ (iii), and as deltas do not have the packing property, (iii) $\Rightarrow$ (i). Suppose now that (i) holds. Let $\mathcal{L} = \mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{k-1}, \mathcal{L}_k = \mathcal{C}$ be a sequence of clutters where for each $i \in [k]$, $\mathcal{L}_i$ is a single identification of $\mathcal{L}_{i-1}$. Let $j \in [k]$ be the smallest index such that $\mathcal{L}_j$ does not have the packing property. Assume that opposite elements e, f of $\mathcal{L}_{j-1}$ are identified to obtain $\mathcal{L}_j$. Then by Proposition 6.10, there exist $L'_e \in \mathcal{L}_{j-1}$ and $K'_f \in b(\mathcal{L}_{j-1})$ such that $e \in L'_e, f \in K'_f$ and $|L'_e \cap K'_f| = 1$. Let L_e (resp. K_f) be the corresponding member of L'_e (resp. K'_f) in $\mathcal{L}$. Then

$$1 \leq |L_e \cap K_f| \leq |L'_e \cap K'_f| = 1,$$

so $|L_e \cap K_f| = 1$, so (ii) holds.

It remains to prove the equivalence of (i') and (ii'). To this end, we need only prove the equivalence of (ii') and the negation of (ii).

Suppose first that (ii') holds. Assume for a contradiction that (ii) also holds for some L_e, K_f. Choose $L_f \in \mathcal{L}$ such that $f \in L_f$ and $L_f \cap K_f = \{f\}$. Since (ii') holds, there exist $L, L' \in \mathcal{L}$ such that $L \cup L' \subseteq (L_e \cup L_f) - \{e, f\}$ and $L \cap L' \subseteq L_e \cap L_f$. But then, as $|K_f \cap L_e| = 1$ and $K_f \cap L_f = \{f\}$, either $K_f \cap L = \emptyset$ or $K_f \cap L' = \emptyset$, a contradiction.

Suppose next that (ii') does not hold for some e, f, L_e, L_f. Let $J := L_e \cap L_f$, $I := \{e, f\} \cup (E(\mathcal{L}) - (L_e \cup L_f))$ and $\mathcal{L}' := \mathcal{L} \setminus I / J$. Then $\nu(\mathcal{L}') = 1$ and as $\mathcal{L}'$ packs, it follows that $\tau(\mathcal{L}') = 1$. Thus there exists a minimal cover K of $\mathcal{L}$ that intersects $(L_e \cup L_f) - \{e, f\}$ exactly once. After possibly relabeling e and f, we may assume that K intersects $L_e - \{e\}$. Since $K \cap L_f \neq \emptyset$, it follows that $f \in K$. Thus, since $|L_e \cap K| = 1$, (i) holds for L_e and $K_f := K$. $\square$

Let us say a few words about this result. Similar to the ideal case, one can prove the equivalence of (i) and (iii) by using the deep Corollary 5.14 instead of Corollary 6.6. The careful reader may notice that conditions (ii) and (iii) in Theorem 6.11 are exactly the same as those in Theorem 6.8 – this is not a coincidence, the reason for it is Corollary 5.15. (Using polyhedral projection, one can obtain fractional analogues of conditions (i') and (ii') for Theorem 6.8; see [1].)

6.5 When identification does not preserve: the max-flow min-cut property

Let $\mathcal{C}$ be a clutter over ground set E. Take an element $g \in E$ and let $\tilde{g}$ be a new copy of g. The *replication* of g is the clutter over ground set $E \cup \{\tilde{g}\}$ with members

$$\mathcal{C} \cup \big\{C \cup \{\tilde{g}\} - \{g\} : C \in \mathcal{C}, g \in C\big\}.$$

Take an integer $n \geq -1$. To *replicate an element n times* is to delete the element if $n = -1$, and to recursively apply replication at the element n times if $n \geq 0$.

Remark 6.12. *Let $\mathcal{C}$ be a clutter with weights $w \in \mathbb{Z}_+^{E(\mathcal{C})}$. Let $\widetilde{\mathcal{C}}$ be the clutter obtained from $\mathcal{C}$ after replicating each element $g \in E(\mathcal{C})$, $w_g - 1$ times. Then*

$$\tau(\mathcal{C}, w) = \tau(\widetilde{\mathcal{C}}) \quad \textit{and} \quad \nu(\mathcal{C}, w) = \nu(\widetilde{\mathcal{C}}).$$

Moreover, if $\mathcal{C}$ has the max-flow min-cut property, then so does $\widetilde{\mathcal{C}}$.

A *replication* of $\mathcal{C}$ is a clutter obtained from $\mathcal{C}$ after repeated applications of element replication. The following remark shows that splitting is in a sense closed under replication:

Remark 6.13. *Suppose $\mathcal{L}$ is a split of $\mathcal{C}$. If $\widetilde{\mathcal{C}}$ is a replication of $\mathcal{C}$, then there is a replication $\widetilde{\mathcal{L}}$ of $\mathcal{L}$ such that $\widetilde{\mathcal{L}}$ is a split of $\widetilde{\mathcal{C}}$.*

Proof. It suffices to prove this for single splits and single element replications. To this end, assume that $\mathcal{L}$ is a single split of $\mathcal{C}$ at element $e \in E(\mathcal{C})$, and that $\widetilde{\mathcal{C}}$ is the replication of some element $g \in E(\mathcal{C})$. If $g \neq e$, then the replication $\widetilde{\mathcal{L}}$ of $\mathcal{L}$ at g is a single split of $\widetilde{\mathcal{C}}$, as desired. Otherwise, $g = e$. In this case, let $\widetilde{\mathcal{L}}$ be the clutter obtained from $\mathcal{L}$ after replicating e and f. Then $\widetilde{\mathcal{C}}$ is the identification of $\widetilde{\mathcal{L}}$ at e, f and at $\tilde{e}, \tilde{f}$, so $\widetilde{\mathcal{L}}$ is a (double) split of $\widetilde{\mathcal{C}}$, as desired. $\square$

We are now ready for the main result of this subsection.

Theorem 6.14. *Let $\mathcal{L}$ be a clutter with the max-flow min-cut property and let $\mathcal{C}$ be an identification of it. Then the following are equivalent:*

(i) *$\mathcal{C}$ does not have the max-flow min-cut property,*

(ii) *for some opposite elements e, f of $\mathcal{L}$ that are identified in $\mathcal{C}$, there exist $L_e \in \mathcal{L}$ and $K_f \in b(\mathcal{L})$ such that $e \in L_e, f \in K_f$ and $|L_e \cap K_f| = 1$,*

(iii) *$\mathcal{C}$ has a delta minor.*

Proof. Since $\mathcal{L}$ also has the packing property, Theorem 6.11 implies (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Suppose now that (i) holds. Then for some $w \in \mathbb{Z}_+^{E(\mathcal{C})}$, $\tau(\mathcal{C}, w) > \nu(\mathcal{C}, w)$. Let $\widetilde{\mathcal{C}}$ be the clutter obtained from $\mathcal{C}$ after replicating each element $g \in E(\mathcal{C})$, $w_g - 1$ times. Then by Remark 6.12, $\tau(\widetilde{\mathcal{C}}) > \nu(\widetilde{\mathcal{C}})$, in particular, $\widetilde{\mathcal{C}}$ does not have the packing property. By Remark 6.13, there is a replication $\widetilde{\mathcal{L}}$ of $\mathcal{L}$ that is a split of $\widetilde{\mathcal{C}}$. By Remark 6.12, $\widetilde{\mathcal{L}}$ has the packing property. Hence by Theorem 6.11, for some opposite elements e, f of $\widetilde{\mathcal{L}}$ that are identified in $\widetilde{\mathcal{C}}$, there exist $L_e \in \widetilde{\mathcal{L}}$ and $\widetilde{K_f} \in b(\widetilde{\mathcal{L}})$ such that $e \in L_e, f \in \widetilde{K_f}$ and $|L_e \cap \widetilde{K_f}| = 1$. Then e, f are also opposite elements of $\mathcal{L}$ that are identified in $\mathcal{C}$. After relabeling elements of $\mathcal{L}$, if necessary, we may assume that $L_e \in \mathcal{L}$. Let K_f be the member corresponding to $\widetilde{K_f}$ in $b(\mathcal{L})$. Then we have that $f \in K_f$ and that $|L_e \cap K_f| = 1$, so (ii) holds. $\square$

6.6 From directed st-paths to st-paths

In §6.1 we saw that the idealness (resp. packing property, MFMC property) of st-path clutters implies that of directed st-path clutters. Here we show the converse by analyzing the conditions of Theorems 6.8, 6.11 and 6.14 for directed st-path clutters.

Let $D = (V, A)$ be a (bi)directed graph with distinguished vertices s, t. Let $e = (v, u)$ and $f = (u, v)$ be opposite arcs in D. Consider a directed st-path L_e containing e and an st-cut K_f containing f. Choose $U \subset V$ such that $s \in U \not\ni t$ and $\delta^+(U) = K_f$. Then $u \in U$ and $v \in V - U$. (See Figure 3.) Since $e \in L_e$, one can

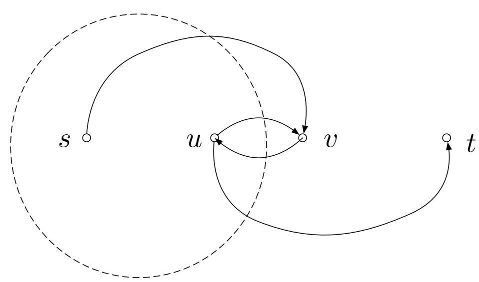


Figure 3: An illustration of $|L_e \cap K_f| \geq 2$.

decompose $L_e - \{e\}$ into a directed sv-path L_e^1 and a disjoint directed ut-path L_e^2. Then $L_e^1 \cap \delta^+(U) \neq \emptyset$ and $L_e^2 \cap \delta^+(U) \neq \emptyset$, implying that

$$(\star) \qquad |L_e \cap K_f| \geq |L_e^1 \cap K_f| + |L_e^2 \cap K_f| \geq 1 + 1 = 2.$$

Now take a directed st-path L_f containing f. Then $L_f - \{f\}$ is the disjoint union of a directed su-path L_f^1 and a directed vt-path L_f^2. (See Figure 4.) Let L be a directed st-path contained in $L_e^1 \cup L_f^2$ and let L' be a directed st-path contained in $L_f^1 \cup L_e^2$. Then

$$(\star\star) \qquad L \cup L' \subseteq (L_e \cup L_f) - \{e, f\} \quad \text{and} \quad L \cap L' \subseteq L_e \cap L_f.$$

Note that $(\star)$ implies that condition (ii) of Theorems 6.8, 6.11 and 6.14 does not hold, and that $(\star\star)$ implies that condition (ii') of Theorem 6.11 does hold for directed st-path clutters. As a result, the converse of Corollary 6.1 does hold:

Corollary 6.15. *The idealness (resp. packing property, max-flow min-cut property) of directed st-path clutters implies the idealness (resp. packing property, max-flow min-cut property) of st-path clutters.*

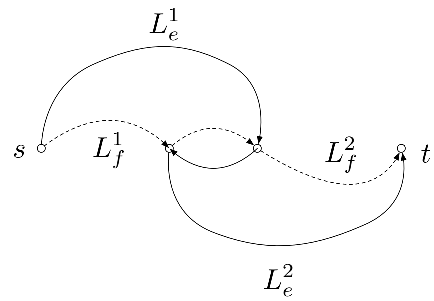


Figure 4: An illustration of L_e and L_f.

There is another way to see (the ideal case of) this corollary. Since an st-path and an st-cut intersect in an odd number of edges, it follows that a clutter of st-paths is binary. This corollary therefore could have been deduced from Corollary 5.6 (which itself uses Lehman's heavy machinery in the background).

7 Conclusion

We conclude by listing all the open problems proposed in the paper, followed by a discussion on more open problems.

(7.1) Is there a minimally non-ideal (resp. minimally non-packing) clutter that has opposite elements?

(7.2) Is there a minimally non-ideal (resp. minimally non-packing) clutter different from a delta with a proper split?

(7.3) When does a clutter of Steiner trees have an ideal split?

(7.4) When does a clutter of matroid circuits have an ideal split?

7.1 On characterizing when a split is ideal

In §6 we characterized when a given identification of an ideal clutter is non-ideal. A natural counterpart is to

(7.5) characterize when a given split $\mathcal{L}$ of a non-ideal clutter $\mathcal{C}$ is ideal.

We found this problem much more difficult to answer than the previous one. One possible reason is that this problem contains the following as a special case: characterize when the clutter of directed Steiner trees of a bidirected graph is ideal. In [3] it is argued why answering this question for directed graphs (as opposed to bidirected graphs) is likely a difficult task, the reason being that such a clutter can contain any arbitrary clutter as a minor. Nonetheless, we can still provide some necessary conditions for $\mathcal{L}$ to be ideal. As we established in Corollary 5.5, a necessary condition is for every mni minor of $\mathcal{C}$ to be a delta. Another necessary condition is provided by Theorem 6.8 (ii). Moreover, as was shown in §3.5, if each element of $\mathcal{C}$ corresponds to at most k elements of $\mathcal{L}$, then another necessary condition is that $Q(\mathcal{C})$ should have integrality gap at most k.

Perhaps an easier task is to:

(7.6) characterize when a *single* split $\mathcal{L}$ of a non-ideal clutter $\mathcal{C}$ is ideal.

Since Δ_3 is the only delta with an ideal single split, every mni minor of $\mathcal{C}$ must be a Δ_3 for $\mathcal{L}$ to be ideal. Moreover, as was shown in the proof of Proposition 6.7, another necessary condition is for all extreme points of $Q(\mathcal{C})$ to be half-integral.

7.2 Minor-closed properties preserved under splitting

We showed in §4 that idealness, the packing and MFMC properties are preserved under splitting. What about other minor-closed properties? Let P be a minor-closed property defined on clutters. We say P is *preserved under splitting* if for any clutter $\mathcal{C}$ that satisfies P, every split of $\mathcal{C}$ also satisfies P. Well, the natural question is:

(7.7) is there a characterization of minor-closed properties preserved under splitting?

A systematic approach to answer this question is as follows. The family of *forbidden minors* for P, denoted by $\mathcal{F}_P$, is the family of minimal clutters that do not satisfy P. Note that a clutter satisfies property P if, and only if, it has no minor in $\mathcal{F}_P$.

Proposition 7.1. *A minor-closed property P is preserved under splitting if, and only if, for every clutter $\mathcal{L}$ with opposite elements e, f the following holds:*

$(\star)$ *if one of $\mathcal{L}, \mathcal{L} \setminus e, \mathcal{L}/f, \mathcal{L} \setminus e/f$ is in $\mathcal{F}_P$, then the identification of $\mathcal{L}$ at e, f has a minor in $\mathcal{F}_P$.*

Proof. Suppose P is preserved under splitting, and assume that one of $\mathcal{L}, \mathcal{L} \setminus e, \mathcal{L}/f, \mathcal{L} \setminus e/f$ is in $\mathcal{F}_P$. In particular, $\mathcal{L}$ does not satisfy P, so neither does its identification $\mathcal{C}$ at e, f. This means that $\mathcal{C}$ has a minor in $\mathcal{F}_P$, proving $(\star)$.

Let us prove the converse. Suppose for a contradiction that $(\star)$ holds, but property P is not preserved under splitting. So there is a clutter $\mathcal{C}$ and a single split $\mathcal{L}$ of it at some element $e \in E(\mathcal{C})$, where $\mathcal{C}$ satisfies P but $\mathcal{L}$ does not. Since $\mathcal{L}$ does not satisfy P, it has a minor in $\mathcal{F}_P$. We may assume that this minor is one of $\mathcal{L}, \mathcal{L} \setminus e, \mathcal{L}/f, \mathcal{L} \setminus e/f$ (if an element g other than e, f is deleted or contracted, then replace both $\mathcal{C}, \mathcal{L}$ by $\mathcal{C} \setminus g, \mathcal{L} \setminus g$ or $\mathcal{C}/g, \mathcal{L}/g$). But then by $(\star)$, $\mathcal{C}$ must have a minor in $\mathcal{F}_P$, a contradiction. □

Minor-closed properties closed under identification are well-understood. We say a family $\mathcal{F}$ of clutters is *split-closed* if for each $\mathcal{C} \in \mathcal{F}$, every split of $\mathcal{C}$ has a minor in $\mathcal{F}$. For instance, as we showed in §5, the following families are split-closed:

(i) the family of minimally non-ideal clutters different from deltas,

(ii) the family of ideal, minimally non-packing clutters.

The following follows immediately from definition:

Remark 7.2. *Let $\mathcal{F}$ be a split-closed family. If clutter $\mathcal{L}$ has no minor in $\mathcal{F}$, neither does any identification of $\mathcal{L}$.*

Note as a corollary of this remark that, a minor-closed property P is closed under identification if, and only if, $\mathcal{F}_P$ is split-closed.

It follows from Remark 3.7 that

(iii) $\{\mathbb{P}_4\} \cup \{\Delta_n : n \geq 3\}$

is another split-closed family. Coincidentally, Seymour [29] proved that this is the family of forbidden minors for the class of binary clutters. Using this fact and Remark 7.2, it is shown in [2] that binary clutters do not have opposite elements. (In fact, they provide a geometric take on opposite elements.)

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