Arc connectivity and submodular flows in digraphs

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[1] Paper available on the speaker's website

k-arc-connected flips

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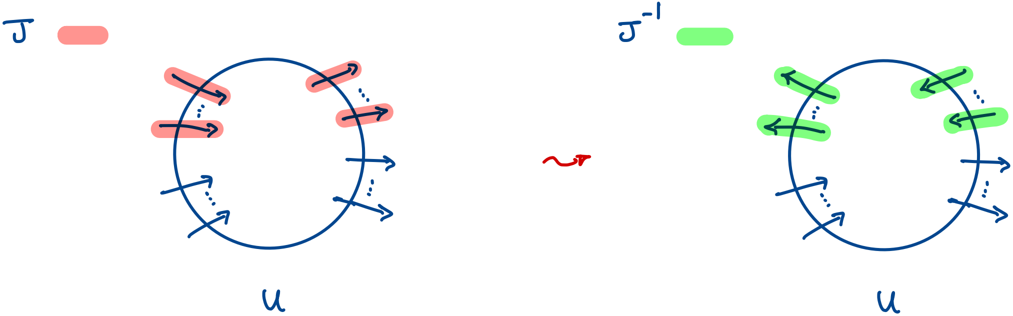
- Let $D = (V, A)$ be a digraph.
- Let $k \geq 1$ be an integer.
- A *k-arc-connected flip* is a $J \subseteq A$ such that $(D \setminus J) \cup J^{-1}$ is (strongly) k-arc-connected.



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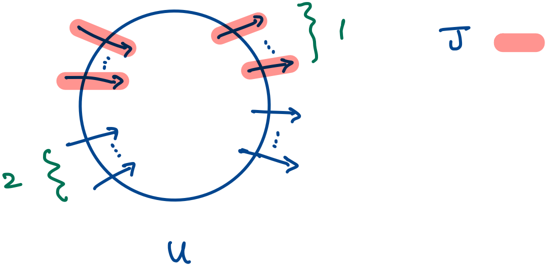
$$|J \cap \delta^+(U)| + |\delta^-(U)| - |J \cap \delta^-(U)| \geq k \qquad \forall U \subsetneq V, U \neq \emptyset.$$



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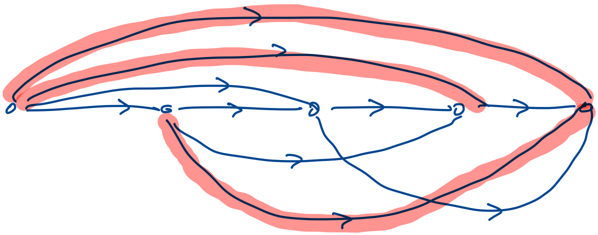
- A 2-arc-connected flip:

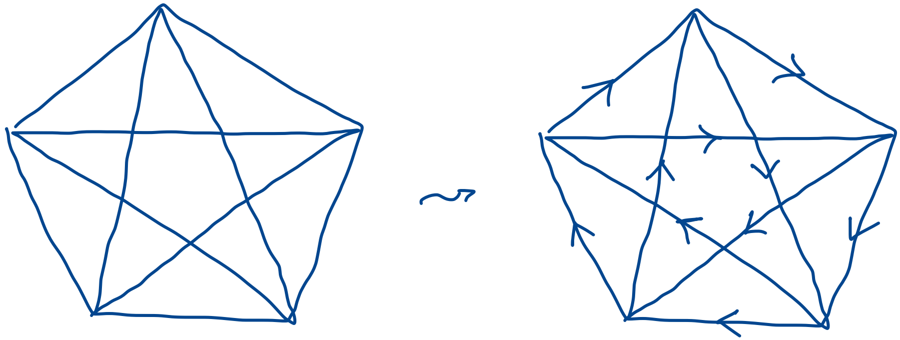


Weak Orientation Theorem

An important theorem in Graph Orientations and Submodular Optimization:

Theorem 1 (Nash-Williams '69)

If the underlying graph of D is $2k$-edge-connected, then D has a k-arc-connected flip.



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An important theorem in Graph Orientations and Submodular Optimization:

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If the underlying graph of D is $2k$-edge-connected, then D has a k-arc-connected flip.

Flash summary of the talk

1. We extend Theorem 1 by finding a k-arc-connected flip whose incidence vector is also a submodular flow.
2. This is made possible by finding capacitated integral solutions to the intersection of two submodular flow systems.

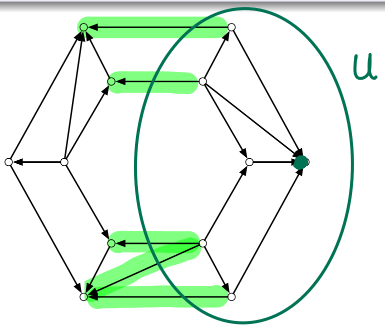
Woodall's conjecture

Dicuts

Let $D = (V, A)$ be a digraph.

Definition

A dicut is a subset of the form $\delta^+(U) \subseteq A$ where $U \neq \emptyset, V$ and $\delta^-(U) = \emptyset$.



Remark

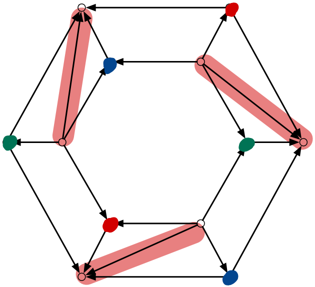
D has a dicut $\Leftrightarrow$ D is not strongly connected.

Dijoins

Let $D = (V, A)$ be a digraph.

Definition

A dijoin is a $J \subseteq A$ that intersects every dicut at least once, i.e. D/J is 1-arc-connected.



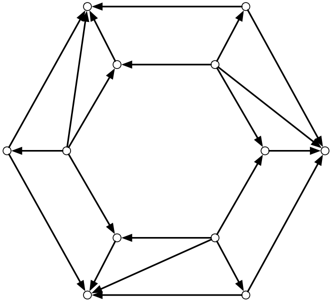
Remark

J is a dijoin $\Leftrightarrow$ $D \cup J^{-1}$ is 1-arc-connected.

Packing dijoins

Remark

Minimum size of a dicut $\geq$ maximum number of pairwise arc disjoint dijoins.



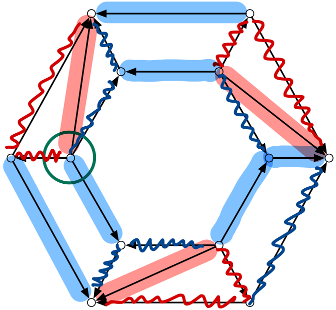
Packing dijoins

Remark

Minimum size of a dicut $\geq$ maximum number of pairwise arc disjoint dijoins.

min = 4

max = 4



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Minimum size of a dicut $\geq$ maximum number of pairwise arc disjoint dijoins.

Conjecture (Woodall 1978)

Minimum size of a dicut $=$ maximum number of pairwise arc disjoint dijoins.

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1. Frank and Tardos 1984: Formulation as a common base packing problem for two matroids.
2. Schrijver 1980, Feofiloff and Younger 1987: Proved for source-sink connected digraphs.
3. Lee and Wakabayashi 2001: Proved for series-parallel digraphs.
4. Mészáros 2018: Proved for digraphs that are $(\tau - 1, 1)$-partition-connected for τ a prime power.
5. A., Cornuéjols and Zlatin 2023: Reduced to nearly-τ-regular bipartite graphs.
6. Other interesting results by Cornuéjols and Guenin (2002), Shepherd and Vetta (2005), Lee and Williams (2006), Chudnovsky, Edwards, Kim, Scott, and Seymour (2016)

k-dijoins

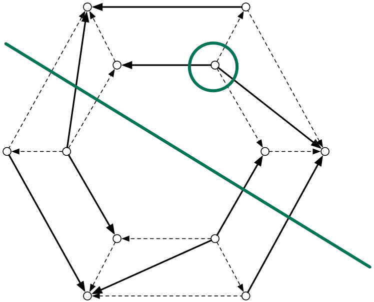
k-dijoins

Let $D = (V, A)$ be a digraph.

Definition

For an integer $k \geq 1$, a k-dijoin is an arc subset that intersects every dicut at least k times.

2-dijoin J :



— J

k-dijoins

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Proposition

The following statements hold:

1. The union of a disjoint k-dijoin and ℓ-dijoin is a $(k + \ell)$-dijoin.

k-dijoins

Let $D = (V, A)$ be a digraph.

Definition

For an integer $k \geq 1$, a k-dijoin is an arc subset that intersects every dicut at least k times.

Proposition

The following statements hold:

1. The union of a disjoint k-dijoin and ℓ-dijoin is a $(k+\ell)$-dijoin.
2. Every k-arc-connected flip J is a k-dijoin.

Proof of 2. Let J be a k-arc-connected flip. For every dicut $\delta^+(U)$,

$$|J \cap \delta^+(U)| = |J \cap \delta^+(U)| + |\delta^-(U)| - |J \cap \delta^-(U)| \geq k.$$

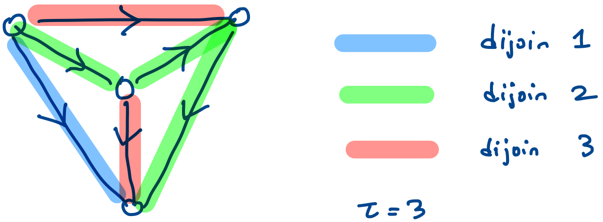
□

Decomposing into a k- and $(\tau - k)$-dijoin

Suppose every dicut of $D = (V, A)$ has size $\geq \tau$, for some integer $\tau \geq 2$.

Conjecture (Woodall 1978)

A can be partitioned into τ dijoins.

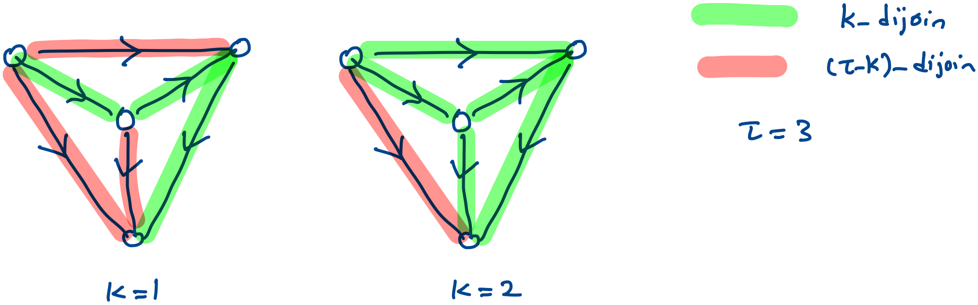


Decomposing into a k- and $(\tau - k)$-dijoin

Suppose every dicut of $D = (V, A)$ has size $\geq \tau$, for some integer $\tau \geq 2$.

Conjecture ("Weak Woodall")

A can be partitioned into a k- and a $(\tau - k)$-dijoin, for all $k \in \{1, \ldots, \tau - 1\}$.



Decomposing into a k- and $(\tau - k)$-dijoin

Suppose every dicut of $D = (V, A)$ has size $\geq \tau$, for some integer $\tau \geq 2$.

Theorem 2 (A., Cornuéjols, Zlatin 2023)

A can be partitioned into a dijoin and a $(\tau - 1)$-dijoin.

Summary

Theorem 1 (Nash-Williams '69)

If the underlying graph of D is $2k$-edge-connected, then D has a k-arc-connected flip.

Theorem 2 (A., Cornuéjols, Zlatin 2023)

If every dicut of $D = (V, A)$ has size $\geq \tau$, then A can be partitioned into a dijoin and a $(\tau - 1)$-dijoin.

Remark

A k-arc-connected flip is a k-dijoin.

A common extension of Theorems 1 and 2

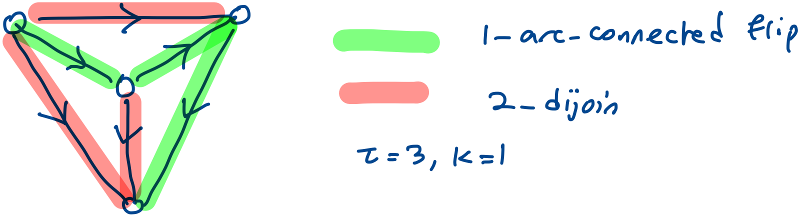
A common extension

Theorem 3 (A., Cornuéjols, Zambelli '23+)

Let $D = (V, A)$ be a digraph such that for some integers $\tau - 1 \geq k \geq 1$ we have

$$|\delta^+(U)| + \left(\frac{\tau}{k} - 1\right) |\delta^-(U)| \geq \tau \qquad \forall U \subsetneq V, U \neq \emptyset.$$

Then A can be partitioned into a k-arc-connected flip and a $(\tau - k)$-dijoin.



A common extension

For $\tau = 2k$ we recover:

Theorem 1 (Nash-Williams '69)

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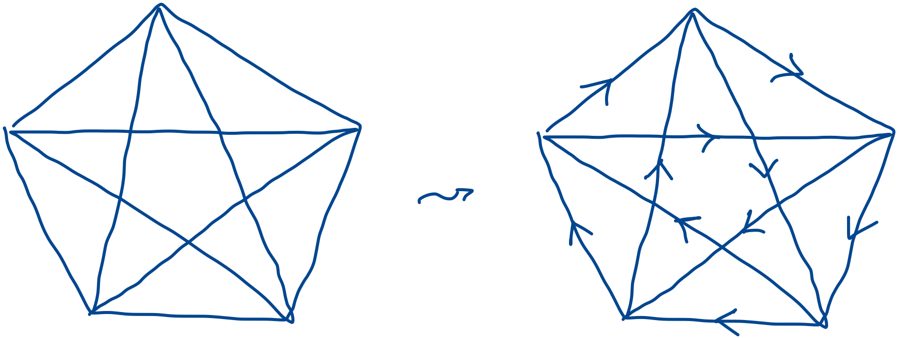

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For $\tau = 2k$ we recover:

Theorem 1 (Nash-Williams '69)

If the underlying graph of D is $2k$-edge-connected, then D has a k-arc-connected flip.

Why? The cut condition

$$|\delta^+(U)| + \left(\frac{\tau}{k} - 1\right) |\delta^-(U)| \geq \tau \qquad \forall U \subsetneq V, U \neq \emptyset.$$

becomes

$$|\delta^+(U)| + |\delta^-(U)| \geq \tau \qquad \forall U \subsetneq V, U \neq \emptyset.$$

A common extension

For $k = 1$ we recover:

Theorem 2 (A., Cornuéjols, Zlatin 2023)

If every dicut of $D = (V, A)$ has size $\geq \tau$, then A can be partitioned into a dijoin and a $(\tau - 1)$-dijoin.

A common extension

For $k = 1$ we recover:

Theorem 2 (A., Cornuéjols, Zlatin 2023)

If every dicut of $D = (V, A)$ has size $\geq \tau$, then A can be partitioned into a dijoin and a $(\tau - 1)$-dijoin.

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$$|\delta^+(U)| + \left(\frac{\tau}{k} - 1\right) |\delta^-(U)| \geq \tau \qquad \forall U \subsetneq V, U \neq \emptyset.$$

becomes

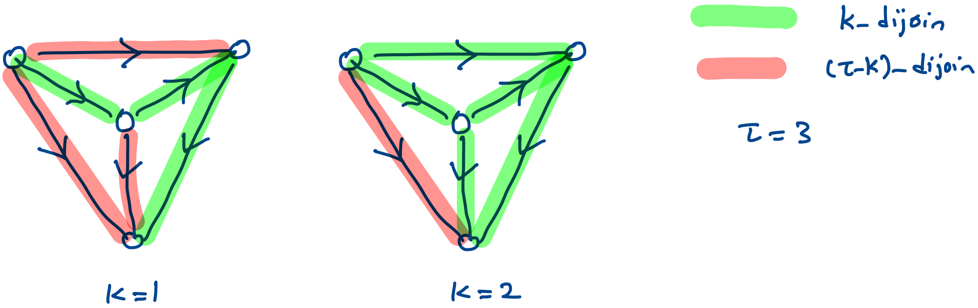
$$|\delta^+(U)| \geq \tau \qquad \forall U \subsetneq V, U \neq \emptyset \text{ s.t. } \delta^-(U) = \emptyset$$

A common extension

"Weak Woodall" is true for τ-edge-connected instances:

Theorem (A., Cornuéjols, Zambelli '23+)

Suppose the underlying graph of $D = (V, A)$ is τ-edge-connected. Then A can be partitioned into a k- and a $(\tau - k)$-dijoin, for all $k \in \{1, \ldots, \tau - 1\}$.



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Proof. We may assume $\tau \geq 2k$. The cut condition holds:

$$|\delta^+(U)| + \underbrace{\left(\frac{\tau}{k} - 1\right)}_{\geq 1} |\delta^-(U)| \geq |\delta^+(U)| + |\delta^-(U)| \geq \tau \qquad \forall U \subsetneq V, U \neq \emptyset.$$

$\therefore$ $\exists$ a partition into a k-arc-connected flip & a $(\tau-k)$-dijoin. $\square$

A common extension

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Let $D = (V, A)$ be a digraph such that for some integers $\tau - 1 \geq k \geq 1$ we have

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Then A can be partitioned into a k-arc-connected flip and a $(\tau - k)$-dijoin.

Let's prove this theorem. We need two ingredients.

Ingredients

Crossing families and submodular functions

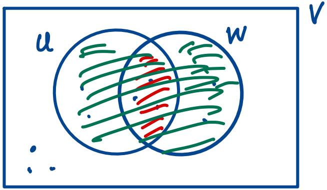
Let $\mathcal{C}$ be a family of subsets of V, and let $f : \mathcal{C} \to \mathbb{Z}$ be a function.

Definition

$\mathcal{C}$ is a crossing family if whenever $U, W \in \mathcal{C}$ and $U \cap W \neq \emptyset, U \cup W \neq V$, then $U \cap W, U \cup W \in \mathcal{C}$.

Definition

f is crossing submodular if whenever $U, W \in \mathcal{C}$ and $U \cap W \neq \emptyset, U \cup W \neq V$, then $f(U \cap W) + f(U \cup W) \leq f(U) + f(W)$.



Crossing families and submodular functions

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Examples

- $\{U \subsetneq V : U \neq \emptyset\}$
- $\{U \subsetneq V : U \neq \emptyset, \delta^-(U) = \emptyset\}$
- $f(U) := |\delta^+(U)|$
- $f(U) := |\delta^+(U)| - \alpha$

Ingredient 1 from Submodular Optimization

Theorem (Edmonds, Frank, Fujishige; see Schrijver 2003, Frank 2011)

Let $f_i : \mathcal{C}_i \to \mathbb{Z}$ be a crossing submodular function, for $i = 1, 2$. Then

$$
\begin{aligned}
& x \in \mathbb{R}^V \\
& \mathbf{1}^\top x = 0 \\
x(U) := {} & \sum_{v \in U} x_v \leq f_1(U) \quad \forall U \in \mathcal{C}_1 \\
& \sum_{v \in U} x_v \leq f_2(U) \quad \forall U \in \mathcal{C}_2
\end{aligned}
$$

is box-totally dual integral, and therefore box-integral. In particular, if it has a fractional solution, it has an integral solution.

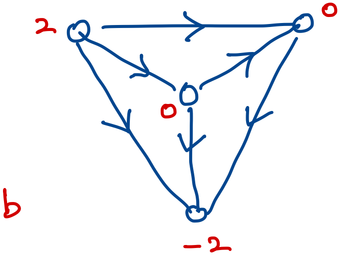
Ingredient 2 from Network Flows

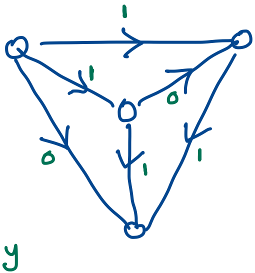
Let $D = (V, A)$ be a digraph, and $b \in \mathbb{Z}^V$ s.t. $1^\top b = 0$.

Definition

A *b-transshipment* is a vector $y \in \mathbb{R}^A$ s.t.

$$y(\delta^+(u)) - y(\delta^-(u)) = b_u \qquad \forall u \in V.$$





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$$y(\delta^+(u)) - y(\delta^-(u)) = b_u \qquad \forall u \in V.$$

Theorem (Hoffman, Gale; see Schrijver 2003)

Take $c, d \in \mathbb{Z}^A$ with $c \leq d$ such that

$$b(U) \leq d(\delta^+(U)) - c(\delta^-(U)) \qquad \forall U \neq \emptyset, V.$$

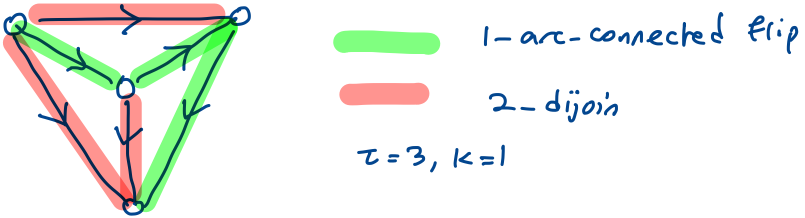
Then there exists a b-transshipment $y^\star \in \mathbb{Z}^A$ such that $c \leq y^\star \leq d$.

Proof of Theorem 3

Theorem 3 (A., Cornuéjols, Zambelli '23+)

A can be decomposed into a k-arc-connected flip and a $(\tau - k)$-dijoin if for every $\emptyset \neq U \subsetneq V$,

$$|\delta^+(U)| + \left(\frac{\tau}{k} - 1\right) |\delta^-(U)| \geq \tau.$$



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$$|\delta^+(U)| + \left(\frac{\tau}{k} - 1\right) |\delta^-(U)| \geq \tau.$$

Proof. We need to find a $0, 1$ vector y such that

- $y(\delta^+(U)) - y(\delta^-(U)) \leq |\delta^+(U)| - k$ for every $\emptyset \neq U \subsetneq V$,
- $y(\delta^+(U)) - y(\delta^-(U)) \leq |\delta^+(U)| - (\tau - k)$ for every dicut $\delta^+(U)$.

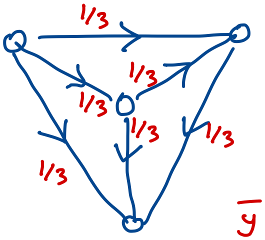
Proof continued

We know

- $|\delta^+(U)| + \left(\frac{\tau}{k} - 1\right)|\delta^-(U)| \geq \tau$ for every $\emptyset \neq U \subsetneq V$,
- Every dicut has size at least τ.

Proof. Let $\bar{y} \in \mathbb{R}^A$ assign $\frac{k}{\tau}$ to every arc. Then

- $\bar{y}(\delta^+(U)) - \bar{y}(\delta^-(U)) \leq |\delta^+(U)| - k$ for every $\emptyset \neq U \subsetneq V$,
- $\bar{y}(\delta^+(U)) - \bar{y}(\delta^-(U)) \leq |\delta^+(U)| - (\tau - k)$ for every dicut $\delta^+(U)$.



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For each vertex v, let $\bar{x}_v := \bar{y}(\delta^+(v)) - \bar{y}(\delta^-(v))$. Then

1. $\mathbf{1}^\top \bar{x} = 0$,
2. $\bar{x}(U) \leq |\delta^+(U)| - k$ for every $\emptyset \neq U \subsetneq V$,
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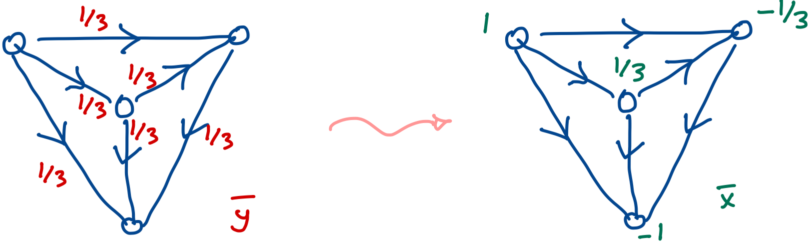

Proof continued

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For each vertex v, let $\bar{x}_v := \bar{y}(\delta^+(v)) - \bar{y}(\delta^-(v))$. Then

1. $1^\top \bar{x} = 0$,
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3. $\bar{x}(U) \leq |\delta^+(U)| - (\tau - k)$ for every dicut $\delta^+(U)$.

Consequence of Ingredient 1

We can make $\bar{x}$ integral!

Proof continued

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- $\bar{y}(\delta^+(U)) - \bar{y}(\delta^-(U)) \leq |\delta^+(U)| - k$ for every $\emptyset \neq U \subsetneq V$,
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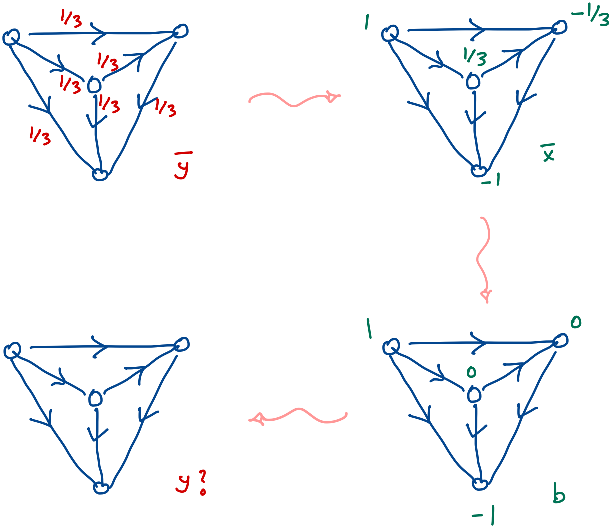
Consequence of Ingredient 1

We can make $\bar{x}$ integral!

But, can we make $\bar{y}$ $0, 1$?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Proof continued

There exists $b \in \mathbb{Z}^V$ such that

1. $1^\top b = 0$,
2. $b(U) \leq |\delta^+(U)| - k$ for every $\emptyset \neq U \subsetneq V$,
3. $b(U) \leq |\delta^+(U)| - (\tau - k)$ for every dicut $\delta^+(U)$

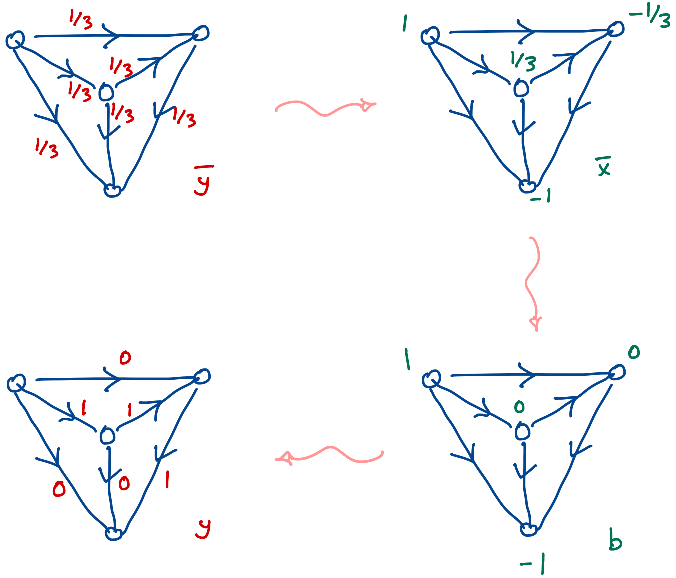
Consequence of Ingredient 2

Since $b(U) \leq |\delta^+(U)|$ for every $\emptyset \neq U \subsetneq V$, there exists a b-transshipment $y^\star \in \{0,1\}^A$, i.e. $y^\star(\delta^+(v)) - y^\star(\delta^-(v)) = b_v$ for every vertex v.

Then,

- $y^\star(\delta^+(U)) - y^\star(\delta^-(U)) \leq |\delta^+(U)| - k$ for every $\emptyset \neq U \subsetneq V$,
- $y^\star(\delta^+(U)) - y^\star(\delta^-(U)) \leq |\delta^+(U)| - (\tau - k)$ for every dicut $\delta^+(U)$,

as required. □



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Digging a little deeper

A more powerful theorem

Theorem 4 (A., Cornuéjols, Zambelli '23+)

Let $D = (V, A)$ be a digraph, and τ, k integers with $\tau - 1 \geq k \geq 1$. Let $f : \mathcal{C} \to \mathbb{Z}$ be a crossing submodular function. Suppose the system

$$\begin{aligned} y(\delta^+(U)) - y(\delta^-(U)) &\leq |\delta^+(U)| - k \qquad && \forall U \subsetneq V, U \neq \emptyset \\ y(\delta^+(U)) - y(\delta^-(U)) &\leq f(U) && \forall U \in \mathcal{C} \end{aligned}$$

is satisfied at $\bar{y} = \frac{k}{\tau}\mathbf{1}$. Then the system has a $0,1$ solution.

A more powerful theorem

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For $f(U) := |\delta^+(U)| - (\tau - k)$ defined on every dicut $\delta^+(U)$, this gives

Theorem 3 (A., Cornuéjols, Zambelli '23+)

A can be decomposed into a k-arc-connected flip and a $(\tau - k)$-dijoin if for every $\emptyset \neq U \subsetneq V$,

$$|\delta^+(U)| + \left(\frac{\tau}{k} - 1\right) |\delta^-(U)| \geq \tau.$$

Applications

For $\tau = 2k$, Theorem 4 gives

Theorem (A., Cornuéjols, Zambelli '23+)

Let $D = (V, A)$ be a digraph whose underlying graph is $2k$-edge-connected. Let $f : \mathcal{C} \to \mathbb{Z}$ be a crossing submodular function such that

$$f(U) \geq \frac{1}{2}(|\delta^+(U)| - |\delta^-(U)|) \qquad \forall U \in \mathcal{C}.$$

Then there is a k-arc-connected flip $J \subseteq A$ such that

$$f(U) \geq |J \cap \delta^+(U)| - |J \cap \delta^-(U)| \qquad \forall U \in \mathcal{C}.$$

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Corollary (Nash-Williams '69)

Every $2k$-edge-connected graph has a k-arc-connected orientation such that at every node, the out- and in-degrees differ by at most one.

Discussion

We find a $0,1$ solution to

$$\begin{aligned} y(\delta^+(U)) - y(\delta^-(U)) &\leq |\delta^+(U)| - k & \forall\, \emptyset \neq U \subsetneq V \\ y(\delta^+(U)) - y(\delta^-(U)) &\leq f(U) & \forall\, U \in \mathcal{C}, \end{aligned}$$

the intersection of two 'submodular flow' systems. This is surprising...

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Folklore facts

Let $D = (V, A)$ be a digraph, and let $f_i : \mathcal{C}_i \to \mathbb{Z}$ be a crossing submodular function, for $i = 1, 2$. Consider the system

$$\begin{aligned} y &\in \mathbb{R}^A \\ y(\delta^+(U)) - y(\delta^-(U)) &\leq f_1(U) \quad \forall U \in \mathcal{C}_1 \\ y(\delta^+(U)) - y(\delta^-(U)) &\leq f_2(U) \quad \forall U \in \mathcal{C}_2. \end{aligned}$$

This system is not necessarily integral, let alone box-totally dual integral. Moreover, finding a $0,1$ solution to this system is an NP-hard task.

A surprising phenomenon

Intersection of two submodular flow systems

- Let $D = (V, A)$ be a digraph.
- Let $f_i : \mathcal{C}_i \to \mathbb{Z}$ be a crossing submodular function, for $i = 1, 2$.

Theorem 5 (A., Cornuéjols, Zambelli '23+)

1. Suppose $\min_{i=1,2} f_i(U) \leq 0$ for all $U \neq \emptyset, V$ s.t. $\delta^+(U) = \delta^-(U) = \emptyset$. Then

$$y(\delta^+(U)) - y(\delta^-(U)) \leq f_1(U) \quad \forall U \in \mathcal{C}_1$$
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is totally dual integral, and hence integral.

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2. Take $c, d \in \mathbb{Z}^A$ satisfying $c \leq d$ and

$$\min_{i=1,2} f_i(U) \leq d(\delta^+(U)) - c(\delta^-(U)) \qquad \forall U \neq \emptyset, V.$$

   Then every nonempty face of the feasible region contains $y^\star \in \mathbb{Z}^A$ with $c \leq y^\star \leq d$.

Applications

Theorem 4 (A., Cornuéjols, Zambelli '23+)

Let $D = (V, A)$ be a digraph, and τ, k integers with $\tau - 1 \geq k \geq 1$. Let $f : \mathcal{C} \to \mathbb{Z}$ be a crossing submodular function. Suppose the system

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is satisfied at $\bar{y} = \frac{k}{\tau}\mathbf{1}$. Then the system has a $0,1$ solution.

Applications

Theorem (Edmonds and Giles 1977)

Let $D = (V, A)$ be a digraph, and $f : \mathcal{C} \to \mathbb{Z}$ a crossing submodular function. The system

$$y(\delta^+(U)) - y(\delta^-(U)) \leq f(U) \qquad \forall U \in \mathcal{C}$$

is box-totally dual integral, and hence box-integral.

A cute application

Theorem (A., Cornuéjols, Zambelli '23+)

Let $D = (V, A)$ be a weakly connected digraph. Let $f_i : \mathcal{C}_i \to \mathbb{Z}$ be a crossing submodular function, for $i = 1, 2$. Then

$$y(\delta^+(U)) - y(\delta^-(U)) \leq f_1(U) \quad \forall U \in \mathcal{C}_1$$
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This system is not necessarily box-integral.

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Proposition

This system is not necessarily box-integral.

Proposition (Goemans and Pan)

This system is not necessarily box-half-integral.

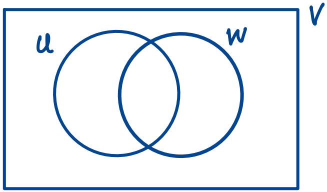
A conjecture

Tree Orientation Conjecture

Let $\mathcal{C}$ be a family of subsets of V.

Definition

$\mathcal{C}$ is a lattice family if $U \cap W, U \cup W \in \mathcal{C}$ for all $U, W \in \mathcal{C}$.



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Tree Orientation Conjecture

Let $T = (V, E)$ be a tree, and let $\mathcal{C}$ be a lattice family over ground set V. Suppose

$$|\delta_T(U)| \geq 2 \qquad \forall U \in \mathcal{C}, U \neq \emptyset, V.$$

Then there is an orientation $\vec{T}$ of T such that

$$\delta^+_{\vec{T}}(U), \delta^-_{\vec{T}}(U) \neq \emptyset \quad \forall U \in \mathcal{C}, U \neq \emptyset, V.$$

Thanks!