Integral bases, perfect matchings, and the Petersen graph

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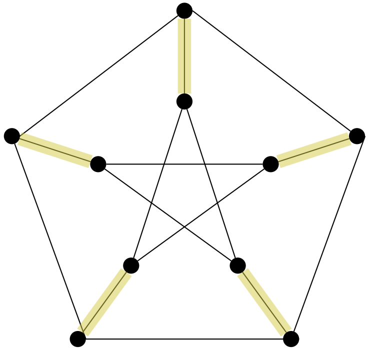
The Petersen graph

Let $G = (V, E)$ be a bridgeless cubic graph.

Fact

G is not necessarily 3-edge-colorable, i.e., E does not necessarily decompose into 3 perfect matchings.



Two conjectures in graph theory

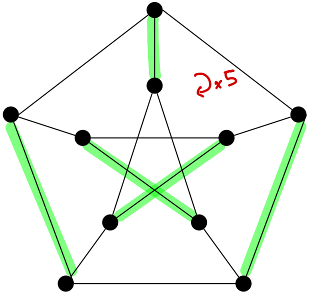
Let $G = (V, E)$ be a bridgeless cubic graph.

The Berge-Fulkerson conjecture (Fulkerson '71)

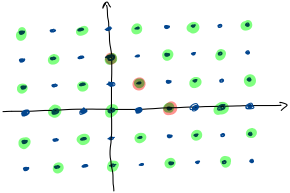
G has six perfect matchings using every edge exactly twice. That is, $(2, \ldots, 2)^\top \in \mathbb{Z}^E$ is an integer conic combination of $\{1_M : M \text{ a perfect matching}\}$.

The 4-flow conjecture (Tutte '66)

If G has no Petersen graph as a minor, then G is 3-edge-colorable.



Dropping nonnegativity



The matching lattice of $G = (V, E)$ is

$$L(G) := \text{ the set of integer linear combinations of } \{1_M : M \subseteq E \text{ perfect matching}\}.$$

Theorem (Seymour '79)

Let $G = (V, E)$ be a bridgeless cubic graph. Then

- $(2, 2, \ldots, 2)^\top \in L(G)$,
- if G has no Petersen graph as a minor, then $(1, 1, \ldots, 1)^\top \in L(G)$!

Proof is graph-theoretic and inductive, but terse.

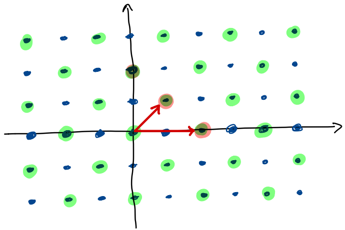
Bases and description of the matching lattice

Theorem

Suppose G is matching-covered, i.e., every edge belongs to some perfect matching. Then

- $(2, 2, \ldots, 2)^\top \in L(G)$ (Lovász '87)
- $(1, 1, \ldots, 1)^\top \in L(G)$ if G is Petersen-free, in particular if G has no Petersen minor (Lovász '87)
- $L(G)$ has a lattice basis in $\{1_M : M \subseteq E \text{ perfect matching}\}$ (Carvalho, Lucchesi, and Murty '02)

Petersen-free = no Petersen brick



Bases and description of the matching lattice

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- $L(G)$ has a lattice basis in $\{1_M : M \subseteq E \text{ perfect matching}\}$ (Carvalho, Lucchesi, and Murty '02)

- The proofs are structural graph-theoretic and inductive.
- Highly technical, long, and rely on many prerequisites in matching theory (3 papers and ≈ 120 pages for the last one).
- The crux in all the proofs is in finding the Petersen graph as a brick, a very special type of a minor.

Our contributions

- simultaneous short proof for both results (≈ 10 pages) that is conceptually simple,
- the first polyhedral proof of the results,
- minimal dependence on matching-theoretic notions,
- direct proof as it does not move to the dual lattice, unlike Lovász.

Our approach

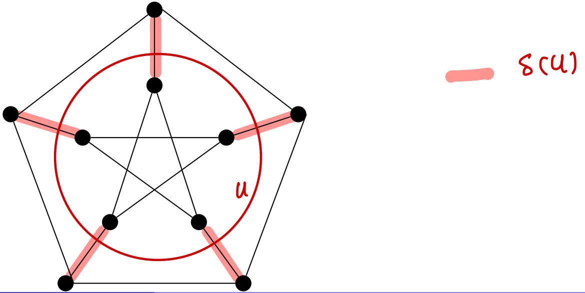
The perfect matching polytope

Theorem (Edmonds '65)

The perfect matching polytope $P(G)$ is described by

$$
\begin{aligned}
x(\delta(v)) &= 1 && v \in V \\
x(\delta(U)) &\geq 1 && U \subset V, 3 \leq |U| \leq |V| - 3 \text{ odd} \\
x_e &\geq 0 && e \in E
\end{aligned}
$$

(odd cut inequality)



Peeling the onion, step I

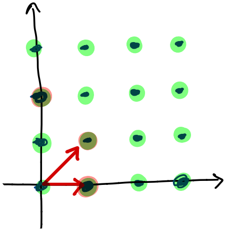
We show the main result follows relatively easily from the following:

The Integral Basis Theorem

Suppose $G = (V, E)$ is Petersen-free. Then the linear hull of $P(G)$ has an integral basis

$$B \subseteq \{1_M : M \text{ a perfect matching}\}$$

that is, every integral vector in $\text{lin}(P(G))$ is an integer linear combination of B.



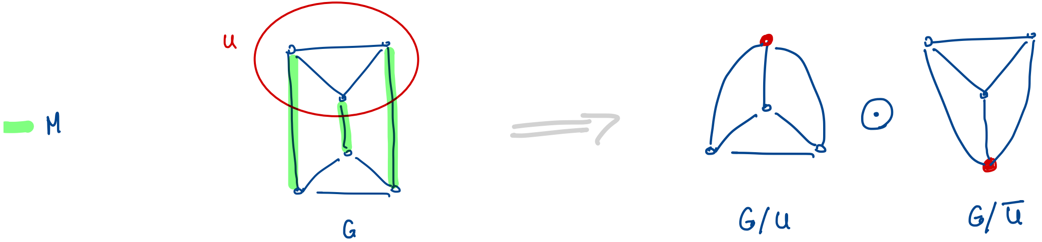
Peeling the onion, step I

The Integral Basis Theorem

Suppose $G = (V, E)$ is Petersen-free. Then the linear hull of $P(G)$ has an integral basis $B \subseteq \{1_M : M \text{ a perfect matching}\}$.

Key Idea

Find a facet-defining inequality $x(\delta(U)) \geq 1$ such that $G/U, G/\overline{U}$ are Petersen-free matching-covered, and there is a perfect matching M such that $|M \cap \delta(U)| = 3$.



Peeling the onion, step II

Let $G = (V, E)$ be a Petersen-free matching-covered graph.

Goal

Find a facet-defining inequality $x(\delta(U)) \geq 1$ such that $G/U, G/\overline{U}$ are Petersen-free matching-covered, and there is a perfect matching M such that $|M \cap \delta(U)| = 3$.

The Intersection Theorem

The above can be guaranteed if $P(G)$ has at least one facet defined by an odd cut inequality, and dimension $|E| - |V| =: d$.

$$\begin{aligned} x(\delta(v)) &= 1 \quad && v \in V \\ x(\delta(U)) &\geq 1 \quad && U \subset V, 3 \leq |U| \leq |V| - 3 \text{ odd} \\ x_e &\geq 0 \quad && e \in E \end{aligned}$$

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Key Ideas

1. If $P(G)$ has a facet of the form $x_e \geq 0$, apply induction to $G \setminus e$ for a suitably chosen e.

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The above can be guaranteed if $P(G)$ has at least one facet defined by an odd cut inequality, and dimension $|E| - |V| =: d$.

Key Ideas

1. If $P(G)$ has a facet of the form $x_e \geq 0$, apply induction to $G \setminus e$ for a suitably chosen e.
2. If $P(G)$ has a $(d-2)$-dimensional face F not described by any $x_e \geq 0$, then apply induction to G/U where $x(\delta(U)) \geq 1$ is a facet 'derived' from F.

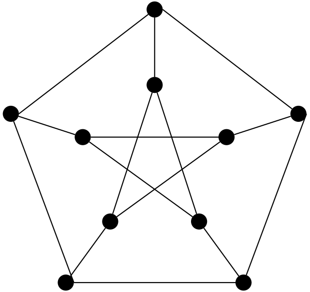
The base case: finding the Petersen graph

The Petersen Graph Lemma (Abdi and Silina '25+)

Suppose G is matching-covered where

- $P(G)$ has at least one facet defined by an odd cut inequality,
- $|M \cap \delta(U)| \neq 3$ for any perfect matching M and any facet-defining inequality $x(\delta(U)) \geq 1$,
- $P(G)$ has dimension $d = |E| - |V|$,
- $P(G)$ has no facet of the form $x_e \geq 0$,
- every $(d-2)$-dimensional face is described by some $x_e \geq 0$,
- minimum degree is at least 3.

Then G is the Petersen graph.



Proof of the Petersen Graph Lemma

The proof

- $P(G)$ has dimension $d = |E| - |V|$,
- $P(G)$ has no facet of the form $x_e \geq 0$,
- every $(d-2)$-dimensional face is described by some $x_e \geq 0$,
- minimum degree ≥ 3.

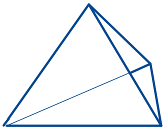
$v := |V|$

$e := |E|$

$f :=$ # facets

$t :=$ # $(d-2)$-faces

Notes:
- $f \geq d+1$
- every facet is adjacent to $\geq d$ facets



$d = 3$

The proof

- $P(G)$ has dimension $d = |E| - |V|$,
- $P(G)$ has no facet of the form $x_e \geq 0$,
- every $(d-2)$-dimensional face is described by some $x_e \geq 0$,
- minimum degree ≥ 3.

$v := |V|$

$e := |E|$

$f := \#$ facets

$t := \#\ (d-2)$-faces

Then

$$e \geq t \geq \frac{fd}{2} \geq \binom{d+1}{2} = \binom{e-v+1}{2} \quad \text{and} \quad 2e \geq 3v.$$

b/c every facet is adj to ≥ d facets

b/c f ≥ d+1

b/c min-deg ≥ 3

Thus, $v \leq 10$, and if $v = 10$ then equality holds throughout.

The proof

- $P(G)$ has at least one facet defined by an odd cut inequality,
- $|M \cap \delta(U)| \neq 3$ for any perfect matching M and any facet-defining inequality $x(\delta(U)) \geq 1$.

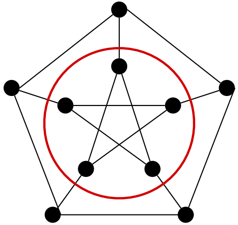
$\Rightarrow$

$$v = 10 \quad \text{and} \quad e = t = \frac{fd}{2} = \binom{d+1}{2} = \binom{e-v+1}{2} \quad \text{and} \quad 2e = 3v.$$

$\Rightarrow e = 15$, $f = 6$, and G is a cubic graph.

...

$\Rightarrow G$ must be



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Thank you!